Spring, 2013 CIS 511

Introduction to the Theory of Computation
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Practice Final Exam

April 15, 2013

Note that this is a **closed-book exam**.
Do the exam in 120 mn.

Solutions will be emailed on April 23.

Problem 1 (10 pts). Given an alphabet Σ, sketch an algorithm to decide whether

$$(R^* + S^*) \cong \Sigma^*,$$

for any two regular expressions R and S over Σ.

Problem 2 (20 pts). Let Σ be an alphabet. Recall that a binary relation, $\sim$, on Σ^*, is *left invariant* iff $u \sim v$ implies that $wu \sim wv$ for all $w \in \Sigma^*$ and *right invariant* iff $u \sim v$ implies that $uw \sim vw$ for all $w \in \Sigma^*$. An equivalence relation on Σ^* that is both left and right-invariant is called a *congruence*. Recall that a congruence satisfies the property: If $u \sim u'$ and $v \sim v'$, then $uv \sim u'v'$ (You **do not** have to prove this). Also recall that there is a version of the Myhill-Nerode theorem that says that a language, L, is regular iff it is the union of equivalence classes of a congruence with a finite number of equivalence classes. (You **do not** have to prove this). Finally, recall that the reversal of a string, $w \in \Sigma^*$, is defined inductively as follows:

$$\begin{aligned} \epsilon^R &= \epsilon \\ (ua)^R &= au^R, \end{aligned}$$

for all $u \in \Sigma^*$ and all $a \in \Sigma$.

(i) Let $\sim$ be a congruence (on Σ^*) and assume that $\sim$ has n equivalence classes. Define $\sim_R$ and $\approx$ by

$$u \sim_R v \quad \text{iff} \quad u^R \sim v^R, \quad \text{for all } u, v \in \Sigma^* \quad \text{and} \quad \approx \,=\, \sim \cap \sim_R .$$

The relation $\approx$ is clearly a congruence (You **do not** have to prove this). Prove that $\approx$ has at most n^2 equivalence classes.

(ii) Given any regular language, L, over Σ^* let

$$L' = \{w \in \Sigma^* \mid ww^R \in L\}.$$

Prove that L' is also regular, using the relation $\approx$ of part (i).
Hint. Use the usual version of the Myhill-Nerode theorem applied to the relation $\approx$.

Problem 3 (20 pts). Prove that the following languages are not regular:

$$\begin{aligned} L_1 &= \{a^m b^n c^m \mid m, n \geq 1\}, \\ L_2 &= \{a^n \mid n \text{ is not a prime}\}. \end{aligned}$$

Problem 4 (10 pts). Give a context-free grammar for the language:

$$L_3 = \{a^m b^m c a^{2n} b^{2n} \mid m, n \geq 1\},$$

where $\Sigma = \{a, b, c\}$.

Problem 5 (20 pts). Give an algorithm to decide whether $L \subseteq \{aa, bb\}^*$, where $L \subseteq \{a, b\}^*$ is any context-free language.

Problem 6 (25 pts). (i) Prove that the following sets are not recursive ($\varphi_1, \varphi_2, \ldots, \varphi_i, \ldots$ is any acceptable indexing of the partial recursive functions):

$$\begin{aligned} A &= \{i \in \mathbb{N} \mid \varphi_i(0) = \varphi_i(1) \quad \text{and} \quad \varphi_i(0), \varphi_i(1) \text{ are both defined}\} \\ B &= \{i \in \mathbb{N} \mid \varphi_i = \varphi_a + \varphi_b\} \\ C &= \{\langle i, j, k\rangle \in \mathbb{N} \mid \varphi_i = \varphi_j + \varphi_k\} \\ D &= \{i \in \mathbb{N} \mid \varphi_i \text{ diverges for exactly one input}\} \end{aligned}$$

where a and b are two fixed natural numbers.

(ii) Prove that A is recursively enumerable.

Problem 7 (15 pts). Consider the version of the tiling problem where an instance of the problem is of the form $((\mathcal{T}, V, H), \widehat{s}, \sigma_0)$, where $(\mathcal{T}, V, H)$ is a tiling system as before, but $\widehat{s}$ is s *in tally notation* and σ_0 is a single tile placed in position $(1, 1)$. Recall that the tally notation of a natural number n is $1 \cdots 1$ with $n + 1$ occurrences of "1". For example, the tally notation for the decimal number 10 is 11111111111.

Prove that this new tiling problem is also $\mathcal{NP}$-complete.