Fall 2021 CIS 511

Introduction to the Theory of Computation
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Homework 6

November 26, 2021; Due December 10, 2021

Problem B1 (20 pts). Let A, B, C, D be the following sets:

$$\begin{aligned}
A &= \{x \in \mathbb{N} \mid \varphi_x \text{ is constant}\},\\
B &= \{\langle x, y\rangle \mid \varphi_x = \varphi_y\},\\
C &= \{x \in \mathbb{N} \mid \varphi_x = \varphi_a\},\\
D &= \{x \in \mathbb{N} \mid \varphi_x \text{ is undefined for all input}\},
\end{aligned}$$

where a is a given natural number. Prove that the above sets are not computable (not recursive).

Problem B2 (40 pts). Given any set, X, for any subset, $A \subseteq X$, recall that the *characteristic function*, χ_A, of A is the function defined so that

$$\chi_A(x) = \begin{cases} 1 & \text{iff } x \in A \\ 0 & \text{iff } x \in X - A. \end{cases}$$

(i) Prove that, for any two subsets, $A, B \subseteq X$,

$$\begin{aligned}
\chi_{A\cap B} &= \chi_A \cdot \chi_B\\
\chi_{A\cup B} &= \chi_A + \chi_B - \chi_A \cdot \chi_B.
\end{aligned}$$

(ii) Prove that the union and the intersection of any two Diophantine sets $A, B \subseteq \mathbb{N}$, is also Diophantine.

(iii) Prove that the union and the intersection of any two listable sets $A, B \subseteq \mathbb{N}$, is also listable.

(iv) Prove that the union and the intersection of any two computable (recursive) sets, $A, B \subseteq \mathbb{N}$, is also a computable set (a recursive set).

Problem B3 (20 pts). (1) Consider the function $rem\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined such that if $n > 0$, then $rem(m, n) = r$ is the remainder of the division of m by n, namely the unique

$r \in \mathbb{N}$ such that $r < n$ and $m = nq + r$ for some $q \in \mathbb{N}$, else $rem(m, 0) = m$. Prove that rem is primitive recursive.

Hint. Use bounded minimization. In your justification, distinguish the cases $m < n$, $m \geq n > 0$, and $n = 0$.

(2) Prove that there is a diophantine polynomial $P(m, n, r, q, v)$ such that

$$r = rem(m, n+1) \quad \text{iff} \quad (\exists q, v)(P(m, n, r, q, v) = 0)$$

for all $m, n, r \in \mathbb{N}$.

Problem B4 (20 pts). Recall that the *floor function* is defined such that for any nonnegative real number x, the floor $\lfloor x \rfloor$ of x is the unique natural number $m \in \mathbb{N}$ such that

$$m \leq x < m + 1.$$

(1) What is the the function f (from $\mathbb{N}$ to $\mathbb{N}$) whose graph $\{(x, y) \in \mathbb{N}^2 \mid y = f(x)\}$ is defined by the polynomial

$$P(x, y, u, v) = (x - y^2 - u)^2 + (x + 1 + v - (y + 1)^2)^2.$$

Recall that this means that

$$\{(x, y) \in \mathbb{N}^2 \mid y = f(x)\} = \{(x, y) \in \mathbb{N}^2 \mid (\exists u, v)(P(x, y, u, v) = 0)\}.$$

See Definition 7.3. What is $f(7)$?

(2) Prove that the subset S of $\mathbb{N}$ defined by the polynomial

$$P(a, y) = a^2 - 4y - 1$$

is the set of natural numbers of the form $4k + 1$ or $4k + 3$, with $k \in \mathbb{N}$.

(3) Prove that S is the set of all nonnegative values taken by the polynomial

$$Q(a, y) = (a + 1)(2 - a^2 + 4y)(a^2 - 4y) - 1,$$

with $a, y \in \mathbb{N}$. How do you obtain the value 7?

Problem B5 (50 pts). Given an undirected graph $G = (V, E)$ and a set $C = \{c_1, \ldots, c_p\}$ of p colors, a *coloring* of G is an assignment of a color from C to each node in V such that no two adjacent nodes share the same color, or more precisely such that for every edge $\{u, v\} \in E$, the nodes u and v are assigned different colors. A k-coloring of a graph G is a coloring using at most k-distinct colors. For example, the graph shown in Figure 1 has a 3-coloring (using green, blue, red).

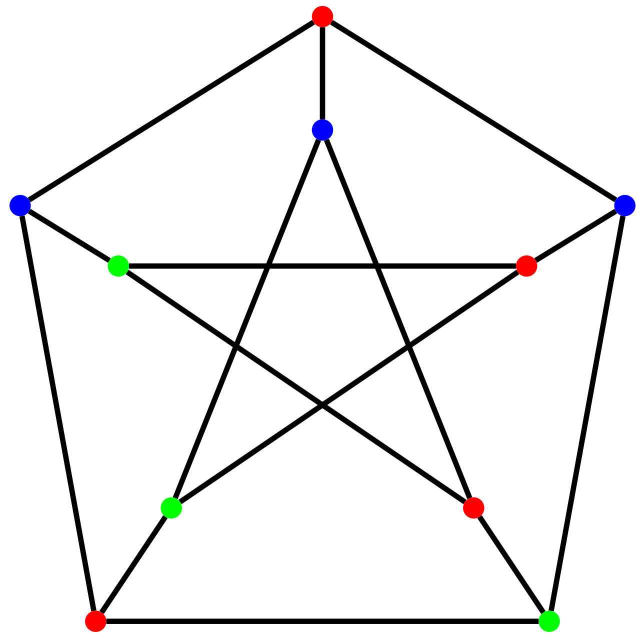

Figure 1: Petersen graph.

The **graph coloring problem** is to decide whether a graph G is k-colorable for a given integer $k \geq 1$.

(1) Give a polynomial reduction from the **graph 3-coloring problem** to the **3-satisfiability problem for propositions in CNF**.

If $|V| = n$, create $n \times 3$ propositional variables x_{ij} with the intended meaning that x_{ij} is true iff node v_i is colored with color j. You need to write sets of clauses to assert the following facts:

1. Every node is colored.
2. No two distinct colors are assigned to the same node.
3. For every edge $\{v_i, v_j\}$, nodes v_i and v_j cannot be assigned the same color.

Beware that it is possible to assert that every node is assigned one and only one color using a proposition in disjunctive normal form, but this is not a correct answer; we want a proposition in conjunctive normal form.

(2) Prove that 2-coloring can be solved deterministically in polynomial time.

Remark: It is known that a graph has a 2-coloring iff its is bipartite, but **do not** use this fact to solve B2(2). *Only use material covered in the notes for CIS511.*

The problem of 3-coloring is actually $\mathcal{NP}$-complete, but this is a bit tricky to prove.

Problem B6 (60 pts). Let A be any $p \times q$ matrix with integer coefficients and let $b \in \mathbb{Z}^p$ be any vector with integer coefficients. The 0-1 *integer programming problem* is to find whether a system of p linear equations in q variables

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1q}x_q &= b_1 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{i1}x_1 + \cdots + a_{iq}x_q &= b_i \\
\vdots \qquad\qquad & \quad \vdots \\
a_{p1}x_1 + \cdots + a_{pq}x_q &= b_p
\end{aligned}
$$

with $a_{ij}, b_i \in \mathbb{Z}$ has any solution $x \in \{0,1\}^q$, that is, with $x_i \in \{0,1\}$. In matrix form, if we let

$$
A = \begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix},
$$

then we write the above system as

$$
Ax = b.
$$

(i) Prove that the 0-1 integer programming problem is in $\mathcal{NP}$.

(ii) Prove that the restricted 0-1 integer programming problem in which the coefficients of A are 0 or 1 and all entries in b are equal to 1 is $\mathcal{NP}$-complete by providing a polynomial-time reduction from the bounded-tiling problem. **Do not try to reduce any other problem to the** 0-1 **integer programming problem**.

Hint. Given a tiling problem, $((\mathcal{T}, V, H), \widehat{s}, \sigma_0)$, create a 0-1-valued variable, x_{mnt}, such that $x_{mnt} = 1$ iff tile t occurs in position (m, n) in some tiling. Write equations or inequalities expressing that a tiling exists and then use "slack variables" to convert inequalities to equations. For example, to express the fact that every position is tiled by a single tile, use the equation

$$
\sum_{t \in \mathcal{T}} x_{mnt} = 1,
$$

for all m, n with $1 \leq m \leq 2s$ and $1 \leq n \leq s$. Also, if you have an inequality such as

$$
2x_1 + 3x_2 - x_3 \leq 5 \qquad (*)
$$

with $x_1, x_2, x_3 \in \mathbb{Z}$, then using a new variable y_1 *taking its values in* $\mathbb{N}$, that is, *nonnegative values*, we obtain the equation

$$
2x_1 + 3x_2 - x_3 + y_1 = 5, \qquad (**)
$$

and the inequality $(*)$ has solutions with $x_1, x_2, x_3 \in \mathbb{Z}$ iff the equation $(**)$ has a solution with $x_1, x_2, x_3 \in \mathbb{Z}$ and $y_1 \in \mathbb{N}$. The variable y_1 is called a *slack variable* (this terminology comes from optimization theory, more specifically, linear programming). For the 0-1-integer programming problem, all variables, including the slack variables, take values in $\{0, 1\}$.

Conclude that the 0-1 integer programming problem is $\mathcal{NP}$-complete.

Problem B7 (20 pts).

(1) Give an example of a Diophantine set which is not computable (recursive).

(2) The family co$\mathcal{NP}$ is the set of complements of languages in $\mathcal{NP}$, namely

$$\mathrm{co}\mathcal{NP} = \{\overline{L} \mid L \in \mathcal{NP}\}.$$

(a) Prove that $\mathcal{P} \subseteq \mathcal{NP} \cap \mathrm{co}\mathcal{NP}$.

(b) Observe that $L \in \mathcal{NP} \cap \mathrm{co}\mathcal{NP}$ iff $L \in \mathcal{NP}$ and $\overline{L} \in \mathcal{NP}$.

Prove that if some language $L \in \mathcal{NP} \cap \mathrm{co}\mathcal{NP}$ is $\mathcal{NP}$-complete, then $\mathcal{NP} = \mathrm{co}\mathcal{NP}$.

Remark: It is not known whether $\mathcal{NP} = \mathrm{co}\mathcal{NP}$, but not likely.

TOTAL: 230 points