$\boxed{\textbf{Spring, 2025} \quad \textbf{CIS 5110}}$

Introduction to the Theory of Computation
Jean Gallier

Homework 5

April 15, 2025; Due May 1, 2025

If the NP-completeness of the tiling problem is not covered before April 24, **SKIP Poblem B5**.

Problem B1 (30 pts). (1) Prove that the extended pairing function $\langle x_1, \ldots, x_n \rangle_n$ defined in the notes (see Definition 6.2 of the proofslambda.pdf) satisfies the equation

$$\langle x_1, \ldots, x_n, x_{n+1} \rangle_{n+1} = \langle x_1, \langle x_2, \ldots, x_{n+1} \rangle_n \rangle.$$

Compute $\langle 2, 5, 7, 17 \rangle_4$ (this integer has 10 digits).

(2) Prove that $\langle x, 0 \rangle = \langle x, 0, \ldots, 0 \rangle_n$ for all $n \geq 2$ and all $x \in \mathbb{N}$.

(3) Prove that

$$\langle \Pi(1, n, z), \ldots, \Pi(n, n, z) \rangle_n = z$$

for all $n \geq 1$ and all $z \in \mathbb{N}$.

Problem B2 (20 pts). Let $f \colon \mathbb{N} \to \mathbb{N}$ be a total computable function. Prove that if f is a bijection, then its inverse f^{-1} is also (total) computable.
Hint. Use monus and minimization.

Problem B3 (20 pts). Let A, B, C, D be the following sets:

$$\begin{aligned}
A &= \{x \in \mathbb{N} \mid \varphi_x \text{ is constant}\}, \\
B &= \{\langle x,\, y \rangle \mid \varphi_x = \varphi_y\}, \\
C &= \{x \in \mathbb{N} \mid \varphi_x = \varphi_a\}, \\
D &= \{x \in \mathbb{N} \mid \varphi_x \text{ is undefined for all input}\},
\end{aligned}$$

where a is a given natural number. Prove that the above sets are not computable (not recursive).

Problem B4 (60 pts).

(1) Give a detailed proof of the equations

$$\mathbf{K}MN \xrightarrow{+}_{\beta} M$$
$$\mathbf{K}_{*}MN \xrightarrow{+}_{\beta} N.$$

(2) Given any two λ-terms M and N, prove rigorously that if $x \notin FV(M)$, then

$$M[x := N] = M.$$

(3) Prove that the Turing combinator $\mathbf{\Theta}$ has no β-normal form.

(4) Prove that the reduction after "so we have the infinite sequence" on Page 272 of proofslambda.pdf is indeed a quasi-leftmost reduction.

Problem B5 (60 pts). Let A be any $p \times q$ matrix with integer coefficients and let $b \in \mathbb{Z}^p$ be any vector with integer coefficients. The 0-1 *integer programming problem* is to find whether a system of p linear equations in q variables

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1q}x_q &= b_1 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{i1}x_1 + \cdots + a_{iq}x_q &= b_i \\
\vdots \qquad\qquad & \quad \vdots \\
a_{p1}x_1 + \cdots + a_{pq}x_q &= b_p
\end{aligned}$$

with $a_{ij}, b_i \in \mathbb{Z}$ has any solution $x \in \{0,1\}^q$, that is, with $x_i \in \{0,1\}$. In matrix form, if we let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix},$$

then we write the above system as

$$Ax = b.$$

(i) Prove that the 0-1 integer programming problem is in $\mathcal{NP}$.

(ii) Prove that the restricted 0-1 integer programming problem in which the coefficients of A are 0 or 1 and all entries in b are equal to 1 is $\mathcal{NP}$-complete by providing a polynomial-time reduction from the bounded-tiling problem. **Do not try to reduce any other problem to the 0-1 integer programming problem**.

Hint. Given a tiling problem, $((\mathcal{T}, V, H), \widehat{s}, \sigma_0)$, create a 0-1-valued variable, x_{mnt}, such that $x_{mnt} = 1$ iff tile t occurs in position (m, n) in some tiling. Write equations or inequalities expressing that a tiling exists and then use "slack variables" to convert inequalities to equations. For example, to express the fact that every position is tiled by a single tile, use the equation

$$\sum_{t \in \mathcal{T}} x_{mnt} = 1,$$

for all m, n with $1 \leq m \leq 2s$ and $1 \leq n \leq s$. Also, if you have an inequality such as

$$2x_1 + 3x_2 - x_3 \leq 5 \tag{$*$}$$

with $x_1, x_2, x_3 \in \mathbb{Z}$, then using a new variable y_1 *taking its values in* $\mathbb{N}$, that is, *nonnegative values*, we obtain the equation

$$2x_1 + 3x_2 - x_3 + y_1 = 5, \tag{$**$}$$

and the inequality $(*)$ has solutions with $x_1, x_2, x_3 \in \mathbb{Z}$ iff the equation $(**)$ has a solution with $x_1, x_2, x_3 \in \mathbb{Z}$ and $y_1 \in \mathbb{N}$. The variable y_1 is called a *slack variable* (this terminology comes from optimization theory, more specifically, linear programming). For the 0-1-integer programming problem, all variables, including the slack variables, take values in $\{0, 1\}$.

Conclude that the 0-1 integer programming problem is $\mathcal{NP}$-complete.

TOTAL: 190 points.