Fall 2021 CIS 511

Introduction to the Theory of Computation
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Homework 4

October 27, 2021; Revised due date: November 12, 2021

Problem B1 (60 pts). (1) Prove that the intersection, $L_1 \cap L_2$, of two regular languages, L_1 and L_2, is regular, **using the Myhill-Nerode characterization** of regular languages.

(2) Let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism, as defined on page 21 of the notes. For any regular language, $L' \subseteq \Delta^*$, prove that

$$h^{-1}(L') = \{w \in \Sigma^* \mid h(w) \in L'\}$$

is regular, **using the Myhill-Nerode characterization** of regular languages.

Proceed as follows: Let $\simeq'$ be a right-invariant equivalence relation on Δ^* of finite index n, such that L' is the union of some of the equivalence classes of $\simeq'$. Let $\simeq$ be the relation on Σ^* defined by

$$u \simeq v \quad \text{iff} \quad h(u) \simeq' h(v).$$

Prove that $\simeq$ is a right-invariant equivalence relation of finite index m, with $m \leq n$, and that $h^{-1}(L')$ is the union of equivalence classes of $\simeq$.

To prove that that the index of $\simeq$ is at most the index of $\simeq'$, use h to define a function $\widehat{h}\colon (\Sigma^*/\simeq) \to (\Delta^*/\simeq')$ from the partition associated with $\simeq$ to the partition associated with $\simeq'$, and prove that $\widehat{h}$ is injective.

Prove that the number of states of any minimal DFA for $h^{-1}(L')$ is at most the number of states of any minimal DFA for L'. Can it be strictly smaller? If so, give an explicit example.

Problem B2 (40 pts). The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Recall that *state equivalence* is the equivalence relation $\equiv$ on Q, defined such that,

$$p \equiv q \quad \text{iff} \quad \forall z \in \Sigma^* (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F),$$

and that *i-equivalence* is the equivalence relation $\equiv_i$ on Q, defined such that,

$$p \equiv_i q \quad \text{iff} \quad \forall z \in \Sigma^*,\ |z| \leq i\ (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

A relation $S \subseteq Q \times Q$ is a *forward closure* iff it is an equivalence relation and whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

We say that a forward closure S is *good* iff whenever $(p, q) \in S$, then *good*(p, q), where *good*(p, q) holds iff either both $p, q \in F$, or both $p, q \notin F$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_{\approx}$ containing R is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q, p) \mid (p, q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$
\begin{aligned}
R_0 &= R_{\approx} \\
R_{i+1} &= (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i,\ a \in \Sigma\})_{\approx}.
\end{aligned}
$$

(1) Prove that $R_{i_0+1} = R_{i_0}$ for some least i_0, and that R_{i_0} is the smallest forward closure containing R.

We denote the smallest forward closure R_{i_0} containing R as $R^{\dagger}$, and call it the *forward closure of* R.

(2) Prove that $p \equiv q$ iff the forward closure $R^{\dagger}$ of the relation $R = \{(p, q)\}$ is good.

Hint. First, prove that if $R^{\dagger}$ is good, then

$$R^{\dagger} \subseteq \equiv .$$

For this, prove by induction that

$$R^{\dagger} \subseteq \equiv_i$$

for all $i \geq 0$.

Then, prove that if $p \equiv q$, then

$$R^{\dagger} \subseteq \equiv .$$

For this, prove that $\equiv$ is an equivalence relation containing $R = \{(p, q)\}$ and that $\equiv$ is forward closed.

Problem B3 (50 pts). Give context-free grammars for the languages

$$
\begin{aligned}
L_1 &= \{xcy \mid x \neq y,\ x, y \in \{a, b\}^*\} \\
L_2 &= \{xcy \mid x \neq y^R,\ x, y \in \{a, b\}^*\}.
\end{aligned}
$$

Hint. Think nondeterministically.

Problem B4 (50 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q \ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p, a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \to \epsilon \quad \text{iff} \quad (S \to \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \overset{n}{\underset{lm}{\Longrightarrow}}_{G_2} w \quad \text{iff} \quad A \overset{n}{\underset{lm}{\Longrightarrow}}_{G} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

TOTAL: 200 points.