Spring, 2009 CIS 511

Introduction to the Theory of Computation
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Homework 4

March 17, 2009; Due March 31, 2009

"A problems" are for practice only, and should not be turned in.

Problem A1. Given any two context-free languages L_1 and L_2 over the same alphabet Σ, prove that $L_1 \cup L_2$ and $L_1 L_2$ are also context-free.

Problem A2. Let Σ and Δ be some alphabets, and let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism. Given any language $L \subseteq \Sigma^*$, recall that

$$h(L) = \{h(w) \in \Delta^* \ \mid \ w \in L\}.$$

Prove that if L is context-free, then $h(L)$ is also context-free.

Problem A3. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is context-free, then L^R is also context-free.

"B problems" must be turned in.

Problem B1 (70 pts). (i) Prove that the conclusion of the pumping lemma holds for the following language L over $\{a, b\}^*$, and yet, L is **not** regular!

$$L = \{w \ \mid \ \exists n \geq 1,\, \exists x_i \in a^+,\, \exists y_i \in b^+,\, 1 \leq i \leq n,\, n \text{ is not prime},\, w = x_1 y_1 \cdots x_n y_n\}.$$

(ii) Consider the following version of the pumping lemma. For any regular language L, there is some $m \geq 1$ so that for every $y \in \Sigma^*$, if $|y| = m$, then there exist $u, x, v \in \Sigma^*$ so that

(1) $y = uxv$;

(2) $x \neq \epsilon$;

(3) For all $z \in \Sigma^*$,

$$yz \in L \quad \text{iff} \quad ux^i vz \in L$$

for all $i \geq 0$.

Prove that this pumping lemma holds.

(iii) Prove that the converse of the pumping lemma in (ii) also holds, i.e., if a language L satisfies the pumping lemma in (ii), then it is regular.

(iv) Consider yet another version of the pumping lemma. For any regular language L, there is some $m \geq 1$ so that for every $y \in \Sigma^*$, if $|y| \geq m$, then there exist $u, x, v \in \Sigma^*$ so that

(1) $y = uxv$;

(2) $x \neq \epsilon$;

(3) For all $\alpha, \beta \in \Sigma^*$,
$$\alpha u \beta \in L \quad \text{iff} \quad \alpha u x^i \beta \in L$$
for all $i \geq 0$.

Prove that this pumping lemma holds.

(v) Prove that the converse of the pumping lemma in (iv) also holds, i.e., if a language L satisfies the pumping lemma in (iv), then it is regular.

Problem B2 (60 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a *trim* DFA. Consider the following procedure:

(1) Form an NFA, N, by reversing all the transitions of D, i.e., there is a transition from p to q on input $a \in \Sigma$ in N iff $\delta(q, a) = p$ in D.

(2) Apply the subset construction to the NFA, N, obtained in (1), taking the start state to be the set F and the only final state of N to be $\{q_0\}$. Then, trim the resulting DFA, obtaining the DFA D^R.

Observe that $L(D^R) = L(D)^R$.

Now, apply the above procedure to D, getting D^R, and apply this procedure again, to get D^{RR}. Prove that D^{RR} is a minimal DFA for $L = L(D)$.

Hint. First prove that if δ_R is the transition function of D^R, then for every $w \in \Sigma^*$ and for every state, T, of D^R,
$$\delta_R^*(T, w) = \{q \in Q \mid \delta^*(q, w^R) \in T\}.$$

Problem B3 (100 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA with n states, say $q_1, \ldots, q_n$, where q_1 is the start state (Note, we denote the start state q_1, not q_0!) We associate with D the $n \times n$ boolean matrix, Δ_D, defined such that
$$\Delta_D(q_i, q_j) = \begin{cases} 1 & \text{if } (\exists a \in \Sigma)(\delta(q_i, a) = q_j), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Delta_D(q_i, q_j) = 1$ iff there is some edge from q_i to q_j (regardless of the label of that edge). Add and multiply matrices treating $\{0, 1\}$ as truth values, i.e.

$$\begin{aligned} 0+0 &= 0 \\ 0+1 &= 1 \\ 1+0 &= 1 \\ 1+1 &= 1 \\ 00 &= 0 \\ 01 &= 0 \\ 10 &= 0 \\ 11 &= 1. \end{aligned}$$

(In other words, $\{0, 1\}$ is the two-element boolean ring).

(a) Prove that Δ_D^k gives the k-step reachability relation on D, i.e., $\Delta_D^k(q_i, q_j) = 1$ iff $\delta^*(q_i, w) = q_j$, for some string $w \in \Sigma^*$ with $|w| = k$ (We set $\Delta_D^0 = I$, the identity matrix).

Prove that there are only finitely many matrices Δ_D^k, where $k \geq 0$. For any $k \geq 0$, let

$$\Delta_D^{*[k]} = I + \Delta_D + \Delta_D^2 + \cdots + \Delta_D^k.$$

Prove that there is a smallest $k \leq n - 1$ so that $\Delta_D^{*[k]} = \Delta_D^{*[k+i]}$ for all $i \geq 1$. Let $\Delta_D^* = \Delta_D^{*[k]}$, for the above k. Prove that $\Delta_D^*(q_i, q_j) = 1$ iff $\delta^*(q_i, w) = q_j$, for some string $w \in \Sigma^*$.

For simplicity of notation, from now on drop the subscript D in Δ_D.

(b) Prove that for any regular language, L, the languages

$$L_{2^n} = \{u \in \Sigma^* \mid (\exists v \in \Sigma^*)(|v| = 2^{|u|} \quad \text{and} \quad uv \in L)\}$$

and

$$L_{2^{2^n}} = \{u \in \Sigma^* \mid (\exists v \in \Sigma^*)(|v| = 2^{2^{|u|}} \quad \text{and} \quad uv \in L)\}$$

are also regular.

Hint. Use the Myhill-Nerode theorem, i.e., find a suitable right-invariant equivalence relation of finite index.

(c) Prove that for any regular language, L, the language

$$L_{n^2} = \{u \in \Sigma^* \mid (\exists v \in \Sigma^*)(|v| = |u|^2 \quad \text{and} \quad uv \in L)\}$$

is also regular.

Hint. Use the Myhill-Nerode theorem.

(d) Let $P(n)$ be any polynomial with nonnegative integer coefficients, say $P(n) = a_0 n^d + a_1 n^{d-1} + \cdots + a_d$. Prove that for every regular language, L, and any polynomial, $P(n)$, as above, the language

$$L_P = \{u \in \Sigma^* \mid (\exists v \in \Sigma^*)(|v| = P(|u|) \quad \text{and} \quad uv \in L)\}$$

is a regular language.

Hint. Use the Myhill-Nerode theorem.

Problem B4 (60 pts). Give context-free grammars for the following languages:

(a) $L_5 = \{wcw^R \mid w \in \{a,b\}^*\}$ (w^R denotes the reversal of w)

(b) $L_6 = \{a^m b^n \mid 1 \leq m \leq n \leq 2m\}$

For any fixed integer $K \geq 2$,

$L_7 = \{a^m b^n \mid 1 \leq m \leq n \leq Km\}$

(c) $L_8 = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\}$

(d) $L_9 = \{a^m b^n a^m b^p \mid m,n,p \geq 1\} \cup \{a^m b^{4n} a^p b^{4n} \mid m,n,p \geq 1\}$

(e) $L_{10} = \{xcy \mid |x| = 2|y|,\, x,y \in \{a,b\}^*\}$

In each case, give a justification of the fact that your grammar generates the desired language.

Problem B5 (40 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q \ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p,a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \to \epsilon \quad \text{iff} \quad (S \to \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \underset{lm}{\overset{n}{\Longrightarrow}}_{G_2} w \quad \text{iff} \quad A \underset{lm}{\overset{n}{\Longrightarrow}}_{G} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

TOTAL: 330 points.