Spring, 2014 CIS 511

Introduction to the Theory of Computation
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Homework 3

February 20, 2014; Due March 6, 2014, *beginning of class*

"A problems" are for practice only, and should not be turned in.

Problem A1. Prove that every finite language is regular.

Problem A2. Sketch an algorithm for deciding whether two regular expressions R, S are equivalent (i,e, whether $\mathcal{L}[R] = \mathcal{L}[S]$).

Problem A3. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is regular, then L^R is also regular.

"B problems" must be turned in.

Problem B1 (60 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Define the relations $\approx$ and $\sim$ on Σ^* as follows:

$$\begin{aligned} x \approx y \quad & \text{if and only if,} \quad \text{for all} \quad p \in Q, \\ & \delta^*(p, x) \in F \quad \text{iff} \quad \delta^*(p, y) \in F, \end{aligned}$$

and

$$x \sim y \quad \text{if and only if,} \quad \text{for all} \quad p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).$$

(a) Show that $\approx$ is a left-invariant equivalence relation and that $\sim$ is an equivalence relation that is both left and right invariant. (A relation R on Σ^* is *left invariant* iff uRv implies that $wuRwv$ for all $w \in \Sigma^*$, and R is *right invariant* iff uRv implies that $uwRvw$ for all $w \in \Sigma^*$.)

(b) Let n be the number of states in Q (the set of states of D). Show that $\approx$ has at most 2^n equivalence classes and that $\sim$ has at most n^n equivalence classes.

(c) Given any language $L \subseteq \Sigma^*$, define the relations λ_L and μ_L on Σ^* as follows:

$$u \, \lambda_L \, v \quad \text{iff,} \quad \text{for all} \quad z \in \Sigma^*, \quad zu \in L \quad \text{iff} \quad zv \in L,$$

and

$$u \, \mu_L \, v \quad \text{iff}, \quad \text{for all} \quad x, y \in \Sigma^*, \quad xuy \in L \quad \text{iff} \quad xvy \in L.$$

Prove that λ_L is left-invariant, and that μ_L is left and right-invariant. Prove that if L is regular, then both λ_L and μ_L have a finite number of equivalence classes.

Hint: Show that the number of classes of λ_L is at most the number of classes of $\approx$, and that the number of classes of μ_L is at most the number of classes of $\sim$.

Problem B2 (80 pts). Recall from class that given any DFA $D = (Q, \Sigma, \delta, q_0, F)$, a *congruence* $\equiv$ *on* D is an equivalence relation $\equiv$ on Q satisfying the following conditions:

(1) If $p \equiv q$, then $\delta(p, a) \equiv \delta(q, a)$, for all $p, q \in Q$ and all $a \in \Sigma$.

(2) If $p \equiv q$ and $p \in F$, then $q \in F$, for all $p, q \in Q$.

(a) Given a congruence $\equiv$ on a DFA D, we define the *quotient DFA* $D/\equiv$ as follows: denoting the equivalence class of a state $p \in Q$ as $[p]$,

$$D/\equiv \; = (Q/\equiv, \Sigma, \delta/\equiv, [q_0], F/\equiv),$$

where

$$\delta/\equiv([p], a) = [\delta(p, a)].$$

Why is $D/\equiv$ well defined? Prove that there is a surjective proper homomorphism $\pi \colon D \to D/\equiv$, and thus, that $L(D) = L(D/\equiv)$ (you may use results from HW1).

(b) Given a DFA D, prove that the state equivalence relation $\equiv_D$ is the coarsest congruence on D (this means that if $\equiv$ is any congruence on D, then $\equiv \subseteq \equiv_D$).

(c) Given two DFA's $D_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, F_1)$ and $D_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, F_2)$, with D_1 trim, prove that the following properties hold.

(1) There is a DFA morphism $h \colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} .$$

(2) There is an F-map $h \colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} \quad \text{and} \quad L(D_1) \subseteq L(D_2);$$

(3) There is a B-map $h \colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} \quad \text{and} \quad L(D_2) \subseteq L(D_1).$$

Conclude that if D_1, D_2 are trim and $L(D_1) = L(D_2)$, then there is a unique surjective proper homomorphism $h\colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} .$$

(you may use results from HW1).

Prove that for any trim DFA D, there is a unique surjective proper homomorphism from D to any minimal DFA D_m accepting $L = L(D)$

(d) Given a regular language L, prove that a minimal DFA D_m for L is characterized by the property that there is unique surjective proper homomorphism $h\colon D \to D_m$ from any trim DFA D accepting L to D_m.

Problem B3 (70 pts). Let L be any regular language over some alphabet Σ. Define the languages

$$\begin{aligned} L^{\infty} &= \bigcup_{k\geq 1}\{w^k \mid w \in L\}, \\ L^{1/\infty} &= \{w \mid w^k \in L, \quad \text{for all } k \geq 1\}, \quad \text{and} \\ \sqrt{L} &= \{w \mid w^k \in L, \quad \text{for some } k \geq 1\}. \end{aligned}$$

Also, for any natural number $k \geq 1$, let

$$L^{(k)} = \{w^k \mid w \in L\},$$

and

$$L^{(1/k)} = \{w \mid w^k \in L\}.$$

(a) Prove that $L^{(1/3)}$ is regular. What about $L^{(3)}$?

(b) Let $k \geq 1$ be any natural number. Prove that there are only finitely many languages of the form $L^{(1/k)} = \{w \mid w^k \in L\}$ and that they are all regular. (In fact, if L is accepted by a DFA with n states, there are at most 2^{n^n} languages of the form $L^{(1/k)}$).

(c) Is $L^{1/\infty}$ regular or not? Is $\sqrt{L}$ regular or not? What about L^{∞}?

Problem B4 (60 pts). Which of the following languages are regular? Justify each answer.

(a) $L_1 = \{wcw \mid w \in \{a,b\}^*\}$

(b) $L_2 = \{xy \mid x, y \in \{a,b\}^* \text{ and } |x| = |y|\}$

(c) $L_3 = \{a^n \mid n \text{ is a prime number}\}$

(d) $L_4 = \{a^m b^n \mid gcd(m,n) = 17\}$.

Problem B5 (50 pts). (a) Prove again that the intersection, $L_1 \cap L_2$, of two regular languages, L_1 and L_2, is regular, **using the Myhill-Nerode characterization** of regular languages.

(b) Let $h\colon \Sigma^* \to \Delta^*$ be a homomorhism, as defined on pages 24-26 of the slides on DFA's and NFA's. For any regular language, $L' \subseteq \Delta^*$, prove that $h^{-1}(L')$ is regular, **using the Myhill-Nerode characterization** of regular languages. Prove that the number of states of any minimal DFA for $h^{-1}(L')$ is at most the number of states of any minimal DFA for L'. Can it be strictly smaller?

Problem B6 (60 pts). The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Recall that *state equivalence* is the equivalence relation $\equiv$ on Q, defined such that,

$$p \equiv q \quad \text{iff} \quad \forall z \in \Sigma^*(\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F),$$

and that *i-equivalence* is the equivalence relation $\equiv_i$ on Q, defined such that,

$$p \equiv_i q \quad \text{iff} \quad \forall z \in \Sigma^*,\ |z| \leq i\ (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

A relation $S \subseteq Q \times Q$ is a *forward closure* iff it is an equivalence relation and whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

We say that a forward closure S is *good* iff whenever $(p, q) \in S$, then *good*(p, q), where *good*(p, q) holds iff either both $p, q \in F$, or both $p, q \notin F$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_\approx$ containing R is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q, p) \mid (p, q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$R_0 = R_\approx$$
$$R_{i+1} = (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i,\ a \in \Sigma\})_\approx.$$

(i) Prove that $R_{i_0+1} = R_{i_0}$ for some least i_0. Prove that R_{i_0} is the smallest forward closure containing R.

We denote the smallest forward closure R_{i_0} containing R as $R^\dagger$, and call it the *forward closure of R*.

(ii) Prove that $p \equiv q$ iff the forward closure $R^\dagger$ of the relation $R = \{(p, q)\}$ is good.

TOTAL: 380 points.