Problem A1. Prove that every finite language is regular.

Problem A2. Sketch an algorithm for deciding whether two regular expressions \( R, S \) are equivalent (i.e., whether \( L[R] = L[S] \)).

Problem A3. Given any language \( L \subseteq \Sigma^* \), let

\[
L^R = \{ w^R \mid w \in L \},
\]

the reversal language of \( L \) (where \( w^R \) denotes the reversal of the string \( w \)). Prove that if \( L \) is regular, then \( L^R \) is also regular.

Problem B1 (60 pts). Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a deterministic finite automaton. Define the relations \( \approx \) and \( \sim \) on \( \Sigma^* \) as follows:

\[
 x \approx y \quad \text{if and only if, for all} \quad p \in Q, \quad \delta^*(p, x) \in F \iff \delta^*(p, y) \in F,
\]

and

\[
 x \sim y \quad \text{if and only if, for all} \quad p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).
\]

(a) Show that \( \approx \) is a left-invariant equivalence relation and that \( \sim \) is an equivalence relation that is both left and right invariant. (A relation \( R \) on \( \Sigma^* \) is left invariant iff \( uRv \) implies that \( uwRvw \) for all \( w \in \Sigma^* \), and \( R \) is right invariant iff \( uRv \) implies that \( uwRvw \) for all \( w \in \Sigma^* \).

(b) Let \( n \) be the number of states in \( Q \) (the set of states of \( D \)). Show that \( \approx \) has at most \( 2^n \) equivalence classes and that \( \sim \) has at most \( n^n \) equivalence classes.

(c) Given any language \( L \subseteq \Sigma^* \), define the relations \( \lambda_L \) and \( \mu_L \) on \( \Sigma^* \) as follows:

\[
 u \lambda_L v \quad \text{iff, for all} \quad z \in \Sigma^*, \quad zu \in L \iff zv \in L,
\]
and
\[ u \mu_L v \iff, \text{ for all } x, y \in \Sigma^*, \quad xuy \in L \iff xyv \in L. \]

Prove that \( \lambda_L \) is left-invariant, and that \( \mu_L \) is left and right-invariant. Prove that if \( L \) is regular, then both \( \lambda_L \) and \( \mu_L \) have a finite number of equivalence classes.

*Hint:* Show that the number of classes of \( \lambda_L \) is at most the number of classes of \( \approx \), and that the number of classes of \( \mu_L \) is at most the number of classes of \( \sim \).

**Problem B2 (70 pts).** Let \( L \) be any regular language over some alphabet \( \Sigma \). Define the languages

\[
L^\infty = \bigcup_{k \geq 1} \{w^k \mid w \in L\},
\]

\[
L^{1/\infty} = \{w \mid w^k \in L, \text{ for all } k \geq 1\}, \quad \text{and}
\]

\[
\sqrt{L} = \{w \mid w^k \in L, \text{ for some } k \geq 1\}.
\]

Also, for any natural number \( k \geq 1 \), let

\[
L^{(k)} = \{w^k \mid w \in L\},
\]

and

\[
L^{(1/k)} = \{w \mid w^k \in L\}.
\]

(a) Prove that \( L^{(1/3)} \) is regular. What about \( L^{(3)} \)?

(b) Let \( k \geq 1 \) be any natural number. Prove that there are only finitely many languages of the form \( L^{(1/k)} = \{w \mid w^k \in L\} \) and that they are all regular. (In fact, if \( L \) is accepted by a DFA with \( n \) states, there are at most \( 2^{n^2} \) languages of the form \( L^{(1/k)} \)).

(c) Is \( L^{1/\infty} \) regular or not? Is \( \sqrt{L} \) regular or not? What about \( L^\infty \)?

**Problem B3 (60 pts).** Which of the following languages are regular? Justify each answer.

(a) \( L_1 = \{wcw \mid w \in \{a,b\}^*\} \)

(b) \( L_2 = \{xy \mid x, y \in \{a,b\}^* \text{ and } |x| = |y|\} \)

(c) \( L_3 = \{a^n \mid n \text{ is a prime number}\} \)

(d) \( L_4 = \{a^m b^n \mid \gcd(m,n) = 17\} \).

**Problem B4 (50 pts).** (a) Prove again that the intersection, \( L_1 \cap L_2 \), of two regular languages, \( L_1 \) and \( L_2 \), is regular, using the Myhill-Nerode characterization of regular languages.

(b) Let \( h : \Sigma^* \to \Delta^* \) be a homomorphism, as defined on pages 24-26 of the slides on DFA’s and NFA’s. For any regular language, \( L' \subseteq \Delta^* \), prove that \( h^{-1}(L') \) is regular, using the Myhill-Nerode characterization of regular languages. Prove that the number of states
of any minimal DFA for $h^{-1}(L')$ is at most the number of states of any minimal DFA for $L'$. Can it be strictly smaller?

**Problem B5 (60 pts).** The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Recall that state equivalence is the equivalence relation $\equiv$ on $Q$, defined such that,

$$p \equiv q \iff \forall z \in \Sigma^* \delta^*(p, z) \in F \iff \delta^*(q, z) \in F,$$

and that $i$-equivalence is the equivalence relation $\equiv_i$ on $Q$, defined such that,

$$p \equiv_i q \iff \forall z \in \Sigma^*, |z| \leq i \delta^*(p, z) \in F \iff \delta^*(q, z) \in F.).$$

A relation $S \subseteq Q \times Q$ is a forward closure iff it is an equivalence relation and whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

We say that a forward closure $S$ is good iff whenever $(p, q) \in S$, then $\text{good}(p, q)$, where $\text{good}(p, q)$ holds iff either both $p, q \in F$, or both $p, q \notin F$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_\approx$ containing $R$ is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q, p) \mid (p, q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$R_0 = R_\approx$$
$$R_{i+1} = (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i, a \in \Sigma\})_\approx.$$

(i) Prove that $R_{i_0 + 1} = R_{i_0}$ for some least $i_0$. Prove that $R_{i_0}$ is the smallest forward closure containing $R$.

We denote the smallest forward closure $R_{i_0}$ containing $R$ as $R^\dagger$, and call it the forward closure of $R$.

(ii) Prove that $p \equiv q$ iff the forward closure $R^\dagger$ of the relation $R = \{(p, q)\}$ is good.

**TOTAL:** 300 points.