$\boxed{\textbf{Spring 2025} \quad \textbf{CIS 5110}}$

Introduction to the Theory of Computation
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Homework 2

February 4, 2025; Due February 27, 2025

Problem B1 (40 pts). Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet of n symbols, with $n \geq 2$.

(1) Construct an NFA with $2n+1$ states accepting the set L_n of strings over Σ such that, every string in L_n has an odd number of a_i, for some $a_i \in \Sigma$. Equivalently, if L_n^i is the set of all strings over Σ with an odd number of a_i, then $L_n = L_n^1 \cup \cdots \cup L_n^n$.

(2) Prove that there is a DFA with 2^n states accepting the language L_n.

(3) Prove that every DFA accepting L_n has at least 2^n states.

Hint. If a DFA D with $k < 2^n$ states accepts L_n, show that there are two strings u, v with the property that, for some $a_i \in \Sigma$, u contains an odd number of a_i's, v contains an even number of a_i's, and D ends in the same state after processing u and v. From this, conclude that D accepts incorrect strings.

Problem B2 (50 pts). Let R be any regular language over some alphabet Σ. Prove that the language

$$L = \{u \in \Sigma^* \mid \exists v \in \Sigma^*,\, uv \in R,\, |u| = |v|\}$$

is regular.
Hint. Think nondeterministically; use a (nonstandard) cross-product construction.

Problem B3 (50 pts). (a) Let $T = \{0, 1, 2\}$, let C be the set of 20 strings of length three over the alphabet T,

$$C = \{u \in T^3 \mid u \notin \{110, 111, 112, 101, 121, 011, 211\}\},$$

let $\Sigma = \{0, 1, 2, c\}$, and consider the language

$$L_M = \{w \in \Sigma^* \mid w = u_1 c u_2 c \cdots c u_n,\, n \geq 1, u_i \in C\}.$$

Prove that L_M is regular (there is a DFA with 7 states).

(b) The language L_M has a geometric interpretation as a certain subset of $\mathbb{R}^3$ (actually, $\mathbb{Q}^3$), as follows: Given any string, $w = u_1 c u_2 c \cdots c u_n \in L_M$, denoting the jth character in u_i

by u_i^j, where $j \in \{1, 2, 3\}$, we obtain three strings

$$\begin{aligned} w^1 &= u_1^1 u_2^1 \cdots u_n^1 \\ w^2 &= u_1^2 u_2^2 \cdots u_n^2 \\ w^3 &= u_1^3 u_2^3 \cdots u_n^3. \end{aligned}$$

For example, if $w = 012c001c222c122$ we have $w^1 = 0021$, $w^2 = 1022$, and $w^3 = 2122$. Now, a string $v \in T^+$ can be interpreted as a decimal real number written in base three! Indeed, if

$$v = b_1 b_2 \cdots b_k, \quad \text{where} \quad b_i \in \{0, 1, 2\} = T \ \ (1 \leq i \leq k),$$

we interpret v as $n(v) = 0.b_1 b_2 \cdots b_k$, i.e.,

$$n(v) = b_1 3^{-1} + b_2 3^{-2} + \cdots + b_k 3^{-k}.$$

Finally, a string, $w = u_1 c u_2 c \cdots c u_n \in L_M$, is interpreted as the point, $(x_w, y_w, z_w) \in \mathbb{R}^3$, where

$$x_w = n(w^1),\ y_w = n(w^2),\ z_w = n(w^3).$$

Therefore, the language, L_M, is the encoding of a set of rational points in $\mathbb{R}^3$, call it M. This turns out to be the rational part of a fractal known as the *Menger sponge*.

Describe recursive rules to create the set M, starting from a unit cube in $\mathbb{R}^3$. Justify as best as you can how these rules are derived from the description of the coordinates of the points of M defined above (which points are omitted, included, ...).

Draw some pictures illustrating this process and showing approximations of the Menger sponge.

Extra Credit (30 points). Write a computer program to draw the Menger sponge (based on the ideas above).

Problem B4 (60 pts). Recall from class that given any DFA $D = (Q, \Sigma, \delta, q_0, F)$, a *congruence* $\equiv$ *on* D is an equivalence relation $\equiv$ on Q satisfying the following conditions:

(1) For all $p, q \in Q$ and all $a \in \Sigma$, if $p \equiv q$, then $\delta(p, a) \equiv \delta(q, a)$.

(2) For all $p, q \in Q$, if $p \equiv q$ and $p \in F$, then $q \in F$.

(a) Given a congruence $\equiv$ on a DFA D, we define the *quotient DFA* $D/\equiv$ as follows: denoting the equivalence class of a state $p \in Q$ as $[p]$,

$$D/\equiv\, = (Q/\equiv, \Sigma, \delta/\equiv, [q_0], F/\equiv),$$

where the transition function $\delta/\equiv$ is given by

$$\delta/\equiv ([p], a) = [\delta(p, a)]$$

for all $p \in Q$ and all $a \in \Sigma$.

Why is $D/\equiv$ well defined? Prove that there is a surjective proper homomorphism $\pi\colon D \to D/\equiv$, and thus, that $L(D) = L(D/\equiv)$ (you may use results from HW1).

(b) Given a DFA $D = (Q, \Sigma, \delta, q_0, F)$, the state equivalence relation $\equiv_D$ is defined such that

$$p \equiv_D q \quad \text{iff} \quad (\forall w \in \Sigma^*)(\delta^*(p, w) \in F \text{ iff } \delta^*(q, w) \in F).$$

It is shown in the class notes that $\equiv_D$ is a congruence, you don't need to prove this fact.

Prove that the state equivalence relation $\equiv_D$ is the coarsest congruence on D (this means that if $\equiv$ is any congruence on D, then $\equiv \subseteq \equiv_D$).

(c) Given two DFA's $D_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, F_1)$ and $D_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, F_2)$, with D_1 trim, prove that the following properties hold.

(1) There is a DFA morphism $h\colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2}.$$

(2) There is a DFA F-map $h\colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} \quad \text{and} \quad L(D_1) \subseteq L(D_2);$$

(3) There is a DFA B-map $h\colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2} \quad \text{and} \quad L(D_2) \subseteq L(D_1).$$

Furthermore, in all three cases, h is surjective iff D_2 is trim. Conclude that if D_1, D_2 are trim and $L(D_1) = L(D_2)$, then there is a unique surjective proper homomorphism $h\colon D_1 \to D_2$ iff

$$\simeq_{D_1} \subseteq \simeq_{D_2}.$$

(you may use results from HW1).

Problem B5 (60 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton.

A relation $S \subseteq Q \times Q$ is a *forward closure* iff it is an equivalence relation and

(1) Whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_\approx$ containing R is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q, p) \mid (p, q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$\begin{aligned} R_0 &= R_\approx \\ R_{i+1} &= (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i,\ a \in \Sigma\})_\approx. \end{aligned}$$

(i) Prove that $R_{i_0+1} = R_{i_0}$ for some least i_0. Prove that R_{i_0} is the smallest forward closure containing R.

We denote the smallest forward closure R_{i_0} containing R as $R^\dagger$, and call it the *forward closure of* R.

In the rest of this problem, all DFA's under consideration use the same alphabet Σ.

(ii) Let $D_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $D_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ be DFA's (not necessarily trim). A triple (D, i_1, i_2) is a *coproduct of* D_1 *and* D_2, where D is a DFA, $i_1 \colon D_1 \to D$ and $i_2 \colon D_2 \to D$ are DFA F-maps, if for every DFA M and any DFA F-maps $f \colon D_1 \to M$ and $g \colon D_2 \to M$, there is a unique DFA F-map $h \colon D \to M$ so that

$$f = h \circ i_1 \quad \text{and} \quad g = h \circ i_2.$$

$$\begin{tikzcd}
D_1 \arrow[d, "i_1"'] \arrow[dr, "f"] & \\
D \arrow[r, "h"] & M \\
D_2. \arrow[u, "i_2"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The above uniqueness condition is called the *the universal mapping property* of coproducts.

Given D_1 and D_2, the disjoint union $D_1 + D_2$ of the two DFA's D_1, D_2 is defined as the state diagram

$$D_1 + D_2 = (Q_1 \cup Q_2, \Sigma, \delta_1 + \delta_2, F_1 \cup F_2),$$

where Q_1 and Q_2 are renamed apart if necessary so that they are disjoint, and where $\delta_1 + \delta_2$ agrees with δ_1 on Q_1, and with δ_2 on Q_2, namely

$$(\delta_1 + \delta_2)(p, a) = \begin{cases} \delta_1(p, a) & \text{if } p \in Q_1 \\ \delta_2(p, a) & \text{if } p \in Q_2. \end{cases}$$

In other words, just put D_1 and D_2 side by side, making sure that Q_1 and Q_2 are disjoing sets, and ignoring that q_{01} and q_{02} are initial states, since this is irrelevant to what follows. Let $F = F_1 \cup F_2$.

We define the DFA $D_1 \coprod D_2$ as the quotient

$$(D_1 + D_2)/\sim, \quad \text{where} \quad \sim = \{(q_{01}, q_{02})\}^\dagger,$$

defined as follows: The set of states of $D_1 \coprod D_2$ is the set of equivalence classes of states of $D_1 + D_2$ modulo the equivalence relation $\sim$, the start state is the equivalence class of q_{01} (which is the same as the equivalence class of q_{02}, since they are identified by $\sim$), the set of final states $\mathcal{F}$ is the set of equivalence classes that contain some state in $F_1 \cup F_2$, and the transition function Δ is defined such that

$$\Delta([p]_\sim, a) = [(\delta_1 + \delta_2)(p, a)]_\sim$$

for all $p \in Q_1 \cup Q_2$ and all $a \in \Sigma$.

Check that $D_1 \coprod D_2$ is indeed a DFA. Define $i_1 \colon D_1 \to D_1 \coprod D_2$ and $i_2 \colon D_2 \to D_1 \coprod D_2$ so that

$$i_1(p_1) = [p_1]_\sim \quad \text{and} \quad i_2(p_2) = [p_2]_\sim$$

for all $p_1 \in Q_1$ and all $p_2 \in Q_2$. Check that i_1 and i_2 are DFA F-maps (since $\sim$ is forward closed).

Let $f \colon D_1 \to M$ and $g \colon D_2 \to M$ be DFA F-maps. Define the relation $\cong$ on the disjoint union $Q_1 \cup Q_2$ by

$$p \cong q \quad \text{iff} \quad \begin{cases} f(p) = f(q) & \text{if } p, q \in Q_1, \\ f(p) = g(q) & \text{if } p \in Q_1,\ q \in Q_2, \\ g(p) = f(q) & \text{if } p \in Q_2,\ q \in Q_1, \\ g(p) = g(q) & \text{if } p, q \in Q_2. \end{cases}$$

Prove that $\cong$ is a forward-closed equivalence relation that contains $\{(q_{0,1}, q_{0,2})\}$. (Proving transitivity involves eight cases). From this, deduce that $\sim \subseteq \cong$.

(iii) Using (ii), prove that the function $h \colon D_1 \coprod D_2 \to M$ defined so that

$$h([p]) = \begin{cases} f(p) & \text{if } p \in Q_1, \\ g(p) & \text{if } p \in Q_2, \end{cases}$$

is a well-defined DFA F-map, and is the unique DFA F-map such that

$$f = h \circ i_1 \quad \text{and} \quad g = h \circ i_2.$$

Thus, $(D_1 \coprod D_2, i_1, i_2)$ is indeed a coproduct of D_1 and D_2.

(iv) Prove or disprove that $L(D_1 \coprod D_2) = L(D_1) \cup L(D_2)$.

(v) (**Extra credit (30 pts)** Given three DFA's D_1, D_2, D_3 and any two DFA F-maps $f \colon D_3 \to D_1$ and $g \colon D_3 \to D_2$, a triple (D, i_1, i_2) is *pushout of $f \colon D_3 \to D_1$ and $g \colon D_3 \to D_2$* if $i_1 \colon D_1 \to D$ and $i_2 \colon D_2 \to D$ are DFA F-maps

$$\begin{tikzcd} D_3 \arrow[r, "f"] \arrow[d, "g"'] & D_1 \arrow[d, "i_1"] \\ D_2 \arrow[r, "i_2"'] & D \end{tikzcd}$$

so that

$$i_1 \circ f = i_2 \circ g,$$

and if for every DFA M and any DFA F-maps $f' \colon D_1 \to M$ and $g' \colon D_2 \to M$

$$\begin{tikzcd} D_3 \arrow[r, "f"] \arrow[d, "g"'] & D_1 \arrow[d, "f'"] \\ D_2 \arrow[r, "g'"'] & M \end{tikzcd}$$

such that

$$f' \circ f = g' \circ g,$$

there is a unique DFA F-map $h\colon D \to M$ so that

$$f' = h \circ i_1 \quad \text{and} \quad g' = h \circ i_2,$$

as illustrated by the following commutative diagram:

$$\begin{tikzcd}
D_3 \arrow[r, "f"] \arrow[d, "g"'] & D_1 \arrow[d, "i_1"'] \arrow[ddr, "f'"] & \\
D_2 \arrow[r, "i_2"] \arrow[drr, "g'"'] & D \arrow[dr, dotted, "h"] & \\
& & M.
\end{tikzcd}$$

The above uniqueness condition is called the *universal mapping property of pushouts*.

Construct the DFA $D_1 \coprod_{D_3} D_2$ as the quotient

$$(D_1 + D_2)/\sim, \quad \text{where} \quad \sim = \{(f(p), g(p)) \in Q_1 \times Q_2 \ \mid p \in Q_3\}^{\dagger},$$

defined in the obvious way, as in (ii).

Check that $D_1 \coprod_{D_3} D_2$ is indeed a DFA. Define $i_1\colon D_1 \to D_1 \coprod_{D_3} D_2$ and $i_2\colon D_2 \to D_1 \coprod_{D_3} D_2$, so that

$$i_1(p_1) = [p_1]_\sim \quad \text{and} \quad i_2(p_2) = [p_2]_\sim$$

for all $p_1 \in Q_1$ and all $p_2 \in Q_2$.

Prove that $(D_1 \coprod_{D_3} D_2, i_1, i_2)$ is indeed a pushout of $f\colon D_3 \to D_1$ and $g\colon D_3 \to D_2$.

(vi) If (D, i_1, i_2) and (D', i_1', i_2') are two pushouts of $f\colon D_3 \to D_1$ and $g\colon D_3 \to D_2$, prove that there is a unique DFA isomorphism between D and D' (this means that there are unique DFA F-maps $h\colon D \to D'$ and $h'\colon D' \to D$ so that $h' \circ h = \mathrm{id}_D$ and $h \circ h' = \mathrm{id}_{D'}$ satisfying the universal mapping property of pushouts).

TOTAL: 260 points + 60 extra credit.