

Introduction to the Theory of Computation

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Homework 1

January 20, 2009; Due February 5, 2009

“A problems” are for practice only, and should not be turned in.

Problem A1. Given an alphabet Σ , prove that the relation \leq_1 over Σ^* defined such that $u \leq_1 v$ iff u is a prefix of v , is a partial ordering. Prove that the relation \leq_2 over Σ^* defined such that $u \leq_2 v$ iff u is a substring of v , is a partial ordering.

Problem A2. Given an alphabet Σ , for any language $L \subseteq \Sigma^*$, prove that $L^{**} = L^*$ and $L^*L^* = L^*$.

Problem A3. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Prove that for all $p \in Q$ and all $u, v \in \Sigma^*$,

$$\delta^*(p, uv) = \delta^*(\delta^*(p, u), v).$$

“B problems” must be turned in.

Problem B1 (30 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Recall that a state $p \in Q$ is *accessible* or *reachable* iff there is some string $w \in \Sigma^*$, such that

$$\delta^*(q_0, w) = p,$$

i.e., there is some path from q_0 to p in D . Consider the following method for computing the set Q_r of reachable states (of D): define the sequence of sets $Q_r^i \subseteq Q$, where

$$Q_r^0 = \{q_0\},$$

$$Q_r^{i+1} = \{q \in Q \mid \exists p \in Q_r^i, \exists a \in \Sigma, q = \delta(p, a)\}.$$

(i) Prove by induction on i that Q_r^i is the set of all states reachable from q_0 using paths of length i (where i counts the number of edges).

Give an example of a DFA such that $Q_r^{i+1} \neq Q_r^i$ for all $i \geq 0$.

(ii) Give an example of a DFA such that $Q_r^i \neq Q_r$ for all $i \geq 0$.

(iii) Change the inductive definition of Q_r^i as follows:

$$Q_r^{i+1} = Q_r^i \cup \{q \in Q \mid \exists p \in Q_r^i, \exists a \in \Sigma, q = \delta(p, a)\}.$$

Prove that there is a smallest integer i_0 such that

$$Q_r^{i_0+1} = Q_r^{i_0} = Q_r.$$

Define the DFA D_r as follows: $D_r = (Q_r, \Sigma, \delta_r, q_0, F \cap Q_r)$, where $\delta_r: Q_r \times \Sigma \rightarrow Q_r$ is the restriction of δ to Q_r . Explain why D_r is indeed a DFA, and prove that $L(D_r) = L(D)$. A DFA is said to be *reachable*, or *trim*, if $D = D_r$.

Problem B2 (20 pts). Given a string w , its reversal w^R is defined inductively as follows: $\epsilon^R = \epsilon$ and $(ua)^R = au^R$, where $a \in \Sigma$ and $u \in \Sigma^*$. Prove that $(uv)^R = v^R u^R$. Prove that $(w^R)^R = w$.

Problem B3 (30 pts). Given any alphabet Σ , prove the following property: for any two strings $u, v \in \Sigma^*$, $uv = vu$ iff there is some $w \in \Sigma^*$ such that $u = w^m$ and $v = w^n$, for some $m, n \geq 0$.

Problem B4 (50 pts). Given any two DFA's $D_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, F_1)$ and $D_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, F_2)$, a *morphism* $h: D_1 \rightarrow D_2$ of DFA's is a function $h: Q_1 \rightarrow Q_2$ satisfying the following two conditions:

- (1) $h(\delta_1(p, a)) = \delta_2(h(p), a)$, for all $p \in Q_1$ and all $a \in \Sigma$;
- (2) $h(q_{0,1}) = q_{0,2}$.

An F -map $h: D_1 \rightarrow D_2$ of DFA's is a morphism satisfying the condition

$$(3a) \quad h(F_1) \subseteq F_2.$$

An F^{-1} -map $h: D_1 \rightarrow D_2$ of DFA's is a morphism satisfying the condition

$$(3b) \quad h^{-1}(F_2) \subseteq F_1.$$

A *proper homomorphism of DFA's* is an F -map of DFA's which is also an F^{-1} -map of DFA's, i.e. it satisfies the condition

$$(3c) \quad h^{-1}(F_2) = F_1.$$

We say that a morphism (resp. F -map, resp. F^{-1} -map) $h: D_1 \rightarrow D_2$ is *surjective* if $h(Q_1) = Q_2$.

(a) Prove that if $f: D_1 \rightarrow D_2$ and $g: D_2 \rightarrow D_3$ are morphisms (resp. F -maps, resp. F^{-1} -maps) of DFAs, then $g \circ f: D_1 \rightarrow D_3$ is also a morphism (resp. F -map, resp. F^{-1} -map) of DFAs.

Prove that if $f: D_1 \rightarrow D_2$ is an F -map that is an isomorphism then it is also an F^{-1} -map, and that if $f: D_1 \rightarrow D_2$ is an F^{-1} -map that is an isomorphism then it is also an F -map.

(b) If $h: D_1 \rightarrow D_2$ is a morphism of DFA's, prove that

$$h(\delta_1^*(p, w)) = \delta_2^*(h(p), w),$$

for all $p \in Q_1$ and all $w \in \Sigma^*$.

As a consequence, prove the following facts:

If $h: D_1 \rightarrow D_2$ is an F -map of DFA's, then $L(D_1) \subseteq L(D_2)$. If $h: D_1 \rightarrow D_2$ is an F^{-1} -map of DFA's, then $L(D_2) \subseteq L(D_1)$. Finally, if $h: D_1 \rightarrow D_2$ is a proper homomorphism of DFA's, then $L(D_1) = L(D_2)$.

(c) Let D_1 and D_2 be DFA's and assume that there is a morphism $h: D_1 \rightarrow D_2$. Prove that h induces a unique surjective morphism $h_r: (D_1)_r \rightarrow (D_2)_r$ (where $(D_1)_r$ and $(D_2)_r$ are the trim DFA's defined in problem B1). This means that if $h: D_1 \rightarrow D_2$ and $h': D_1 \rightarrow D_2$ are DFA morphisms, then $h(p) = h'(p)$ for all $p \in (Q_1)_r$, and the restriction of h to $(D_1)_r$ is surjective onto $(D_2)_r$. Moreover, if $L(D_1) = L(D_2)$, prove that h induces a unique surjective proper homomorphism $h_r: (D_1)_r \rightarrow (D_2)_r$.

(d) Relax the condition that a DFA morphism $h: D_1 \rightarrow D_2$ maps $q_{0,1}$ to $q_{0,2}$ (so, it is possible that $h(q_{0,1}) \neq q_{0,2}$), and call such a function a *weak morphism*. We have an obvious notion of *weak F-map*, *weak F⁻¹-map* and *weak proper homomorphism* (by imposing condition (3a) or condition (3b), or (3c)). For any language, $L \subseteq \Sigma^*$ and any fixed string, $u \in \Sigma^*$, let $D_u(L)$, also denoted L/u (called the *(left) derivative of L by u*), be the language

$$D_u(L) = \{v \in \Sigma^* \mid uv \in L\}.$$

Prove the following facts, **assuming that D_2 is trim**: If $h: D_1 \rightarrow D_2$ is a weak F -map of DFA's, then $L(D_1) \subseteq D_u(L(D_2))$, for some suitable $u \in \Sigma^*$. If $h: D_1 \rightarrow D_2$ is a weak F^{-1} -map of DFA's, then $D_u(L(D_2)) \subseteq L(D_1)$, for the same u as above. Finally, if $h: D_1 \rightarrow D_2$ is a weak proper homomorphism of DFA's, then $L(D_1) = D_u(L(D_2))$, for the same u as above.

Suppose there is a weak morphism $h: D_1 \rightarrow D_2$. What can you say about the restriction of h to $(D_1)_r$? What can you say about surjectivity? (you may need to consider $(D_2)_r$ with respect to a **different** start state). What happens (and what can you say) if D_2 is **not** trim?

Problem B5 (40 pts). (a) For any language $L \subseteq \{a\}^*$, prove that if $L = L^*$, then there is a finite language $S \subseteq L$ such that $L = S^*$. Prove that L is regular.

(b) Let $L \subseteq \{a\}^*$ be any infinite regular language. Prove that there is a finite set $F \subseteq \{a\}^*$, and some strings $a^m, a^{p_1}, \dots, a^{p_k}$, and $a^q \neq \epsilon$, with $0 \leq p_1 < p_2 < \dots < p_k < q$, such that

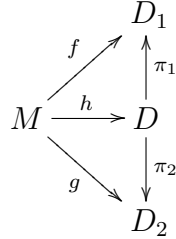
$$L = F \cup \bigcup_{i=1}^k a^{m+p_i} \{a^q\}^*.$$

Problem B6 (50 pts). (a) Given any two DFA's D_1 and D_2 , prove that there is a DFA D and two DFA F^{-1} -maps $\pi_1: D \rightarrow D_1$ and $\pi_2: D \rightarrow D_2$ such that the following *universal*

mapping property of products holds: For any DFA M and any two DFA F^{-1} -maps $f: M \rightarrow D_1$ and $g: M \rightarrow D_2$, there is a *unique* DFA F^{-1} -map $h: M \rightarrow D$ such that

$$f = \pi_1 \circ h \quad \text{and} \quad g = \pi_2 \circ h,$$

as shown in the diagram below:



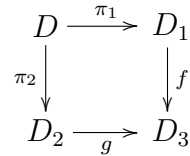
Moreover, prove that π_1 and π_2 are surjective. Prove that D is unique up to a DFA F^{-1} -map isomorphism. This means that if D' is another DFA and if there are two DFA F^{-1} -maps $\pi'_1: D' \rightarrow D_1$ and $\pi'_2: D' \rightarrow D_2$ such that the universal mapping property of products holds, then there are two unique DFA F^{-1} -maps $\varphi: D \rightarrow D'$ and $\varphi': D' \rightarrow D$ so that $\varphi' \circ \varphi = \text{id}_D$, $\pi_1 = \pi'_1 \circ \varphi$, $\pi_2 = \pi'_2 \circ \varphi$, $\varphi \circ \varphi' = \text{id}_{D'}$, $\pi'_1 = \pi_1 \circ \varphi'$ and $\pi'_2 = \pi_2 \circ \varphi'$. What is the language accepted by D' ?

Remark: We call D the *product of D_1 and D_2* and we denote it by $D_1 \amalg D_2$.

(b) Given any three DFA's D_1 , D_2 , and D_3 and any two DFA F^{-1} -maps $f: D_1 \rightarrow D_3$ and $g: D_2 \rightarrow D_3$, prove that there is a DFA D and two DFA F^{-1} -maps $\pi_1: D \rightarrow D_1$ and $\pi_2: D \rightarrow D_2$ such that

$$f \circ \pi_1 = g \circ \pi_2,$$

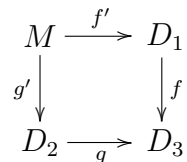
as in the diagram below



and the following *universal mapping property of fibred products* holds: for any DFA M and any two DFA F^{-1} -maps $f': M \rightarrow D_1$ and $g': M \rightarrow D_2$ such that

$$f \circ f' = g \circ g',$$

as in the diagram below



there is a *unique* DFA F^{-1} -map $h: M \rightarrow D$ such that

$$f' = \pi_1 \circ h \quad \text{and} \quad g' = \pi_2 \circ h.$$

Prove that D is unique up to a DFA F^{-1} -map isomorphism. This means that if D' is another DFA and if there are two DFA F^{-1} -maps $\pi'_1: D' \rightarrow D_1$ and $\pi'_2: D' \rightarrow D_2$ such that

$$f \circ \pi'_1 = g \circ \pi'_2$$

and the universal mapping property of fibred products holds, then there are two unique DFA F^{-1} -maps $\varphi: D \rightarrow D'$ and $\varphi': D' \rightarrow D$ so that $\varphi' \circ \varphi = \text{id}_D$, $\pi_1 = \pi'_1 \circ \varphi$, $\pi_2 = \pi'_2 \circ \varphi$, $\varphi \circ \varphi' = \text{id}_{D'}$, $\pi'_1 = \pi_1 \circ \varphi'$ and $\pi'_2 = \pi_2 \circ \varphi'$.

Remark: We denote D by $D_1 \prod_{D_3} D_2$ and call it a *fibred product of D_1 and D_2 over D_3* , or a *pullback of D_1 and D_2 over D_3* .

Letting T denote any one-state DFA accepting \emptyset (this single state is rejecting), observe that there is a unique DFA F^{-1} -map from every DFA D to T . Use this to show that if $D_1 \prod D_2$ is the product DFA arising in (a), then

$$D_1 \prod D_2 = D_1 \prod_T D_2.$$

Extra Credit (40 points). Redo questions (a) and (b) for F -maps instead of F^{-1} -maps.

Remark: If we dualize (b), i.e., turn the arrows around, we get the notion of *fibred coproduct* or *pushout*. It can be shown that fibred coproducts exist, both for F -maps and F^{-1} -maps, but this is tricky.

TOTAL: 220 + 40 points.