Spring, 2013 CIS 511

Introduction to the Theory of Computation
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Final Exam

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Note that this is a **closed-book exam**
Read all the questions **before** starting solving any of them!

Problem 1 (10 pts). Given an alphabet Σ, sketch an algorithm to decide whether

$$R^*S^* = \Sigma^*,$$

for any two regular expressions R and S over Σ.

Problem 2 (20 pts). Let Σ be an alphabet. Recall that a binary relation, $\sim$, on Σ^*, is *left invariant* iff $u \sim v$ implies that $wu \sim wv$ for all $w \in \Sigma^*$ and *right invariant* iff $u \sim v$ implies that $uw \sim vw$ for all $w \in \Sigma^*$. An equivalence relation on Σ^* that is both left and right-invariant is called a *congruence*. Recall that a congruence satisfies the property: If $u \sim u'$ and $v \sim v'$, then $uv \sim u'v'$ (You **do not** have to prove this).

Given any regular language, L, over Σ^* let

$$L^{1/4} = \{w \in \Sigma^* \mid wcwdwcw \in L\},$$

where $c, d \in \Sigma$ are some given letters. Prove that $L^{1/4}$ is also regular.

Problem 3 (25 pts).

Consider the language (over $\Sigma = \{a, b\}$)

$$L_1 = \{w \in \{a, b\}^* \mid \#(a) = \#(b)\}$$

consisting of all strings having an equal number of a's and b's and the language

$$L_1' = \{w \in \{a, b\}^* \mid \#(b) > \#(a)\}$$

consisting of all strings having strictly more b's than a's.

(1) Prove that every nonempty string $w \in L_1$ is of the form

(1) $w = aub$, where $u \in L_1$ ($u = \epsilon$ is allowed);

(2) $w = bua$, where $u \in L_1$ ($u = \epsilon$ is allowed);

(3) $w = uv$, where $u, v \in L_1$, with $u, v \neq \epsilon$.

and that every nonempty string $w \in L_1'$ is of the form

(1) $w = bu$, where $u \in L_1 \cup L_1'$ ($u = \epsilon$ is allowed);

(2) $w = uv$, where $u \in L_1$ and $v \in L_1'$, with $u \neq \epsilon$.

(2) Using the above, give a context-free grammar for L_1'.

Problem 4 (25 pts). Prove that the following languages are not context-free:

$$\begin{aligned} L_1 &= \{u_1\#v_1\#u_2\#v_2 \mid |u_1| = |u_2|,\ |v_1| = |v_2|,\ u_1, u_2, v_1, v_2 \in \{a, b, c, d\}^+\}, \\ L_2 &= \{a^{n^2} \mid n \geq 1\}. \end{aligned}$$

Hint. To prove L_1 non context-free, you may want to consider the intersection of L_1 with a well chosen regular language.

Problem 5 (15 pts). Let $\{\varphi_i\}$ be an acceptable indexing of the partial recursive functions (over $\mathbb{N}$).

(1) Prove that the following sets are not recursive:

$$\begin{aligned} &A = \{i \in \mathbb{N} \mid \varphi_i(0) = \varphi_a(0) \quad \text{and} \quad \varphi_i(0), \varphi_a(0) \ \text{are both defined}\} \\ &B = \{i \in \mathbb{N} \mid \varphi_i(0) = \varphi_a(0) \quad \text{and} \quad \varphi_i(1) = \varphi_a(1)\}, \\ &C = \{\langle i, j\rangle \in \mathbb{N} \mid \varphi_i(0) = \varphi_j(0) \quad \text{and} \quad \varphi_i(1) = \varphi_j(1)\}, \end{aligned}$$

for some given partial recursive function, φ_a.

(2) Prove that A is recursively enumerable.

Problem 6 (25 pts). (i) Given any context-free language, $L \subseteq \{a, b\}^*$, is the following problem decidable:

$L \subseteq a^*b^*a^*b^*$?

(ii) If $R \subseteq \{a\}^*$ is a regular language and $L \subseteq \Sigma^*$ is any context-free language, with $a \in \Sigma$, is it decidable whether

$$R \subseteq L?$$

What if R is any regular language (not necessarily over the alphabet $\{a\}$)?