**Spring, 2009 CIS 511**

# Introduction to the Theory of Computation
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## Final Exam

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Note that this is an **open-book exam**
**Read** all the questions **before** starting solving any of them!

**Problem 1 (15 pts).** Given any regular expression, $S$, over $\Sigma$, define $S^R$ recursively as follows:

$$\begin{aligned} a_i^R &= a_i \\ \epsilon^R &= \epsilon \\ \emptyset^R &= \emptyset \\ (S_1 + S_2)^R &= (S_1^R + S_2^R) \\ (S_1 \cdot S_2)^R &= (S_2^R \cdot S_1^R) \\ (S_1^*)^R &= (S_1^R)^*. \end{aligned}$$

Prove that

$$\mathcal{L}[S^R] = (\mathcal{L}[S])^R,$$

where $(\mathcal{L}[S])^R$ denotes the reversal of the language $\mathcal{L}[S]$, that is,
$(\mathcal{L}[S])^R = \{w^R \in \Sigma^* \mid w \in \mathcal{L}[S]\}$. You may use without proof the fact that

$$(uv)^R = v^R u^R,$$

for any two strings, $u, v$.

(Recall, $\mathcal{L}[S]$ is the regular language denoted by the regular expression $S$.)

**Problem 2 (20 pts).** Let $\Sigma$ be an alphabet. Recall that a binary relation, $\sim$, on $\Sigma^*$, is *left invariant* iff $u \sim v$ implies that $wu \sim wv$ for all $w \in \Sigma^*$ and *right invariant* iff $u \sim v$ implies that $uw \sim vw$ for all $w \in \Sigma^*$. An equivalence relation on $\Sigma^*$ that is both left and right-invariant is called a *congruence*. Recall that a congruence satisfies the property: If $u \sim u'$ and $v \sim v'$, then $uv \sim u'v'$ (You **do not** have to prove this).

Given any regular language, $L$, over $\Sigma^*$ let

$$L^{1/3} = \{w \in \Sigma^* \mid wcwdw \in L\},$$

where $c, d \in \Sigma$ are some given letters. Prove that $L^{1/3}$ is also regular.

**Problem 3 (25 pts).**

Consider the language (over $\Sigma = \{a, b\}$)

$$L_1 = \{w \in \{a, b\}^* \mid \#(a) = \#(b)\}$$

consisting of all strings having an equal number of $a$'s and $b$'s and the language

$$L_1' = \{w \in \{a, b\}^* \mid \#(b) > \#(a)\}$$

consisting of all strings having strictly more $b$'s than $a$'s.

(1) Prove that every nonempty string $w \in L_1$ is of the form

(1) $w = aub$, where $u \in L_1$ ($u = \epsilon$ is allowed);

(2) $w = bua$, where $u \in L_1$ ($u = \epsilon$ is allowed);

(3) $w = uv$, where $u, v \in L_1$, with $u, v \neq \epsilon$.

and that every nonempty string $w \in L_1'$ is of the form

(1) $w = bu$, where $u \in L_1 \cup L_1'$ ($u = \epsilon$ is allowed);

(2) $w = uv$, where $u \in L_1$ and $v \in L_1'$, with $u \neq \epsilon$.

(2) Using the above, give a context-free grammar for $L_1'$.

**Problem 4 (25 pts).** Prove that the following languages are not context-free:

$$\begin{aligned} L_1 &= \{u\#v\#u\#v \mid u, v \in \{a, b, c, d\}^+\}, \\ L_2 &= \{a^{n^2} \mid n \geq 1\}. \end{aligned}$$

*Hint*. To prove $L_1$ non context-free, you may want to consider the intersection of $L_1$ with a well chosen regular language.

**Problem 5 (10 pts).** Let $\{\varphi_i\}$ be an acceptable indexing of the partial recursive functions (over $\mathbb{N}$). Prove that the following problems are undecidable:

(1) $\varphi_i(0) = \varphi_a(0)$ and $\varphi_i(1) = \varphi_a(1)$, for some given partial recursive function, $\varphi_a$.

(2) $\varphi_i(0) = \varphi_j(0)$ and $\varphi_i(1) = \varphi_j(1)$, for any two partial recursive functions, $\varphi_i$ and $\varphi_j$.

**Problem 6 (25 pts).** (i) Given any context-free language, $L \subseteq \{a, b\}^*$, is the following problem decidable:

$L \subseteq a^*b^*$?

(ii) If $R \subseteq \{a\}^*$ is a regular language and $L \subseteq \Sigma^*$ is any context-free language, with $a \in \Sigma$, is it decidable whether

$$R \subseteq L?$$

What if $R$ is any regular language (not necessarily over the alphabet $\{a\}$)?