Summer 1, 2006 CIS 511

Introduction to the Theory of Computation
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Homework 3

June 6, 2006; Due June 15, 2006

"A problems" are for practice only, and should not be turned in.

Problem A1. Given any two context-free languages L_1 and L_2 over the same alphabet Σ, prove that $L_1 \cup L_2$ and $L_1 L_2$ are also context-free.

Problem A2. Let Σ and Δ be some alphabets, and let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism. Given any language $L \subseteq \Sigma^*$, recall that

$$h(L) = \{h(w) \in \Delta^* \mid w \in L\}.$$

Prove that if L is context-free, then $h(L)$ is also context-free.

Problem A3. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is context-free, then L^R is also context-free.

"B problems" must be turned in.

Problem B1 (40 pts). The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Recall that *state equivalence* is the equivalence relation $\equiv$ on Q, defined such that,

$$p \equiv q \quad \text{iff} \quad \forall z \in \Sigma^*(\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

and that *i-equivalence* is the equivalence relation $\equiv_i$ on Q, defined such that,

$$p \equiv_i q \quad \text{iff} \quad \forall z \in \Sigma^*,\ |z| \leq i\ (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

A relation $S \subseteq Q \times Q$ is a *forward closure* iff it is an equivalence relation and whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

We say that a forward closure S is *good* iff whenever $(p,q) \in S$, then $good(p,q)$, where $good(p,q)$ holds iff either both $p,q \in F$, or both $p,q \notin F$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_{\approx}$ containing R is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q,p) \mid (p,q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$
\begin{aligned}
R_0 &= R_{\approx} \\
R_{i+1} &= (R_i \cup \{(\delta(p,a), \delta(q,a)) \mid (p,q) \in R_i,\ a \in \Sigma\})_{\approx}.
\end{aligned}
$$

(i) Prove that $R_{i_0+1} = R_{i_0}$ for some least i_0. Prove that R_{i_0} is the smallest forward closure containing R.

We denote the smallest forward closure R_{i_0} containing R as $R^{\dagger}$, and call it the *forward closure of R*.

(ii) Prove that $p \equiv q$ iff the forward closure $R^{\dagger}$ of the relation $R = \{(p,q)\}$ is good.

Problem B2 (30 pts). (i) A context-free grammar $G = (V, \Sigma, P, S)$ is called an *extended right-linear grammar* iff its productions are of the form

$$
\begin{aligned}
A &\longrightarrow \alpha B \\
A &\longrightarrow w
\end{aligned}
$$

where $A, B \in N = V - \Sigma$, $\alpha \in \Sigma^*$, and $w \in \Sigma^*$. Note that chain rules $A \longrightarrow B$ are allowed as well as ϵ-rules $A \longrightarrow \epsilon$.

(i) Prove that a language is regular iff it is generated by an extended right-linear grammar.

(ii) A context-free grammar $G = (V, \Sigma, P, S)$ is called an *extended left-linear grammar* iff its productions are of the form

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A &\longrightarrow B\alpha \\
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where $A, B \in N = V - \Sigma$, $\alpha \in \Sigma^*$, and $w \in \Sigma^*$. Note that chain rules $A \longrightarrow B$ are allowed as well as ϵ-rules $A \longrightarrow \epsilon$.

Prove that a language is regular iff it is generated by an extended left-linear grammar.

Problem B3 (60 pts). Give context-free grammars for the following languages:

(a) $L_5 = \{wcw^R \mid w \in \{a,b\}^*\}$ (w^R denotes the reversal of w)

(b) $L_6 = \{a^m b^n \mid 1 \leq m \leq n \leq 2m\}$

For any fixed integer $K \geq 2$,

$L_7 = \{a^m b^n \mid 1 \leq m \leq n \leq Km\}$

(c) $L_8 = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\}$

(d) $L_9 = \{a^m b^n a^m b^p \mid m, n, p \geq 1\} \cup \{a^m b^{4n} a^p b^{4n} \mid m, n, p \geq 1\}$

(e) $L_{10} = \{xcy \mid |x| = 2|y|,\ x, y \in \{a, b\}^*\}$

In each case, give a justification of the fact that your grammar generates the desired language.

Problem B4 (40 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q\ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p, a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \to \epsilon \quad \text{iff} \quad (S \to \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \overset{n}{\underset{lm\ G_2}{\Longrightarrow}} w \quad \text{iff} \quad A \overset{n}{\underset{lm\ G}{\Longrightarrow}} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

Problem B5 (40 pts). Give context-free grammars for the languages

$$\begin{aligned} L_1 &= \{xcy \mid x \neq y,\ x, y \in \{a, b\}^*\} \\ L_2 &= \{xcy \mid \ x \neq y^R,\ x, y \in \{a, b\}^*\}. \end{aligned}$$

TOTAL: 210 points.