Spring, 2004 CIS 511

Introduction to the Theory of Computation
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Solutions to the Practice Final Exam

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Problem 1 (10 pts). The regular expression R^*S^* can be converted to an NFA, N, accepting the language denoted by R^*S^*, using a well-known algorithm. Then, N can be converted to DFA, D, using the subset construction. Since $L(D) = \Sigma^*$ is equivalent to $\overline{L(D)} = \emptyset$, we can construct the DFA, $\overline{D}$, accepting $\overline{L(D)}$, by swapping final and rejecting states, and then, we just have to test whether $\overline{D}$ accepts the empty language. However, $\overline{D}$ accepts nothing iff there is no path from the start state to any final state, and this can be easily decided (graph reachability).

Problem 2 (20 pts). The language $L_1 = \{ww^R \mid w \in R\}$ may be regular, for example, if R is finite or if Σ consists of a single letter. We claim that if $R = \{a, b\}^*$, then L_1 is not regular. We proceed by contradiction using Myhill-Nerode. If L_1 were regular, there would be some right-invariant equivalence relation, $\cong$, of finite index, say n, so that L_1 is the union of classes of $\cong$. Since there are infinitely many strings of the form

$$ab, a^2b, \ldots, a^nb,$$

we must have

$$a^ib \cong a^jb$$

for some $i < j$. But then, by right-invariance, we get

$$a^ibba^i \cong a^jbba^i,$$

with $i < j$. Since $a^ibba^i \in L_1$, we should also have $a^jbba^i \in L_1$, which is absurd.

Let $D = (Q, \Sigma, \delta_R, q_0, F)$ be a DFA accepting R. We construct an NFA, N, accepting $L_2 = \{w \in \Sigma^* \mid ww^R \in R\}$, as follows:

$$N = ((Q \times Q) \cup \{s\}, \Sigma, \delta, s, \{(p, p) \mid p \in Q\}),$$

where δ is given by:

$$(q_0, p) \in \delta(s, \epsilon) \quad \text{for all } p \in F,$$

and

$$(q_1, q_2) \in \delta((p_1, p_2), a) \quad \text{iff} \quad \delta_R(p_1, a) = q_1 \quad \text{and} \quad \delta_R(q_2, a) = p_2,$$

for every $a \in \Sigma$.

Problem 3 (15 pts). (i)

$$\begin{aligned} S &\longrightarrow XcY \\ X &\longrightarrow aXb \\ X &\longrightarrow ab \\ Y &\longrightarrow aaYbb \\ Y &\longrightarrow aabb \end{aligned}$$

is a CFG for
$$L_3 = \{a^m b^m c a^{2n} b^{2n} \mid m, n \geq 1\}.$$

(ii) A *deterministic* PDA accepting L_3 by empty stack is easily constructed.

Problem 4 (20 pts). We prove that

$$L_4 = \{a^m b^n c a^{2m} b^{2n} \mid m, n \geq 1\}$$

is not context-free by contradiction, using Ogden's lemma. If $K > 1$ is the constant of Ogden's lemma, we pick $w = a^K b^K c a^{2K} b^{2K}$, with the middle b's and a's distinguished. Then, the cases are pretty much as in the class notes on context-free languages, page 59-60.

Proving that
$$L_5 = \{a^n \mid n \text{ is not a prime}\}$$
is not context-free is a little more tricky. We use the fact proved as a homework that a contex-free language over a one-letter alphabet is actually regular. Were L_5 context-free, then it would be regular. But then, its complement, L_6, would also be regular. However, we can easily show using the pumping lemma for the regular languages that

$$L_6 = \{a^n \mid n \text{ is a prime}\}$$

is not regular. Thus, in the end, L_5 is not context-free.

Problem 5 (20 pts). Pick any $a \in \Sigma$, let G be a linear context-free grammar and let $L = L(G)$. We want to prove that $L/a = \{w \in \Sigma^* \mid wa \in L\}$ is still linear context-free. One should be aware that ϵ-rules, which *are allowed* in linear CFG's, cause a little bit of trouble. We construct a new grammar, G_a, from G, by adding new nonterminals of the form $[A/a]$ and new productions

$$[A/a] \longrightarrow \alpha, \quad \text{if} \quad A \longrightarrow \alpha a \in P \quad \text{or} \quad A \longrightarrow \alpha a B \in P \quad \text{with} \quad B \stackrel{+}{\Longrightarrow} \epsilon$$

and
$$[A/a] \longrightarrow u[B/a], \quad \text{if} \quad A \longrightarrow uB \in P.$$

The start symbol of G_a is $[S/a]$. We claim that $L(G_a) = L(G)/a = L/a$.

Claim 1: $L/a \subseteq L(G_a)$. For this, we prove by induction on the length of derivations in G that if $S \stackrel{+}{\Longrightarrow} \alpha a$, then $[S/a] \stackrel{+}{\Longrightarrow} \alpha$ in G_a and if $S \stackrel{+}{\Longrightarrow} \alpha B$, then $[S/a] \stackrel{+}{\Longrightarrow} \alpha[B/a]$ in G_a, with $\alpha \in V^*$. Note that for linear context-free grammars, all derivations are leftmost derivations (and rightmost derivations).

If $S \stackrel{1}{\Longrightarrow} \alpha a$, then by construction, the production $[S/a] \longrightarrow \alpha$ is in G_a; similarly, if $S \stackrel{1}{\Longrightarrow} \alpha B$, then by construction, the production $[S/a] \longrightarrow \alpha[B/a]$ is in G_a. Thus, the base step holds.

Now, assume the induction hypothesis holds for n and consider a derivation $S \stackrel{n+1}{\Longrightarrow} \alpha a$ or $S \stackrel{n+1}{\Longrightarrow} \alpha B$ of length $n+1$, with $n \geq 1$. There are four cases:

Case 1. The derivation is of the form $S \stackrel{n}{\Longrightarrow} wA\beta a \Longrightarrow w\gamma\beta a$, where $w \in \Sigma^*$ and $\beta, \gamma \in V^*$

By the induction hypothesis, there is a derivation $[S/a] \stackrel{+}{\Longrightarrow} wA\beta$ in G_a. Thus, we get the derivation

$$[S/a] \stackrel{+}{\Longrightarrow} wA\beta \Longrightarrow w\gamma\beta \quad \text{in } G_a.$$

Case 2. The derivation is of the form $S \stackrel{n}{\Longrightarrow} wA \Longrightarrow wuBva$, where $u, v, w \in \Sigma^*$.

By the induction hypothesis, there is a derivation $[S/a] \stackrel{+}{\Longrightarrow} w[A/a]$ in G_a and by construction, there is a production $[A/a] \longrightarrow uBv$ in G_a. Thus, we get the derivation

$$[S/a] \stackrel{+}{\Longrightarrow} w[A/a] \Longrightarrow wuBv \quad \text{in } G_a.$$

Case 3. The derivation is of the form $S \stackrel{n}{\Longrightarrow} waA \Longrightarrow wa$, where $w \in \Sigma^*$. This is the case when the last step is an ϵ-rule, $A \longrightarrow \epsilon$.

In this case, there is a first derivation step during which a appears and the above derivation must be of the form

$$S \stackrel{+}{\Longrightarrow} w_1A_1 \Longrightarrow w_1w_2aB \stackrel{*}{\Longrightarrow} w_1w_2aA \Longrightarrow w_1w_2a,$$

where $w = w_1w_2$. Thus, $B \stackrel{+}{\Longrightarrow} \epsilon$. By the induction hypothesis, there is a derivation $[S/a] \stackrel{+}{\Longrightarrow} w_1[A_1/a]$ in G_a and by construction, there is a production $[A_1/a] \longrightarrow w_2$ in G_a. Thus, we get the derivation

$$[S/a] \stackrel{+}{\Longrightarrow} w_1[A_1/a] \Longrightarrow w_1w_2 \quad \text{in } G_a.$$

Case 4. The derivation is of the form $S \stackrel{n}{\Longrightarrow} wA \Longrightarrow wuB$, where $u, w \in \Sigma^*$.

By the induction hypothesis, there is a derivation $S \stackrel{+}{\Longrightarrow} w[A/a]$ in G_a and by construction, there is a production $[A/a] \longrightarrow u[B/a]$ in G_a. Thus, we get the derivation

$$S \stackrel{+}{\Longrightarrow} w[A/a] \Longrightarrow wu[B/a] \quad \text{in } G_a.$$

Claim 2: $L(G_a) \subseteq L/a$. For this, we prove by induction on the length of derivations in G_a that if $[S/a] \stackrel{+}{\Longrightarrow} \alpha$, then $S \stackrel{+}{\Longrightarrow} \alpha a$, and if $[S/a] \stackrel{+}{\Longrightarrow} \alpha[B/a]$, then $S \stackrel{+}{\Longrightarrow} \alpha B$, with $\alpha \in V^*$.

If $[S/a] \stackrel{1}{\Longrightarrow} w$, then by construction, either the production $S \longrightarrow wa$ is in G or the production $S \longrightarrow waB$ is in G with $B \stackrel{+}{\Longrightarrow} \epsilon$. In the first case, $S \stackrel{1}{\Longrightarrow} wa$ and in the second case,

$$S \Longrightarrow waB \stackrel{+}{\Longrightarrow} wa.$$

Thus, the base step holds.

Now, assume the induction hypothesis holds for n and consider a derivation $[S/a] \stackrel{n+1}{\Longrightarrow} \alpha$ or $S \stackrel{n+1}{\Longrightarrow} \alpha[B/a]$ of length $n+1$, with $n \geq 1$.

Due to the form of the productions involving the new nonterminals $[A/a]$, an easy induction shows that for any derivation in G_a of the form $[S/a] \stackrel{+}{\Longrightarrow} \alpha[B/a]\beta$, we must have $\beta = \epsilon$. There are three cases.

Case 1. The derivation is of the form $[S/a] \stackrel{n}{\Longrightarrow} wA\beta \Longrightarrow w\gamma\beta$, where $w \in \Sigma^*$ and $\beta, \gamma \in V^*$

By the induction hypothesis, there is a derivation $S \stackrel{+}{\Longrightarrow} wA\beta a$ in G. Thus, we get the derivation

$$S \stackrel{+}{\Longrightarrow} wA\beta a \Longrightarrow w\gamma\beta a \quad \text{in } G.$$

Case 2. The derivation is of the form $[S/a] \stackrel{n}{\Longrightarrow} w[A/a] \Longrightarrow w\alpha$, where $w \in \Sigma^*$ and $\alpha \in V^*$.

By the induction hypothesis, there is a derivation $S \stackrel{+}{\Longrightarrow} wA$ in G and by construction, either there is a production $A \longrightarrow \alpha a$ in G or a production $A \longrightarrow \alpha a B$ in G with $B \stackrel{+}{\Longrightarrow} \epsilon$. Thus, we get the derivation

$$S \stackrel{+}{\Longrightarrow} wA \Longrightarrow w\alpha a \quad \text{in } G$$

or a derivation

$$S \stackrel{+}{\Longrightarrow} wA \Longrightarrow w\alpha aB \stackrel{+}{\Longrightarrow} w\alpha a \quad \text{in } G.$$

Case 3. The derivation is of the form $[S/a] \stackrel{n}{\Longrightarrow} w[A/a] \Longrightarrow w\alpha[B/a]$, where $w \in \Sigma^*$ and $\alpha \in V^*$.

By the induction hypothesis, there is a derivation $S \stackrel{+}{\Longrightarrow} wA$ in G and by construction, there is a production $A \longrightarrow \alpha B$ in G. Thus, we get the derivation

$$S \stackrel{+}{\Longrightarrow} wA \Longrightarrow w\alpha B \quad \text{in } G.$$

Putting Claim 1 and Claim 2 together, we get $L(G_a) = L/a$.

Problem 6 (25 pts). First, assume that M halts only for finitely inputs. Is so, the language L_M is finite, thus context-free. Now, assume that M accepts infinitely many inputs and that

L_M is context-free. Then, M accepts arbitrarily long strings and so, if K is the constant of the pumping lemma, there is some proper halting ID

$$w = w_1\#w_2^R\#w_3\#\cdots$$

with $|w_1|, |w_2| > K$. Let w_2 be marked. Then, in any decomposition $uvxyz$ of w, as x must contain some marked occurrence and vxy contains at most K marked occurrences, it is impossible that at the same time v contains symbols in w_1 and that y contain symbols in w_3, as vxy would contain too many marked occurrences. But then, when pumping up v and y (recall, $uv^ixy^iz \in L_M$, for all $i \geq 0$), either w_1 remains intact or w_3 remains intact while w_2 gets strictly bigger. As the TM is deterministic, this is impossible: there can't be a valid move from w_1 to the new w_2 or from the new w_2 to w_3. Therefore, L_M can't be context-free and if L_M is context-free, then it is finite.

(ii) Given any TM, M, first modify M so that every computation makes at least two moves on every input. This can be done by adding two extra states. We know that $L = \overline{L_M}$ is context-free and from (i), M halts on finitely many inputs iff $L_M = \overline{L}$ is context-free. However, it is undecidable whether a TM accpts a finite language (by Rice's theorem, as it is undecidable whether a partial function has a finite domain). Therefore, deciding for any CFG, G, whether $\overline{L(G)}$ is context-free must also be undecidable.