Spring, 2004 CIS 511

Introduction to the Theory of Computation

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Final Exam

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Note that this is a **closed-book exam**

Read all the questions **before** starting solving any of them!

If you are taking the WPE, please write your ID number on **all** your solution sheets.

Problem 1 (15 pts). Let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism (i.e., $h(uv) = h(u)h(v)$, for all $u, v \in \Sigma^*$). Given any regular expression, R, over Σ, extend h so that $h(\text{“(”}) = \text{“(”}$, $h(\text{“)”}) = \text{“)”}$, $h(+) = +$, $h(\cdot) = \cdot$, $h(*) = *$, $h(\text{“}\epsilon\text{”}) = \text{“}\epsilon\text{”}$ and $h(\text{“}\emptyset\text{”}) = \text{“}\emptyset\text{”}$. Prove that $h(R)$ is a regular expression (over Δ) such that

$$\mathcal{L}[h(R)] = h(\mathcal{L}[R]).$$

(Recall, $\mathcal{L}[S]$ is the regular language denoted by the regular expression S.)

Problem 2 (20 pts). Let Σ be an alphabet. Recall that a binary relation, $\sim$, on Σ^*, is *left invariant* iff $u \sim v$ implies that $wu \sim wv$ for all $w \in \Sigma^*$ and *right invariant* iff $u \sim v$ implies that $uw \sim vw$ for all $w \in \Sigma^*$. An equivalence relation on Σ^* that is both left and right-invariant is called a *congruence*. Recall that a congruence satisfies the property: If $u \sim u'$ and $v \sim v'$, then $uv \sim u'v'$ (You **do not** have to prove this).

Given any regular language, L, over Σ^* let

$$L^{1/3} = \{w \in \Sigma^* \mid www \in L\}.$$

Prove that $L^{1/3}$ is also regular.

Problem 3 (25 pts). Which of the following two languages are context-free. Justify your answer carefully:

$$\begin{aligned} L_2 &= \{xcy \mid 2|x| = |y|,\ x, y \in \{a, b\}^*\} \\ L_3 &= \{xxcx \mid x \in \{a, b\}^*\}. \end{aligned}$$

Problem 4 (30 pts). A context-free grammar, $G = (V, \Sigma, P, S)$, is *linear* iff for every production $(A \to \alpha) \in P$,

$$\alpha \in \Sigma^* N \Sigma^* \cup \Sigma^*,$$

where $N = V - \Sigma$. A language, L, is a *linear context-free language* iff there is some linear context-free grammar, G, such that $L = L(G)$.

(i) Let $L = L(G)$ be any linear-context-free language and $R = L(D)$ be any regular language where $G = (V, \Sigma, P, S)$ and $D = (Q, \Sigma, \delta, q_0, F)$ is a DFA. Construct a linear context-free grammar, G', as follows: The nonterminals are triples $(p, A, q) \in Q \times N \times Q$ or S_0, with S_0 a new symbol and the productions are:

$$S_0 \longrightarrow (q_0, S, f), \quad \text{for every } f \in F;$$

$$(p, A, q) \longrightarrow u \quad \text{iff} \quad A \longrightarrow u \in P \quad \text{and} \quad \delta^*(p, u) = q,$$

for all $p, q \in Q$;

$$(p, A, q) \longrightarrow u(r, B, s)v \quad \text{iff} \quad A \longrightarrow uBv \in P, \quad \delta^*(p, u) = r \quad \text{and} \quad \delta^*(s, v) = q,$$

for all $p, q, r, s \in Q$;

Prove
Claim 1:

$$(p, A, q) \stackrel{n}{\Longrightarrow} x(r, B, s)y \quad \text{iff} \quad A \stackrel{n}{\Longrightarrow} xBy, \quad \delta^*(p, x) = r \quad \text{and} \quad \delta^*(s, y) = q,$$

for all $n \geq 0$, all $x, y \in \Sigma^*$, all $p, q, r, s \in Q$ and all $A, B \in N$, and
Claim 2:

$$(p, A, q) \stackrel{n}{\Longrightarrow} w \quad \text{iff} \quad A \stackrel{n}{\Longrightarrow} w \quad \text{and} \quad \delta^*(p, w) = q,$$

for all $n \geq 1$, all $w \in \Sigma^*$, all $p, q \in Q$ and all $A \in N$.

(ii) Prove that $L \cap R$ is linear context-free.

Problem 5 (10 pts). Let $\{\varphi_i\}$ be an acceptable indexing of the partial recursive functions (over $\mathbb{N}$). Prove that the following problems are undecidable:

(1) The partial recursive function, φ_i, satisfies $\varphi_i(0) = 1$.

(2) The partial recursive function, φ_i, is undefined for a finite number of input values (including none).

Problem 6 (20 pts). Prove that for every recursively enumerable set, L, over an alphabet Σ, there exist two context-free languages L_1 and L_2 over some possibly larger alphabet Δ, and a homomorphism $h\colon \Delta^* \to \Sigma^*$, so that

$$L = h(L_1 \cap L_2).$$