Spring, 2003 CIS 511

Introduction to the Theory of Computation

Final Exam

May 2, 2003

Note that this is a **closed-book exam**
Read all the questions **before** starting solving any of them!
If you are taking the WPE, please write your ID number on **all** your solution sheets.

Problem 1 (15 pts). Let Σ be an alphabet and let $L_1, L_2 \subseteq \Sigma^*$ be languages so that L_1 is *not* regular but L_2 *is* regular.

(i) Assume $L_1 \cap L_2$ is finite. Prove that $L_1 \cup L_2$ is not regular.

(ii) Now, allow $L_1 \cap L_2$ to be infinite (still under the assumption that L_1 is *not* regular but L_2 *is* regular). Give an example where $L_1 \cup L_2$ is regular.

Problem 2 (20 pts). Let Σ be an alphabet. Recall that a binary relation, $\sim$, on Σ^*, is *left invariant* iff $u \sim v$ implies that $wu \sim wv$ for all $w \in \Sigma^*$ and *right invariant* iff $u \sim v$ implies that $uw \sim vw$ for all $w \in \Sigma^*$. A equivalence relation on Σ^* that is both left and right-invariant is called a *congruence*. Recall that a congruence satisfies the property: If $u \sim u'$ and $v \sim v'$, then $uv \sim u'v'$ (You **do not** have to prove this). Also recall that there is a version of the Myhill-Nerode theorem that says that a language, L, is regular iff it is the union of equivalence classes of a congruence with a finite number of equivalence classes. (You **do not** have to prove this). Finally, recall that the reversal of a string, $w \in \Sigma^*$, is defined inductively as follows:

$$\begin{aligned} \epsilon^R &= \epsilon \\ (ua)^R &= au^R, \end{aligned}$$

for all $u \in \Sigma^*$ and all $a \in \Sigma$.

(i) Let $\sim$ be a congruence (on Σ^*) and assume that $\sim$ has n equivalence classes. Define $\sim_R$ and $\approx$ by

$$u \sim_R v \quad \text{iff} \quad u^R \sim v^R, \quad \text{for all } u, v \in \Sigma^* \quad \text{and} \quad \approx \,=\, \sim \cap \sim_R .$$

The relation $\approx$ is clearly a congruence (You **do not** have to prove this). Prove that $\approx$ has at most n^2 equivalence classes.

(ii) Given any regular language, L, over Σ^* let

$$L' = \{w \in \Sigma^* \mid ww^R \in L\}.$$

Prove that L' is also regular, using the relation $\approx$ of part (i).
Hint. Use the usual version of the Myhill-Nerode theorem applied to the relation $\approx$.

Problem 3 (25 pts). (i) Give context-free grammars for the languages

$$L_3 = \{a^m b^n c^p \mid m \neq n,\ m, n, p \geq 1\}$$

and

$$L_4 = \{a^m b^n c^p \mid n \neq p,\ m, n, p \geq 1\}.$$

(ii) Prove that the language $L_5 = L_3 \cup L_4$ is context-free, and yet, $L_6 = \{a, b, c\}^* - L_5$ is not context-free.

Problem 4 (25 pts). Let $f\colon \mathbb{N} \to \mathbb{N}$ be any function such that for some (large) $n_0 \in \mathbb{N}$, we have $f(n+1) - f(n) \geq n + 1$, for all $n \geq n_0$.

(i) Prove that the language (over the one-letter alphabet $\{a\}$)

$$L_7 = \{a^{f(n)} \mid n \geq 0\}$$

is not context-free.

(ii) Are the following languages context-free?

$$\begin{aligned} L_8 &= \{a^{n(n+1)/2} \mid n \geq 0\} \\ L_9 &= \{a^{n!} \mid n \geq 0\}. \end{aligned}$$

Justify your answers carefully.

Problem 5 (10 pts). Let A and B be two recursively enumerable sets.

(i) Prove that $A \cup B$ and $A \cap B$ are also recursively enumerable.

(ii) Give an example of an r.e. set whose complement is not r.e.

Problem 6 (25 pts). Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be a given fixed acceptable indexing of the partial recursive functions.

(i) Let $a \in \mathbb{N}$ be any fixed natural number and consider the set

$$S_a = \{i \in \mathbb{N} \mid \varphi_i(i) = a\}.$$

Prove that S_a is a recursively enumerable set (for short, r.e. set) that is not recursive.
Hint. You may use the fact that the equality predicate, $m = n$, is primitive recursive.

(ii) Find two disjoint r.e. nonrecursive sets A and B such that there is *no* recursive set, R, so that $A \subseteq R$ and $B \cap R = \emptyset$ (we say that A and B are *recursively inseparable*).