Spring, 2003 CIS 511

Introduction to the Theory of Computation

Homework 6

April 17, 2003; Due April 28

"A problems" are for practice only, and should not be turned in.

Problem A1. Prove that every context-free language is a recursive set.

Problem A2. Consider the definition of the Kleene T-predicate given in the notes in Definition 5.4.1.

(i) Verify that $T(x, y, z)$ holds iff x codes a RAM program, y is an input, and z codes a halting computation of P_x on input y.

(ii) Verify that the Kleene normal form holds:

$$\varphi_x(y) = \mathrm{Res}[\min z(T(x, y, z))].$$

"B problems" must be turned in.

Problem B1 (30 pts). Given $\Sigma = \{a\}$, give a Turing machine accepting

$$L = \{a^{2^n} \mid n \geq 1\}.$$

Problem B2 (30 pts). *Ackermann's function* A is defined recursively as follows:

$$\begin{aligned} A(0,\, y) &= y + 1, \\ A(x+1,\, 0) &= A(x,\, 1), \\ A(x+1,\, y+1) &= A(x,\, A(x+1, y)). \end{aligned}$$

Prove that

$$\begin{aligned} A(0,\, x) &= x + 1, \\ A(1,\, x) &= x + 2, \\ A(2,\, x) &= 2x + 3, \\ A(3,\, x) &= 2^{x+3} - 3, \end{aligned}$$

and

$$A(4, x) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2^{16}}}}}}\right\}^{x} - 3,$$

with $A(4,0) = 16 - 3 = 13$. Equivalently (and perhaps less confusing)

$$A(4,x) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2^2}}}}}\right\}^{x+3} - 3.$$

Problem B3 (20 pts). Prove that the following properties of partial recursive functions are undecidable.

(a) A partial recursive function is a constant function.

(b) Two partial recursive functions φ_x and φ_y are identical.

(c) A partial recursive function φ_x is equal to a given partial recursive function φ_a.

(d) A partial recursive function diverges for all input.

Problem B4 (40 pts). Prove that a RAM program with $p \geq 2$ registers can be simulated by a RAM program with a single register, by encoding the contents $r_1, \ldots, r_p$ of the p registers as the string

$$r_1\#r_2\#\cdots\#r_p,$$

using a new marker $\#$.

Hint: Begin by simulating a RAM program with two registers using a single register. Another (painful) solution is to simulate a Turing machine using a RAM with a single variable. In this case, the variable will represent the tape uav as $av\#u$. Some care must be exercised at both ends of the tape. Then, go from RAM to TM to RAM with a single register!

Conclude that the halting problem for RAM programs with one register is undecidable.

Problem B5 (50 pts). A *linear context-free grammar* is a context-free grammar whose productions are of the form either

$$\begin{aligned} A &\longrightarrow uBv, \quad \text{or} \\ A &\longrightarrow u, \end{aligned}$$

where A, B are nonterminals and $u, v \in \Sigma^*$. A language is *linear context-free* iff it is generated by some linear context-free grammar.

(a) Prove that every regular language is linear context-free. Prove that if L is a linear context-free language, then for every $a \in \Sigma$, the language $L/a = \{w \in \Sigma^* \mid wa \in L\}$ is also linear context-free.

Hint. Construct a grammar using some new nonterminals, $[A/a]$, and new productions

$$[A/a] \longrightarrow \alpha, \quad \text{if} \quad A \longrightarrow \alpha a \in P \quad \text{or} \quad A \longrightarrow \alpha a B \in P \quad \text{with} \quad B \stackrel{+}{\Longrightarrow} \epsilon$$

and

$$[A/a] \longrightarrow u[B/a], \quad \text{if} \quad A \longrightarrow uB \in P,$$

(b) Prove that it is undecidable whether a context-free language, L, is linear context-free.

TOTAL: 170 points.