Spring, 2003 CIS 511

Introduction to the Theory of Computation

Homework 4

March 18, 2003; Due April 1, beginning of class

"A problems" are for practice only, and should not be turned in.

Problem A1. Given any two context-free languages L_1 and L_2 over the same alphabet Σ, prove that $L_1 \cup L_2$ and L_1L_2 are also context-free.

Problem A2. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is context-free, then L^R is also context-free.

"B problems" must be turned in.

Problem B1 (60 pts). (1) Let $(A, \leq)$ be an ω-complete partial order. Assume we have m functions, $f_i\colon A^{m+n} \to A$, and n functions, $g_j\colon A^n \to A$, where $m, n \geq 1$ and where A^n and A^{m+n} are ω-complete partial orders under the componentwise ordering (i.e., $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ iff $a_i \leq b_i$, for $i = 1, \ldots, n$). Define the functions, $F\colon A^{m+n} \to A^{m+n}$ and $G\colon A^n \to A^n$, by

$$F_i(a_1, \ldots, a_{m+n}) = \begin{cases} f_i(a_1, \ldots, a_{m+n}) & \text{if } 1 \leq i \leq m \\ g_{i-m}(a_{m+1}, \ldots, a_{m+n}) & \text{if } m+1 \leq i \leq m+n, \end{cases}$$

and

$$G_j(a_1, \ldots, a_n) = g_j(a_1, \ldots, a_n), \quad \text{for } j = 1, \ldots, n.$$

If the f_i's and the g_j's are ω-continuous, check that F and G are ω-continuous. Thus, F has a least fixed point, say

$$\gamma = (\alpha', \beta') = (\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n)$$

and G has a least fixed point, say $\beta = (\beta_1, \ldots, \beta_n)$. We can also consider the function, $F_\beta\colon A^m \to A^m$, given by

$$F_{\beta,i}(a_1, \ldots, a_m) = f_i(a_1, \ldots, a_m, \beta_1, \ldots, \beta_n), \quad \text{for } i = 1, \ldots, m.$$

Check that F_β is ω-continuous. Thus, F_β has a least fixed point, say $\alpha = (\alpha_1, \ldots, \alpha_m)$. Prove that $\alpha' = \alpha$ and $\beta' = \beta$.

This shows that the least fixed point of F can be computed in two stages: First, compute the least fixed point of G; then, substitute the component of this least fixed point for the last n arguments of F and compute the least fixed point of the resulting function, F_β.

(2) Given any two alphabets Σ, Δ, a *substitution* is a function $\tau\colon \Sigma \to 2^{\Delta^*}$ assigning some language $\tau(a) \subseteq \Delta^*$ to every symbol $a \in \Sigma$. A substitution $\tau\colon \Sigma \to 2^{\Delta^*}$ is extended to a map $\tau\colon 2^{\Sigma^*} \to 2^{\Delta^*}$ by first extending τ to strings using the following definition

$$\tau(\epsilon) = \{\epsilon\},$$
$$\tau(ua) = \tau(u)\tau(a),$$

where $u \in \Sigma^*$ and $a \in \Sigma$, and then to languages by letting

$$\tau(L) = \bigcup_{w \in L} \tau(w),$$

for every language $L \subseteq \Sigma^*$.

For example, let $\tau\colon \Sigma \to 2^{\Sigma^*}$ be the substitution defined such that $\tau(a) = \{\epsilon, a\}$ for every $a \in \Sigma$. Explain (in words) what $\tau(L)$ is.

In general, prove, **using B1 (1)**, that if L is a context-free language and if $\tau(a)$ is a context-free language for every $a \in \Sigma$, then $\tau(L)$ is also a context-free language. Deduce from the above that the context-free languages are closed under homomorphisms.

(3) Are the regular languages closed under substitution by regular languages?

(4) Prove again that if L is a context-free language and if $\tau(a)$ is a context-free language for every $a \in \Sigma$, then $\tau(L)$ is also a context-free language, this time, using CFG's and derivations.

Problem B2 (20 pts). Given any set A, recall that we denote the power set of A by 2^A. We define the partial order $\leq$ on $2^A \times 2^A$ as follows: For all $A_1, A_2, B_1, B_2 \subseteq A$,

$$(A_1, A_2) \leq (B_1, B_2) \quad \text{iff} \quad A_1 \subseteq B_1 \quad \text{and} \quad A_2 \subseteq B_2.$$

(i) Check that the least upper bound of every ω-chain $(X_i, Y_i)_{i \geq 0}$ in $2^A \times 2^A$ is

$$\Big(\bigcup_{i \geq 0} X_i, \bigcup_{i \geq 0} Y_i\Big).$$

(ii) Now, assume that $A = \Sigma^*$, where Σ is some finite alphabet. Prove that union, concatenation and intersection (functions $2^{\Sigma^*} \times 2^{\Sigma^*} \longrightarrow 2^{\Sigma^*}$) are ω-continuous. What about complementation ? (a function $2^{\Sigma^*} \longrightarrow 2^{\Sigma^*}$).

Extra Credit (20 pts). Prove that the function Φ_G associated with a context-free grammar G (see the notes, page 41-42) is ω-continuous.

Problem B3 (60 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Consider the following procedure:

(1) Form an NFA, N, by reversing all the transitions of D, i.e., there is a transition from p to q on input $a \in \Sigma$ in N iff $\delta(q, a) = p$ in D.

(2) Apply the subset construction to the NFA, N, obtained in (1), taking the start state to be the set F and the only final state of N to be $\{q_0\}$. Then, trim the resulting DFA, obtaining the DFA D^R.

Observe that $L(D^R) = L(D)^R$.

Now, apply the above procedure to D, getting D^R, and apply this procedure again, to get D^{RR}. Prove that D^{RR} is a minimal DFA for $L = L(D)$.

Hint. First prove that if δ_R is the transition function of D^R, then for every $w \in \Sigma^*$ and for every state, T, of D^R,

$$\delta_R^*(T, w) = \{q \in Q \mid \delta^*(q, w^R) \in T\}.$$

Problem B4 (60 pts). Give context-free grammars for the following languages:

(a) $L_5 = \{ww^R \mid w \in \{a, b\}^*\}$ (w^R denotes the reversal of w)

(b) $L_6 = \{a^m b^n \mid 1 \leq m \leq n \leq 3m\}$

For any fixed integer $K \geq 2$,

$L_7 = \{a^m b^n \mid 1 \leq m \leq n \leq Km\}$

(c) $L_8 = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\}$

(d) $L_9 = \{a^m b^n a^m b^p \mid m, n, p \geq 1\} \cup \{a^m b^{3n} a^p b^{3n} \mid m, n, p \geq 1\}$

(e) $L_{10} = \{xcy \mid |x| = 3|y|,\ x, y \in \{a, b\}^*\}$

In each case, give a justification of the fact that your grammar generates the desired language.

Problem B5 (40 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q\ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p, a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \rightarrow \epsilon \quad \text{iff} \quad (S \rightarrow \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \underset{lm}{\overset{n}{\Longrightarrow}}_{G_2} w \quad \text{iff} \quad A \underset{lm}{\overset{n}{\Longrightarrow}}_{G} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

Problem B6 (40 pts). Give context-free grammars for the languages

$$\begin{aligned} L_1 &= \{xcy \mid x \neq y,\ x, y \in \{a, b\}^*\} \\ L_2 &= \{xcy \mid \ x \neq y^R,\ x, y \in \{a, b\}^*\}. \end{aligned}$$

TOTAL: 280 points.