Spring, 2004 CIS 511

Introduction to the Theory of Computation
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Homework 4

March 4, 2004; Due March 25, 2004

"A problems" are for practice only, and should not be turned in.

Problem A1. Given any two context-free languages L_1 and L_2 over the same alphabet Σ, prove that $L_1 \cup L_2$ and $L_1 L_2$ are also context-free.

Problem A2. Let Σ and Δ be some alphabets, and let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism. Given any language $L \subseteq \Sigma^*$, recall that

$$h(L) = \{h(w) \in \Delta^* \mid w \in L\}.$$

Prove that if L is context-free, then $h(L)$ is also context-free.

Problem A3. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is context-free, then L^R is also context-free.

"B problems" must be turned in.

Problem B1 (70 pts). (i) Prove that the conclusion of the pumping lemma holds for the following language L over $\{a, b\}^*$, and yet, L is **not** regular!

$$L = \{w \mid \exists n \geq 1, \exists x_i \in a^+, \exists y_i \in b^+, 1 \leq i \leq n, n \text{ is not prime}, w = x_1 y_1 \cdots x_n y_n\}.$$

(ii) Consider the following version of the pumping lemma. For any regular language L, there is some $m \geq 1$ so that for every $y \in \Sigma^*$, if $|y| = m$, then there exist $u, x, v \in \Sigma^*$ so that

(1) $y = uxv$;

(2) $x \neq \epsilon$;

(3) For all $z \in \Sigma^*$,

$$yz \in L \quad \text{iff} \quad ux^i vz \in L$$

for all $i \geq 0$.

Prove that this pumping lemma holds.

(iii) Prove that the converse of the pumping lemma in (ii) also holds, i.e., if a language L satisfies the pumping lemma in (ii), then it is regular.

(iv) Consider yet another version of the pumping lemma. For any regular language L, there is some $m \geq 1$ so that for every $y \in \Sigma^*$, if $|y| \geq m$, then there exist $u, x, v \in \Sigma^*$ so that

(1) $y = uxv$;

(2) $x \neq \epsilon$;

(3) For all $\alpha, \beta \in \Sigma^*$,

$$\alpha u \beta \in L \quad \text{iff} \quad \alpha u x^i \beta \in L$$

for all $i \geq 0$.

Prove that this pumping lemma holds.

(v) Prove that the converse of the pumping lemma in (iv) also holds, i.e., if a language L satisfies the pumping lemma in (iv), then it is regular.

Problem B2 (40 pts). The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Recall that *state equivalence* is the equivalence relation $\equiv$ on Q, defined such that,

$$p \equiv q \quad \text{iff} \quad \forall z \in \Sigma^* (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

and that *i-equivalence* is the equivalence relation $\equiv_i$ on Q, defined such that,

$$p \equiv_i q \quad \text{iff} \quad \forall z \in \Sigma^*,\ |z| \leq i\ (\delta^*(p, z) \in F \quad \text{iff} \quad \delta^*(q, z) \in F).$$

A relation $S \subseteq Q \times Q$ is a *forward closure* iff it is an equivalence relation and whenever $(p, q) \in S$, then $(\delta(p, a), \delta(q, a)) \in S$, for all $a \in \Sigma$.

We say that a forward closure S is *good* iff whenever $(p, q) \in S$, then *good*(p, q), where *good*(p, q) holds iff either both $p, q \in F$, or both $p, q \notin F$.

Given any relation $R \subseteq Q \times Q$, recall that the smallest equivalence relation $R_\approx$ containing R is the relation $(R \cup R^{-1})^*$ (where $R^{-1} = \{(q, p) \mid (p, q) \in R\}$, and $(R \cup R^{-1})^*$ is the reflexive and transitive closure of $(R \cup R^{-1})$). We define the sequence of relations $R_i \subseteq Q \times Q$ as follows:

$$R_0 = R_\approx$$
$$R_{i+1} = (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i,\ a \in \Sigma\})_\approx.$$

(i) Prove that $R_{i_0+1} = R_{i_0}$ for some least i_0. Prove that R_{i_0} is the smallest forward closure containing R.

We denote the smallest forward closure R_{i_0} containing R as $R^\dagger$, and call it the *forward closure of* R.

(ii) Prove that $p \equiv q$ iff the forward closure $R^\dagger$ of the relation $R = \{(p, q)\}$ is good.

Problem B3 (60 pts). Give context-free grammars for the following languages:

(a) $L_5 = \{wcw^R \mid w \in \{a,b\}^*\}$ (w^R denotes the reversal of w)

(b) $L_6 = \{a^m b^n \mid 1 \leq m \leq n \leq 2m\}$

For any fixed integer $K \geq 2$,

$L_7 = \{a^m b^n \mid 1 \leq m \leq n \leq Km\}$

(c) $L_8 = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\}$

(d) $L_9 = \{a^m b^n a^m b^p \mid m,n,p \geq 1\} \cup \{a^m b^{4n} a^p b^{4n} \mid m,n,p \geq 1\}$

(e) $L_{10} = \{xcy \mid |x| = 2|y|,\ x,y \in \{a,b\}^*\}$

In each case, give a justification of the fact that your grammar generates the desired language.

Problem B4 (40 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q \ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p,a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \to \epsilon \quad \text{iff} \quad (S \to \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \overset{n}{\underset{lm}{\Longrightarrow}}_{G_2} w \quad \text{iff} \quad A \overset{n}{\underset{lm}{\Longrightarrow}}_{G} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

Problem B5 (40 pts). Give context-free grammars for the languages

$$\begin{aligned} L_1 &= \{xcy \mid x \neq y,\, x, y \in \{a, b\}^*\} \\ L_2 &= \{xcy \mid x \neq y^R,\, x, y \in \{a, b\}^*\}. \end{aligned}$$

Problem B6 (50 pts). Let $L \subseteq \{a\}^*$ be a context-free language. Prove that L is actually a regular language. Proceed as follows. If L is finite, this is obvious, thus, assume that L is infinite. Let $L = L(G)$, for some CFG G.

(i) Let $K > 1$ be the constant of the pumping lemma for G, and let $r = K!$. Prove the following fact: for every $w \in L$, if $|w| \geq K$, then

$$\{wa^{rn} \mid n \geq 0\} \subseteq L.$$

(ii) For every i such that $0 \leq i < r$, let

$$L_i = \{a^n \mid a^n \in L,\, n \geq K,\, n \equiv i \bmod r\}.$$

Clearly,

$$L = \{a^n \mid a^n \in L,\, n < K\} \cup \bigcup_{i=0}^{r-1} L_i.$$

If $L_i \neq \emptyset$, let z_i be the shortest string in L_i. Prove that

$$L_i = \{z_i a^{rm} \mid m \geq 0\}.$$

Conclude that L is regular.

(iii) Prove that it is decidable whether $L_i = \emptyset$ (i.e., describe (concisely) an algorithm).

(iv) Given a context-free language L over $\{a, b\}$, prove that it is decidable whether $\{a\}^* \subseteq L$ (i.e., describe (concisely) an algorithm).

TOTAL: 300 points.