Fall 2002 CIS 510 Curves and Surfaces

Homework 3

November 12, 2002; Due December 3, beginning of class

You may work in groups of 2 to 4 students. Please, write up your solutions as clearly and concisely as possible. Be rigorous!

"A problems" are for practice only, and should not be turned in.

Problem A1. Compute a rectangular net for the surface defined by the equation

$$z = x^3 - 3xy^2$$

with respect to the affine frames $(-1, 1)$ and $(-1, 1)$. Compute a triangular net for the same surface with respect to the affine frame $((1, 0), (0, 1), (0, 0))$.

Note that z is the real part of the complex number $(u + iv)^3$. This surface is called a monkey *saddle*.

Problem A2. Compute a rectangular net for the surface defined by the equation

$$z = 1/6(x^3 + y^3)$$

with respect to the affine frames $(-1, 1)$ and $(-1, 1)$. Compute a triangular net for the same surface with respect to the affine frame $((1, 0), (0, 1), (0, 0))$.

"B problems" must be turned in.

Problem B1 (30 pts). Recall that the polar form of the monomial $u^h v^k$ with respect to the bidegree $\langle p, q\rangle$ (where $h \leq p$ and $k \leq q$) is

$$f_{h,k}^{p,q} = \frac{1}{\binom{p}{h}\binom{q}{k}} \sum_{\substack{I\subseteq\{1,\ldots,p\},|I|=h \\ J\subseteq\{1,\ldots,q\},|J|=k}} \left(\prod_{i\in I} u_i\right)\left(\prod_{j\in J} v_j\right).$$

Letting $\sigma_{h,k}^{p,q} = \binom{p}{h}\binom{q}{k} f_{h,k}^{p,q}$, prove that we have the following recurrence equations:

$$\sigma_{h,k}^{p,q} = \begin{cases} \sigma_{h,k}^{p-1,q-1} + u_p\sigma_{h-1,k}^{p-1,q-1} + v_q\sigma_{h,k-1}^{p-1,q-1} + u_p v_q\sigma_{h-1,k-1}^{p-1,q-1} & \text{if } 1 \leq h \leq p \text{ and } 1 \leq k \leq q, \\ \sigma_{0,k}^{p,q-1} + v_q\sigma_{0,k-1}^{p,q-1} & \text{if } h = 0 \leq p \text{ and } 1 \leq k \leq q, \\ \sigma_{h,0}^{p-1,q} + u_p\sigma_{h-1,0}^{p-1,q} & \text{if } 1 \leq h \leq p \text{ and } k = 0 \leq q, \\ 1 & \text{if } h = k = 0,\ p \geq 0, \text{ and } q \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, prove that $f_{h,k}^{p,q}$ can be computed directly using the recurrence formula

$$f_{h,k}^{p,q} = \frac{(p-h)(q-k)}{pq}\, f_{h,k}^{p-1,q-1} + \frac{h(q-k)}{pq} u_p\, f_{h-1,k}^{p-1,q-1} + \frac{(p-h)k}{pq} v_q\, f_{h,k-1}^{p-1,q-1} + \frac{hk}{pq} u_p v_q\, f_{h-1,k-1}^{p-1,q-1},$$

where $1 \leq h \leq p$ and $1 \leq k \leq q$,

$$f_{0,k}^{p,q} = \frac{(q-k)}{q}\, f_{0,k}^{p,q-1} + \frac{k}{q} v_q\, f_{0,k-1}^{p,q-1},$$

where $h = 0 \leq p$ and $1 \leq k \leq q$, and

$$f_{h,0}^{p,q} = \frac{(p-h)}{p}\, f_{h,0}^{p-1,q} + \frac{h}{p} u_p\, f_{h-1,0}^{p-1,q},$$

where $1 \leq h \leq p$ and $k = 0 \leq q$.

Show that for any $(u_1, \ldots, u_p, v_1, \ldots, v_q)$, computing all the polar values

$$f_{h,k}^{i,j}(u_1, \ldots, u_p, v_1, \ldots, v_q),$$

where $1 \leq h \leq i$, $1 \leq k \leq j$, $1 \leq i \leq p$, and $1 \leq j \leq q$, can be done in time $O(p^2 q^2)$.

Problem B2 (30 pts). Using the result of problem B1, write a computer program for computing the control points of a rectangular surface patch defined parametrically.

Problem B3 (30 pts). (1) Prove that the polar form of the monomial $u^h v^k$ with respect to the total degree m (where $h + k \leq m$) can be expressed as

$$f_{h,k}^{m} = \frac{1}{\binom{m}{h}\binom{m-h}{k}} \sum_{\substack{I \cup J \subseteq \{1,\ldots,m\} \\ |I|=h, |J|=k, I\cap J=\emptyset}} \left(\prod_{i\in I} u_i\right)\left(\prod_{j\in J} v_j\right).$$

Letting $\sigma_{h,k}^{m} = \binom{m}{h}\binom{m-h}{k} f_{h,k}^{m}$, prove that we have the following recurrence equations:

$$\sigma_{h,k}^{m} = \begin{cases} \sigma_{h,k}^{m-1} + u_m \sigma_{h-1,k}^{m-1} + v_m \sigma_{h,k-1}^{m-1} & \text{if } h,k \geq 0 \text{ and } 1 \leq h+k \leq m, \\ 1 & \text{if } h = k = 0 \text{ and } m \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, prove that $f_{h,k}^{m}$ can be computed directly using the recurrence formula

$$f_{h,k}^{m} = \frac{(m-h-k)}{m}\, f_{h,k}^{m-1} + \frac{h}{m} u_m\, f_{h-1,k}^{m-1} + \frac{k}{m} v_m\, f_{h,k-1}^{m-1},$$

where $h, k \geq 0$ and $1 \leq h + k \leq m$.

(2) Show that for any $((u_1, v_1), \ldots, (u_m, v_m))$, computing all the polar values

$$f^i_{h,k}((u_1, v_1), \ldots, (u_m, v_m)),$$

where $h, k \geq 0$, $1 \leq h + k \leq i$, and $1 \leq i \leq m$, can be done in time $O(m^3)$.

Problem B4 (30 pts). Using the result of problem B3, write a computer program for computing the control points of a triangular surface patch defined parametrically.

Problem B5 (20 pts). Compute a rectangular net for the surface defined by the equation

$$z = x^4 - 6x^2y^2 + y^4$$

with respect to the affine frames $(-1, 1)$ and $(-1, 1)$. Compute a triangular net for the same surface with respect to the affine frame $((1, 0), (0, 1), (0, 0))$.

Note that z is the real part of the complex number $(u + iv)^4$. This surface is a more complex kind of monkey *saddle*.

Problem B6 (20 pts). Compute a rectangular net for the surface defined by

$$\begin{aligned} x &= u(u^2 + v^2), \\ y &= v(u^2 + v^2), \\ z &= u^2v - v^3/3, \end{aligned}$$

with respect to the affine frames $(0, 1)$ and $(0, 1)$. Compute a triangular net for the same surface with respect to the affine frame $((1, 0), (0, 1), (0, 0))$.

Explain what happens for $(u, v) = (0, 0)$.

Problem B7 (100 pts). The purpose of this problem is to show how rational curves can be handled in terms of control points.

(1) Given an affine space $\mathcal{E}$, for any hyperplane H in $\mathcal{E}$ and any point a_0 not in H, the *central projection (or conic projection, or perspective projection) of center* a_0 *onto* H is the partial map p defined as follows: For every point x not in the hyperplane passing through a_0 and parallel to H, we define $p(x)$ as the intersection of the line defined by a_0 and x with the hyperplane H.

Assume that $\mathcal{E}$ has dimension $n + 1$ and that $(a_0, (e_1, \ldots, e_{n+1}))$ is an affine frame for $\mathcal{E}$. Let H_0 be the hyperplane of equation $x_{n+1} = 0$. We want to determine the coordinates of the central projection $p\colon (\mathcal{E} - H_0) \to \mathcal{E}$ of a point $x \in \mathcal{E} - H_0$ onto the hyperplane H of equation $x_{n+1} = 1$ (the center of projection being a_0). If

$$x = a_0 + x_1e_1 + \cdots + x_ne_n + x_{n+1}e_{n+1},$$

assuming that $x_{n+1} \neq 0$ prove that the coordinates of $p(x)$ are

$$p(x) = \Big(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\Big).$$

Note that $p(x)$ is undefined when $x_{n+1} = 0$, i.e., when $x \in H_0$.

In view of the above, it is natural to define the map $\pi\colon (\mathbb{A}^{n+1} - H_0) \to \mathbb{A}^n$ by

$$\pi(x_1, \ldots, x_n, x_{n+1}) = \Big(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\Big).$$

(2) A *rational curve of degree* m in $\mathbb{A}^n$ is specified by n fractions, say

$$x_1(t) = \frac{F_1(t)}{F_{n+1}(t)}, \quad x_2(t) = \frac{F_2(t)}{F_{n+1}(t)}, \quad \ldots, \quad x_n(t) = \frac{F_n(t)}{F_{n+1}(t)},$$

where $F_1(X), \ldots, F_{n+1}(X)$ are polynomials of degree at most m.

For example, the rational curve

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}$$

defines a unit circle in $\mathbb{A}^2$.

We can view a rational curve F in $\mathbb{A}^n$ as some appropriate projection $\pi(\widehat{F})$ of a polynomial curve $\widehat{F}$ in $\mathbb{A}^{n+1}$, back onto $\mathbb{A}^n$.

For example, we can view F as the polynomial curve $\widehat{F}$ in $\mathbb{A}^3$ given by

$$x = 1 - t^2, \quad y = 2t, \quad z = 1 + t^2.$$

If we project this curve $\widehat{F}$ using π onto $\mathbb{A}^2$, we get the previous circle F.

Given a rational curve F in $\mathbb{A}^n$ of degree m specified by

$$x_1(t) = \frac{F_1(t)}{F_{n+1}(t)}, \quad \ldots, \quad x_n(t) = \frac{F_n(t)}{F_{n+1}(t)},$$

define the polynomial curve $\widehat{F}$ of degree m in $\mathbb{A}^{n+1}$ as the curve defined by

$$x_1(t) = F_1(t),, \quad \ldots, \quad x_n(t) = F_n(t), \quad x_{n+1} = F_{n+1}(t).$$

The control points $(b_0, \ldots, b_m)$ of $\widehat{F}$ *in* $\mathbb{A}^{n+1}$ are called the *control points of the rational curve*. Observe that b_i is either of the form

$$b_i = (a_i, w_i),$$

or

$$b_i = (a_i, 0),$$

where $a_i \in \mathbb{A}^n$ and $w_i \neq 0$, $w_i \in \mathbb{R}$. In the first case, w_i is called the *weight* of b_i, and b_i is a *weighted control point*, and in the second case, b_i is called a *control vector*.

Show that the control points (in $\mathbb{A}^3$) of the circle

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}$$

w.r.t the frame $[0, 1]$ are

$$(1, 0, 1),\ (1, 1, 1), (0, 2, 2),$$

and w.r.t to the frame $[-1, 1]$ are

$$(0, -2, 2), (2, 0, 0), (0, 2, 2).$$

What do you observe in the second case?

(3) The de Casteljau algorithm can be adapted to draw rational curve segments as follows. Given a rational curve F of degree m in $\mathbb{A}^n$, compute the control points of $\widehat{F}$ in $\mathbb{A}^{n+1}$ with respect to $[r, s]$, apply the de Casteljau algorithm to $\widehat{F}$ (w.r.t $[r, s]$), and project the resulting approximation onto $\mathbb{A}^n$ using π. Note that points in H_0 are sent to infinity.

Apply this method to the following curves

(a) The circle w.r.t. $[0, 1]$ and $[-1, 1]$.

(b) The *Lemniscate of Bernoulli*, defined by the polynomials

$$x = \frac{t+t^3}{1+t^4}$$
$$y = \frac{t-t^3}{1+t^4}$$

w.r.t. $[0, 1]$ and $[-1, 1]$.

(c) The rose given by

$$x = \frac{4t(1-t^2)^2}{(1+t^2)^3}$$
$$y = \frac{8t^2(1-t^2)}{(1+t^2)^3}$$

w.r.t. $[0, 1]$ and $[-1, 1]$.

(d) The *Viviani window*, given by

$$x = \frac{2t-2t^3}{(1+t^2)^2}$$
$$y = \frac{4t^2}{(1+t^2)^2}$$
$$z = \frac{1-t^4}{(1+t^2)^2}$$

w.r.t. $[0, 1]$ and $[-1, 1]$.

For more examples, you may want to consult Chapter 22 of my (other) book at http://www.cis.upenn.edu/ jean/gbooks/geom2.html

(4) **Extra Credit**. If the rational curve, F, is closed, find a simple an efficient method to draw the entire curve as the union of two curve segments, F_1 and F_2, where the control points of F_2 are obtained in a simple way from the control points of F_1.

TOTAL: 260.