Fall 2002 CIS 510 Curves and Surfaces

Homework 2

October 17, 2002; Due November 5, beginning of class

You may work in groups of 2 to 4 students. Please, write up your solutions as clearly and concisely as possible. Be rigorous!

"B problems" must be turned in.

Problem B1 (60 pts). (i) Write a computer program taking a polynomial $F(X)$ of degree n (in one variable X), and returning its m-polar form f, where $n \leq m$. You may use *Mathematica* of any other language with symbolic facilities.

Estimate the complexity of your algorithm.

(ii) Let

$$f_k^m = \frac{1}{\binom{m}{k}} \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ |I|=k}} \left(\prod_{i \in I} t_i \right).$$

and $\sigma_k^m = \binom{m}{k} f_k^m$. Prove the following recurrence equations:

$$\sigma_k^m = \begin{cases} \sigma_k^{m-1} + t_m \sigma_{k-1}^{m-1} & \text{if } 1 \leq k \leq m; \\ 1 & \text{if } k = 0 \text{ and } m \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Does this remind you of the Pascal triangle for computing binomial coefficients?

Alternatively, show that f_k^m can be computed directly using the recurrence formula

$$f_k^m = \frac{(m-k)}{m} f_k^{m-1} + \frac{k}{m} t_m \, f_{k-1}^{m-1},$$

where $1 \leq k \leq m$.

(iii) Use part (ii) to design an efficient algorithm computing the control points $b_i = f(r^{m-i}, s^i)$ (where $0 \leq i \leq m$) w.r.t. some interval $[r, s]$, of a polynomial curve defined parametrically by $x = p_1(t), y = p_2(t)$, where $p_1(t)$ and $p_2(t)$ are polynomials, and the maximum of the degrees of p_1 and p_2 is m.
Hint. A polynomial is a sum of monomials, so we are reduced to monomials, i.e., what you need is to compute efficiently $\sigma_k^m(r^{m-i}, s^i)$. Use the above recurrence relations to do so.

Implement this algorithm.

After that, you can use your program from Homework 1, question B1, to display polynomial curves (over some interval $[r, s]$) given parametrically (using the subdivision version of the de Casteljau algorithm).

Problem B2 (50 pts). (Lagrange interpolation) Consider the following interpolation problem: Given a sequence $(\alpha_1, \ldots, \alpha_{m+1})$ of pairwise distinct scalars in $\mathbb{R}$ and any sequence $(\beta_1, \ldots, \beta_{m+1})$ of scalars in $\mathbb{R}$, where the β_j are not necessarily distinct, find a polynomial $P(X)$ of degree at most m such that

$$P(\alpha_1) = \beta_1, \ldots, P(\alpha_{m+1}) = \beta_{m+1}.$$

(a) First, prove that if such a polynomial exists, then it is unique.
Hint. If two polynomials P_1, P_2 of degree at most m solve the interpolation problem, then $P_1 - P_2$ is a polynomial of degree at most m that has $m + 1$ distinct roots.

(b) Consider the following so-called *Lagrange polynomials*:

$$L_i(X) = \frac{(X - \alpha_1) \cdots (X - \alpha_{i-1})(X - \alpha_{i+1}) \cdots (X - \alpha_{m+1})}{(\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \cdots (\alpha_i - \alpha_{m+1})}.$$

Note that $L(\alpha_i) = 1$ and that $L(\alpha_j) = 0$, for all $j \neq i$. Prove that

$$P(X) = \beta_1 L_1 + \cdots + \beta_{m+1} L_{m+1}$$

is the unique solution to the Lagrange interpolation problem. Prove that the polynomials $(L_1, \ldots, L_{m+1})$ form a basis of the vector space of all polynomials of degree $\leq m$.

(c) It has been observed that Lagrange interpolants oscillate quite badly as their degree increases, and thus, this makes them undesirable as a stable method for interpolation. A standard example due to Runge, is the function

$$f(x) = \frac{1}{1 + x^2},$$

in the interval $[-5, +5]$.

Assuming a uniform distribution of points on the curve in the interval $[-5, +5]$, as the degree of the Lagrange interpolant increases, the interpolant shows wilder and wilder oscillations around the points $x = -5$ and $x = +5$.

Plot the Lagrange interpolants of degree $1, 2, 3,\ 4, 5, 10,\ 14, 18, 22$ over $[-5, +5]$. When do you observe the *Runge phenomenon*.

Problem B3 (100 pts). (C^2-continuous cubic spline interpolation in Hermite form) Let $\mathcal{E}$ be some affine space, usually $\mathbb{A}^2$ or $\mathbb{A}^3$. Divide the interval $[0, s]$ $(s > 0)$ into $m \geq 2$ subintervals of equal length δ, so that $s = m\delta$, and consider the following interpolation problem: Given $m+1$ data points $x_0, \ldots, x_m$ in $\mathcal{E}$, find a (uniform) C^2-continuous cubic spline curve F (piecewise polynomial curve consisting of cubic curve segments joining with C^2-continuity) such that

$$F(i\delta) = x_i, \quad \text{for } i = 0, \ldots, m.$$

We can adapt the cubic Hermite form discussed in Homework 1 (B3) to solve this problem. Given any two points a_0, a_1 in $\mathcal{E}$ and any two vectors $\overrightarrow{u_0}$ and $\overrightarrow{u_1}$ in $\overrightarrow{\mathcal{E}}$, show that the unique cubic F such that

$$\begin{aligned} F(0) &= a_0, \\ F(\delta) &= a_1, \\ F'(0) &= \overrightarrow{u_0}, \\ F'(\delta) &= \overrightarrow{u_1}, \end{aligned}$$

is given by

$$F(t) = a_0 H_0(t/\delta) + \delta \overrightarrow{u_0} H_1(t/\delta) + \delta \overrightarrow{u_1} H_2(t/\delta) + a_1 H_3(t/\delta),$$

where H_0, H_1, H_2, H_3 are the Hermite polynomials (of degree 3)

$$\begin{aligned} H_0 &= 2t^3 - 3t^2 + 1, \\ H_1 &= t^3 - 2t^2 + t, \\ H_2 &= t^3 - t^2, \\ H_3 &= -2t^3 + 3t^2. \end{aligned}$$

In terms of control points, F can be expressed by

$$F(t) = b_0 B_0(t/\delta) + b_1 B_1(t/\delta) + b_2 B_2(t/\delta) + b_3 B_3(t/\delta),$$

where

$$b_0 = a_0, \quad b_1 = a_0 + \frac{\delta}{3}\overrightarrow{u_0}, \quad b_2 = a_1 - \frac{\delta}{3}\overrightarrow{u_1}, \quad b_3 = a_1,$$

and $B_0 = (1-t)^3$, $B_1 = 3t(1-t)^2$, $B_2 = 3t^2(1-t)$, $B_3 = t^3$ are the Bernstein polynomials over $[0, 1]$.

To find a solution to the interpolation problem, we assume that the interpolating spline consists of m cubic segments $F_0, \ldots, F_{m-1}$, where

$$F_i(t) = x_i H_0((t - i\delta)/\delta) + \delta \overrightarrow{u_i} H_1((t - i\delta)/\delta) + \delta \overrightarrow{u_{i+1}} H_2((t - i\delta)/\delta) + x_{i+1} H_3((t - i\delta)/\delta),$$

with $0 \le i \le m-1$ and $i\delta \le t \le (i+1)\delta$. The problem reduces to finding the $m+1$ vectors $\overrightarrow{u_0}, \ldots, \overrightarrow{u_m}$.

(a) Prove that

$$F_i''(i\delta) = \frac{1}{\delta^2}(-6x_i - 4\delta\overrightarrow{u_i} - 2\delta\overrightarrow{u_{i+1}} + 6x_{i+1}),$$

and

$$F_i''((i+1)\delta) = \frac{1}{\delta^2}(6x_i + 2\delta\overrightarrow{u_i} + 4\delta\overrightarrow{u_{i+1}} - 6x_{i+1}).$$

Prove that the condition for insuring C^2-continuity at $t = (i+1)\delta$ between F_i and F_{i+1} is

$$\overrightarrow{u_i} + 4\overrightarrow{u_{i+1}} + \overrightarrow{u_{i+2}} = \frac{3}{\delta}(x_{i+2} - x_i),$$

where $0 \leq i \leq m-2$.

(b) From (a), we are led to a system of $m-1$ equations in the $m+1$ unknowns $\overrightarrow{u_0}, \ldots, \overrightarrow{u_m}$. As such, the problem is underdetermined. To determine it completely, we need some *end conditions*. A simple type of end conditions are the so-called *clamped end conditions*, which consist in assuming that $\overrightarrow{u_0}$ and $\overrightarrow{u_m}$ are prescribed at both ends of the spline curve. Assuming clamped end conditions, show that $\overrightarrow{u_0}, \ldots, \overrightarrow{u_m}$ are solutions of the linear system

$$\begin{pmatrix} 1 & & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & 0 & & \\ & & & \ddots & & & & \\ & & 0 & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{u_0} \\ \overrightarrow{u_1} \\ \overrightarrow{u_2} \\ \vdots \\ \overrightarrow{u_{m-2}} \\ \overrightarrow{u_{m-1}} \\ \overrightarrow{u_m} \end{pmatrix} = \begin{pmatrix} \overrightarrow{w_0} \\ \overrightarrow{w_1} \\ \overrightarrow{w_2} \\ \vdots \\ \overrightarrow{w_{m-2}} \\ \overrightarrow{w_{m-1}} \\ \overrightarrow{w_m} \end{pmatrix}$$

where $\overrightarrow{w_i} = 3(x_{i+1} - x_{i-1})/\delta$ for $i = 1, \ldots, m-1$, and $\overrightarrow{w_0}$ and $\overrightarrow{w_m}$ are some prescribed vectors.

(c) Prove that the above matrix is invertible. Look up methods for solving tridiagonal linear systems efficiently (try *Numerical Recipes*).

(d) Using Homework 1, we can switch back to the Bézier form of cubic curve segments, i.e., we have

$$F_i(t) = x_i B_0((t - i\delta)/\delta) + b_{i,1}B_1((t - i\delta)\delta) + b_{i,2}B_2((t - i\delta)/\delta) + x_{i+1}B_3((t - i\delta)/\delta),$$

with $0 \leq i \leq m-1$ and $i\delta \leq t \leq (i+1)\delta$, where

$$b_{i,1} = x_i + \frac{\delta}{3}\overrightarrow{u_i}, \quad b_{i,2} = x_{i+1} - \frac{\delta}{3}\overrightarrow{u_{i+1}},$$

and $B_0 = (1-t)^3$, $B_1 = 3t(1-t)^2$, $B_2 = 3t^2(1-t)$, $B_3 = t^3$ are the Bernstein polynomials over $[0, 1]$. Thus, letting $\overrightarrow{v_i} = \delta\overrightarrow{u_i}$, we have

$$b_{i,1} = x_i + \frac{1}{3}\overrightarrow{v_i}, \quad b_{i,2} = x_{i+1} - \frac{1}{3}\overrightarrow{v_{i+1}},$$

and it is preferable to solve the system

$$\begin{pmatrix} 1 & & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & 0 & & \\ & & & \ddots & & & & \\ & & 0 & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{v_0} \\ \overrightarrow{v_1} \\ \overrightarrow{v_2} \\ \vdots \\ \overrightarrow{v_{m-2}} \\ \overrightarrow{v_{m-1}} \\ \overrightarrow{v_m} \end{pmatrix} = \begin{pmatrix} \overrightarrow{w_0} \\ \overrightarrow{w_1} \\ \overrightarrow{w_2} \\ \vdots \\ \overrightarrow{w_{m-2}} \\ \overrightarrow{w_{m-1}} \\ \overrightarrow{w_m} \end{pmatrix}$$

where $\overrightarrow{w_i} = 3(x_{i+1} - x_{i-1})$ for $i = 1, \ldots, m-1$, and $\overrightarrow{w_0}$ and $\overrightarrow{w_m}$ are some prescribed vectors.

Write a computer program for finding a cubic spline interpolating $m+1$ data points under the hypothesis of clamped end-conditions. In particular, use the above method to interpolate the function

$$f(x) = \frac{1}{1+x^2},$$

in the interval $[0, +5]$.

More specifically, write a computer program using the de Casteljau algorithm to plot the cubic interpolants corresponding to $3, 5, 7,\ 9, 11,$ and 13 points, assuming a uniform distribution of these points on the interval $[0, 5]$. Compare your results with those of Problem B2. How do you initialize the tangent vectors are 0 and 5?

(e) We can also look for a closed spline curve interpolating $x_0, \ldots, x_m$. In this case, we have the extra conditions that

$$x_m = x_0 \quad \text{and} \quad \overrightarrow{u_m} = \overrightarrow{u_0}.$$

This time, we have m distinct points $x_0, \ldots, x_{m-1}$ and again, m curve segments $F_0, \ldots, F_{m-1}$. By writing the C^2-continuity conditions between F_i and F_{i+1} for $i = 0, \ldots, m-2$, and between F_{m-1} and F_0, we obtain a system of m equations in the m unknowns $\overrightarrow{u_0}, \ldots, \overrightarrow{u_{m-1}}$. Show that the equation asserting C^2-continuity between F_{m-1} and F_0 is

$$4\overrightarrow{u_0} + \overrightarrow{u_1} + \overrightarrow{u_{m-1}} = \frac{3}{\delta}(x_1 - x_{m-1})/\delta.$$

Prove that $\overrightarrow{u_0}, \ldots, \overrightarrow{u_{m-1}}$ are solutions of the linear system

$$\begin{pmatrix} 4 & 1 & & & & & & 1 \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & 0 & & \\ & & & \ddots & & & & \\ & & 0 & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ 1 & & & & & & 1 & 4 \end{pmatrix} \begin{pmatrix} \overrightarrow{u_0} \\ \overrightarrow{u_1} \\ \overrightarrow{u_2} \\ \vdots \\ \overrightarrow{u_{m-2}} \\ \overrightarrow{u_{m-1}} \end{pmatrix} = \begin{pmatrix} \overrightarrow{w_0} \\ \overrightarrow{w_1} \\ \overrightarrow{w_2} \\ \vdots \\ \overrightarrow{w_{m-2}} \\ \overrightarrow{w_{m-1}} \end{pmatrix}$$

where $\overrightarrow{w_i} = 3(x_{i+1} - x_{i-1})/\delta$ for $i = 1, \ldots, m-2$, $\overrightarrow{w_0} = 3(x_1 - x_{m-1})/\delta$, and $\overrightarrow{w_{m-1}} = 3(x_0 - x_{m-2})/\delta$.

Why is this matrix invertible?

(Extra credit) (40 pts) Write a computer program for finding a closed cubic spline interpolating m points.

(f) (Extra credit) (30 pts) We can also consider the slightly more general problem of finding a *nonuniform* interpolating spline. This time, we subdivide the interval $[0, s]$ into m intervals $[s_i, s_{i+1}]$ of possibly different lengths. Thus, we assume that we have an increasing sequence

$$s_0 = 0 < s_1 < s_2 < \cdots < s_{m-1} < s_m = s.$$

We let $\Delta_i = s_{i+1} - s_i$, $0 \leq i \leq m-1$. The interpolation problem is to find a (nonuniform) C^2-continuous cubic spline curve F such that

$$F(s_i) = x_i$$

for $i = 0, \ldots, m$. The original problem is the special case where $\Delta_i = \delta$ is constant. Show that F_i is given by

$$F_i(t) = x_i H_0((t - s_i)/\Delta_i) + \delta \overrightarrow{u_i} H_1((t - s_i)/\Delta_i) + \delta \overrightarrow{u_{i+1}} H_2((t - s_i)/\Delta_i) + x_{i+1} H_3((t - s_i)/\Delta_i),$$

with $0 \leq i \leq m-1$ and $s_i \leq t \leq s_{i+1}$. Prove that the condition for insuring C^2-continuity at $t = s_{i+1}$ between F_i and F_{i+1} is

$$\Delta_{i+1} \overrightarrow{u_i} + 2(\Delta_i + \Delta_{i+1}) \overrightarrow{u_{i+1}} + \Delta_i \overrightarrow{u_{i+2}} = 3 \left(\frac{\Delta_i (x_{i+2} - x_{i+1})}{\Delta_{i+1}} + \frac{\Delta_{i+1}(x_{i+1} - x_i)}{\Delta_i} \right)$$

where $0 \leq i \leq m-2$.

Write the matrix of the linear system of equations, assuming clamped end conditions. Why is the matrix invertible?

(g) (Extra credit) (40 pts) Apply the method to other interpolation problems for curves in $\mathbb{A}^2$, for example, drawing letters of some alphabet.

Problem B4 (20 pts). Let $F\colon \mathbb{A} \to \mathcal{E}$ be a polynomial cubic curve. Prove that the polar form $f\colon \mathbb{A}^3 \to \mathcal{E}$ of F can be expressed as

$$f(u, v, w) = \frac{1}{24} \left[27F\left(\frac{u + v + w}{3}\right) - F(u + v - w) - F(u + w - v) - F(v + w - u) \right]$$

TOTAL: 230 + 30 points.