UPR-1011-T

On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective

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Abstract

In this writing we shall address certain beautiful inter-relations between the construction of 4-dimensional supersymmetric gauge theories and resolution of algebraic singularities, from the perspective of String Theory. We review in some detail the requisite background in both the mathematics, such as orbifolds, symplectic quotients and quiver representations, as well as the physics, such as gauged linear sigma models, geometrical engineering, Hanany-Witten setups and D-brane probes.

We investigate aspects of world-volume gauge dynamics using D-brane resolutions of various Calabi-Yau singularities, notably Gorenstein quotients and toric singularities. Attention will be paid to the general methodology of constructing gauge theories for these singular backgrounds, with and without the presence of the NS-NS B-field, as well as the T-duals to brane setups and branes wrapping cycles in the mirror geometry. Applications of such diverse and elegant mathematics as crepant resolution of algebraic singularities, representation of finite groups and finite graphs, modular invariants of affine Lie algebras, etc. will naturally arise. Various viewpoints and generalisations of McKay's Correspondence will also be considered.

The present work is a transcription of excerpts from the first three volumes of the author's PhD thesis which was written under the direction of Prof. A. Hanany - to whom he is much indebted - at the Centre for Theoretical Physics of MIT, and which, at the suggestion of friends, he posts to the ArXiv pro hac vice; it is his sincerest wish that the ensuing pages might be of some small use to the beginning student.

[1]Research supported at various stages under the gracious patronage of the CTP and the LNS of MIT under the U.S. Department of Energy cooperative research agreement #DE-FC02-94ER40818, the KITP of UCSB under NSF grant PHY94-07194, the Dept. of Physics of UPenn under #DE-FG02-95ER40893, an NSF Graduate Fellowship, the Presidential Fellowship of MIT, as well as the C. Reed Fund.

Præfatio et Agnitio

Forsan et haec olim meminisse iuvabit. *Vir. Aen. I.1.203*

N*ot that I merely owe this title to the font, my education, or the clime wherein I was born, as being bred up either to confirm those principles my parents instilled into my understanding, or by a general consent proceed in the religion of my country; but having, in my riper years and confirmed judgment, seen and examined all, I find myself obliged, by the principles of grace, and the law of mine own reason, to embrace no other name but this.*

So wrote Thomas Browne in *Religio Medici* of his conviction to his Faith. Thus too let me, with regard to that title of "Physicist," of which alas I am most unworthy, with far less wit but with equal devotion, confess my allegiance to the noble Cause of Natural Philosophy, which I pray that in my own riper years I shall embrace none other. Therefore prithee gentle reader, bear with this fond fool as he here leaves his rampaging testimony to your clemency.

Some nine years have past and gone, since when the good Professor H. Verlinde, of Princeton, first re-embraced me from my straying path, as Saul was upon the road to Damascus - for, Heaven forbid, that in the even greater folly of my youth I had once blindly fathomed to be my destiny the more pragmatic career of an Engineer (pray mistake me not, as I hold great esteem for this Profession, though had I pursued her my own heart and soul would have been greatly misplaced indeed) - to the Straight and Narrow path leading to Theoretical Physics, that Holy Grail of Science.

I have suffered, wept and bled sweat of labour. Yet the divine Bach reminds us in the Passion of Our Lord according to Matthew, "*Ja! Freilich will in uns das Fleisch und Blut zum Kreuz gezwungen sein; Je mehr es unsrer Seele gut, Je herber geht es ein.*" Ergo, I too have rejoiced, laughed and shed tears of jubilation. Such is the nature of Scientific Research, and indeed the grand *Principia Vitæ*. These past half of a decade has been constituted of thousands of nightly lucubrations, each a battle, each *une petite mort*, each with its *te Deum* and *Non Nobis Domine*. I carouse to these five years past, short enough to be one day deemed a mere passing period, long enough to have earned some silvery strands upon my idle rank.

And thus commingled, the *fructus labori* of these years past, is the humble work I shall present in the ensuing pages. I beseech you o gentle reader, to indulge its length, I regret to confess that what I lack in content I can only supplant with volume, what I lack in wit I can only distract with loquacity. To that great Gaussian principle of *Pauca sed Matura* let me forever bow in silent shame.

Yet the poorest offering does still beseech painstaking preparation and the lowliest work, a helping hand. How blessed I am, to have a flight souls aiding me in bearing the great weight!

For what is a son, without the wings of his parent? How blessed I am, to have my dear mother and father, my aunt DaYi and grandmother, embrace me with four-times compounded love! Every fault, a tear, every wrong, a guiding hand and every triumph, an exaltation.

For what is Dante, without his Virgil? How blessed I am, to have the perspicacious guidance of the good Professor Hanany, who in these 4 years has taught me so much! His ever-lit lamp and his ever-open door has been a beacon for home amidst the nightly storms of life and physics. In addition thereto, I am indebted to Professors Zwiebach, Freedman and Jaffe, together with all my honoured Professors and teachers, as well as the ever-supportive staff: J. Berggren, R. Cohen, S. Morley and E. Sullivan at the Centre for Theoretical Physics, to have brought me to my intellectual manhood.

For what is Damon, without his Pythias? How blessed I am, to have such multitudes of friends! I drink to their health! To the Ludwigs: my brother, mentor and colleague in philosophy and mathematics, J. S. Song and his JJFS; my brother and companion in wine and Existentialism, N. Moeller and his Marina. To my collaborators: my colleagues and brethren, B. Feng, I. Ellwood, A. Karch, N. Prezas and A. Uranga. To my brothers in Physics and remembrances past: I. Savonije and M. Spradlin, may that noble Nassau-Orange thread bind the colourless skeins of our lives. To my Spiritual counsellors: M. Serna and his ever undying passion for Physics, D. Matheu and his Franciscan soul, L. Pantelidis and his worldly wisdom, as well as the Schmidts and the Domesticity which they symbolise. To the fond memories of one beauteous adventuress Ms. M. R. Warden, who once wept with me at the times of sorrow and danced with me at the moments of delight. And to you all my many dear beloved friends whose names, though I could not record here, I shall each and all engrave upon my heart.

And so composed is a fledgling, through these many years of hearty battle, and amidst blood, sweat and tears was formed another grain of sand ashore the Vast Ocean of Unknown. Therefore at this eve of my reception of the title *Doctor Philosophiae*, though I myself could never dream to deserve to be called either "learned" or a "philosopher," I shall fast and pray, for henceforth I shall bear, as Atlas the weight of Earth upon his shoulders, the name "Physicist" upon my soul. And so I shall prepare for this my initiation into a Brotherhood of Dreamers, as an incipient neophyte intruding into a Fraternity of Knights, accoladed by the sword of *Regina Mathematica*, who dare to uphold that Noblest calling of *"Sapere Aude"*.

Let me then embrace, not with merit but with homage, not with arms eager but with knees bent, and indeed not with a mind deserving but with a heart devout, naught else but this dear cherished Title of "Physicist."

I call upon ye all, gentle readers, my brothers and sisters, all the Angels and Saints, and Mary, ever Virgin, to pray for me, *Dei Sub Numine*, as I dedicate this humble work and my worthless self,

Ad Catharinae Sanctae Alexandriae et Ad Majorem Dei Gloriam...

De Singularitatis Algebraicæ, Graphicæ Finitatis, & Theorica Mensuræ Branæ Dirichletiensis: Aspectus Theoricæ Chordæ, cum digressi super theorica campi chordae. Libellus in Quattuor Partibus, sub Auspicio CTP et LNS, MIT, atque DOE et NSF, sed potissimum, Sub Numine Dei.

Invocatio et Apologia

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We live in an Age of Dualism. The Absolutism which has so long permeated through Western Thought has been challenged in every conceivable fashion: from philosophy to politics, from religion to science, from sociology to aesthetics. The ideological conflicts, so often ending in tragedy and so much a theme of the twentieth century, had been intimately tied with the recession of an archetypal norm of undisputed Principles. As we enter the third millennium, the Zeitgeist is already suggestive that we shall perhaps no longer be victims but beneficiaries, that the uncertainties which haunted and devastated the proceeding century shall perhaps serve to guide us instead.

Speaking within the realms of Natural Philosophy, beyond the wave-particle duality or the Principle of Equivalence, is a product which originated in the 60's and 70's, a product which by now so well exemplifies a dualistic philosophy to its very core.

What I speak of, is the field known as String Theory, initially invented to explain the dual-resonance behaviour of hadron scattering. The dualism which I emphasise is more than the fact that the major revolutions of the field, string duality and D-branes, AdS/CFT Correspondence, etc., all involve dualities in a strict sense, but more so the fact that the essence of the field still remains to be defined. A chief theme of this writing shall be the dualistic nature of String theory as a scientific endeavour: it has thus far no experimental verification to be rendered physics and it has thus far no rigorous formulations to be considered mathematics. Yet String theory has by now inspired so much activity in both physics and mathematics that, to quote C. N. Yang in the early days of Yang-Mills theory, its beauty alone certainly merits our attention.

I shall indeed present you with breath-taking beauty; in Books I and II, I shall carefully guide the readers, be them physicists or mathematicians, to a preparatory journey to the requisite mathematics in Liber I and to physics in Liber II. These two books will attempt to review a tiny fraction of the many subjects developed in the last few decades in both fields in relation to string theory. I quote here a saying of E. Zaslow of which I am particularly fond, though it applies to me far more appropriately: in the Book on mathematics I shall be the physicist and the Book on physics, I the mathematician, so as to beg the reader to forgive my inexpertise in both.

Books III and IV shall then consist of some of my work during my very enjoyable stay at the Centre for Theoretical Physics at MIT as a graduate student. I regret that I shall tempt the readers with so much elegance in the first two books and yet lead them to so humble a work, that the journey through such a beautiful garden would end in such a witless swamp. And I take the opportunity to apologise again to the reader for the excruciating length, full of sound and fury and signifying nothing. Indeed as Saramago points out that the shortness of life is so incompatible with the verbosity of the world.

Let me speak no more and let our journey begin. Come then, ye Muses nine, and with strains divine call upon mighty Diane, that she, from her golden quiver may draw the arrow, to pierce my trembling heart so that it could bleed the ink with which I shall hereafter compose this my humble work...

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De Singularitatis Algebraicæ, Graphicæ Finitatis, & Theorica Mensuræ Branæ Dirichletiensis: Aspectus Theoricæ Chordæ

Chapter 1

INTROIT

The two pillars of twentieth century physics, General Relativity and Quantum Field Theory, have brought about tremendous progress in Physics. The former has described the macroscopic, and the latter, the microscopic, to beautiful precision. However, the pair, in and of themselves, stand incompatible. Standard techniques of establishing a quantum theory of gravity have met uncancellable divergences and unrenormalisable quantities.

As we enter the twenty-first century, a new theory, born in the mid-1970's, has promised to be a candidate for a Unified Theory of Everything. The theory is known as **String Theory**, whose basic tenet is that all particles are vibrational modes of strings of Plankian length. Such elegant structure as the natural emergence of the graviton and embedding of electromagnetic and large N dualities, has made the theory more and more attractive to the theoretical physics community. Moreover, concurrent with its development in physics, string theory has prompted enormous excitement among mathematicians. Hitherto unimagined mathematical phenomena such as Mirror Symmetry and orbifold cohomology have brought about many new directions in algebraic geometry and representation theory.

Promising to be a Unified Theory, string theory must incorporate the Standard Model of interactions, or minimally supersymmetric extensions thereof. The purpose of this work is to study various aspects of a wide class of gauge theories arising from string theory in the background of singularities, their dynamics, moduli spaces, duality transformations etc. as well as certain branches of associated mathematics. We will investigate how these gauge theories, of various supersymmetry and in various dimensions, arise as low-energy effective theories associated with hypersurfaces in String Theory known as D-branes.

It is well-known that the initial approach of constructing the real world from String Theory had been the compactification of the 10 dimensional superstring or the 10(26) dimensional heterotic string on Calabi-Yau manifolds of complex dimension three. These are complex manifolds described as algebraic varieties with Ricci-flat curvature so as to preserve supersymmetry. The resulting theories are $\mathcal{N} = 1$ supersymmetric gauge theories in 4 dimensions that would be certain minimal extensions of the Standard Model.

This paradigm has been widely pursued since the 1980's. However, we have a host of Calabi-Yau threefolds to choose from. The inherent length-scale of the superstring and deformations of the world-sheet conformal field theory, made such violent behaviour as topology changes in space-time natural. These changes connected vast classes of manifolds related by, notably, mirror symmetry. For the physics, these mirror manifolds which are markedly different mathematical objects, give rise to the same conformal field theory.

Physics thus became equivalent with respect to various different compactifications. Even up to this equivalence, the plethora of Calabi-Yau threefolds (of which there is still yet no classification) renders the precise choice of the compactification difficult to select. A standing problem then has been this issue of "vacuum degeneracy."

Ever since Polchinski's introduction of D-branes into the arena in the Second String Revolution of the mid-90's, numerous novel techniques appeared in the construction of gauge theories of various supersymmetries, as low-energy effective theories of the ten dimensional superstring and eleven dimensional M-theory (as well as twelve dimensional F-theory).

The natural existence of such higher dimensional surfaces from a theory of strings proved to be crucial. The Dp-branes as well as Neveu-Schwarz (NS) 5-branes are carriers of Ramond-Ramond and NS-NS charges, with electromagnetic duality (in

10-dimensions) between these charges (forms). Such a duality is well-known in supersymmetric field theory, as exemplified by the four dimensional Montonen-Olive Duality for $\mathcal{N} = 4$, Seiberg-Witten for $\mathcal{N} = 2$ and Seiberg's Duality for $\mathcal{N} = 1$. These dualities are closely associated with the underlying S-duality in the full string theory, which maps small string coupling to the large.

Furthermore, the inherent winding modes of the string includes another duality contributing to the dualities in the field theory, the so-called T-duality where small compactification radii are mapped to large radii. By chains of applications of S and T dualities, the Second Revolution brought about a unification of the then five disparate models of consistent String Theories: types I, IIA/B, Heterotic $E_8 \times E_8$ and Heterotic Spin$(32)/\mathbb{Z}_2$.

Still more is the fact that these branes are actually solutions in 11-dimensional supergravity and its dimensional reduction to 10. Subsequently proposals for the enhancement for the S and T dualities to a full so-called U-Duality were conjectured. This would be a symmetry of a mysterious underlying M-theory of which the unified string theories are but perturbative limits. Recently Vafa and collaborators have proposed even more intriguing dualities where such U-duality structure is intimately tied with the geometric structure of blow-ups of the complex projective 2-space, viz., the del Pezzo surfaces.

With such rich properties, branes will occupy a central theme in this writing. We will exploit such facts as their being BPS states which break supersymmetry, their dualisation to various pure geometrical backgrounds and their ability to probe sub-stringy distances. We will investigate how to construct gauge theories empowered with them, how to realise dynamical processes in field theory such as Seiberg duality in terms of toric duality and brane motions, how to study their associated open string states in bosonic string field theory as well as many interesting mathematics that emerge.

We will follow the thread of thought of the trichotomy of methods of fabricating low-energy effective super-Yang-Mills theories which soon appeared in quick succession in 1996, after the D-brane revolution.

One method was very much in the geometrical vein of compactification: the so-named **geometrical engineering** of Katz-Klemm-Lerche-Vafa. With branes of various dimensions at their disposal, the authors wrapped (homological) cycles in the Calabi-Yau with branes of the corresponding dimension. The supersymmetric cycles (i.e., cycles which preserve supersymmetry), especially the middle dimensional 3-cycles known as Special Lagrangian submanifolds, play a crucial rôle in Mirror Symmetry.

In the context of constructing gauge theories, the world-volume theory of the wrapped branes are described by dimensionally reduced gauge theories inherited from the original D-brane and supersymmetry is preserved by the special properties of the cycles. Indeed, at the vanishing volume limit gauge enhancement occurs and a myriad of supersymmetric Yang-Mills theories emerge. In this spirit, certain global issues in compactification could be addressed in the analyses of the local behaviour of the singularity arising from the vanishing cycles, whereby making much of the geometry tractable.

The geometry of the homological cycles, together with the wrapped branes, determine the precise gauge group and matter content. In the language of sheafs, we are studying the intersection theory of coherent sheafs associated with the cycles. We will make usage of these techniques in the study of such interesting behaviour as "toric duality."

The second method of engineering four dimensional gauge theories from branes was to study the world-volume theories of configurations of branes in 10 dimensions. Heavy use were made especially of the D4 brane of type IIA, placed in a specific position with respect to various D-branes and the solitonic NS5-branes. In the limit of low energy, the world-volume theory becomes a supersymmetric gauge theory in 4-dimensions.

Such configurations, known as **Hanany-Witten setups**, provided intuitive realisations of the gauge theories. Quantities such as coupling constants and beta functions were easily visualisable as distances and bending of the branes in the setup. Moreover, the configurations lived directly in the flat type II background and the intricacies involved in the curved compactification spaces could be avoided altogether.

The open strings stretching between the branes realise as the bi-fundamental and adjoint matter of the resulting theory while the configurations are chosen judiciously to break down to appropriate supersymmetry. Motions of the branes relative to each other correspond in the field theory to moving along various Coulomb and Higgs branches of the Moduli space. Such dynamical processes as the Hanany-Witten Effect of brane creation lead to important string theoretic realisations of Seiberg's duality.

We shall too take advantage of the insights offered by this technique of brane setups which make quantities of the product gauge theory easily visualisable.

The third method of engineering gauge theories was an admixture of the above two, in the sense of utilising both brane dynamics and singular geometry. This became known as the **brane probe** technique, initiated by Douglas and Moore. Stacks of parallel D-branes were placed near certain local Calabi-Yau manifolds; the world-volume theory, which would otherwise be the uninteresting parent $U(n)$ theory in flat space, was projected into one with product gauge groups, by the geometry of the singularity on the open-string sector.

Depending on chosen action of the singularity, notably orbifolds, with respect to the $SU(4)$ R-symmetry of the parent theory, various supersymmetries can be achieved. When we choose the singularity to be $SU(3)$ holonomy, a myriad of gauge theories of $\mathcal{N}=1$ supersymmetry in 4-dimensions could be thus fabricated given local structures of the algebraic singularities. The moduli space, as solved by the vacuum conditions of D-flatness and F-flatness in the field theory, is then by construction, the Calabi-Yau singularity. In this sense space-time itself becomes a derived concept, as realised by the moduli space of a D-brane probe theory.

As Maldacena brought about the Third String Revolution with the AdS/CFT conjecture in 1997, new light shone upon these probe theories. Indeed the $SU(4)$ R-symmetry elegantly manifests as the $SO(6)$ isometry of the 5-sphere in the $AdS_5 \times S^5$ background of the bulk string theory. It was soon realised by Kachru, Morrison, Silverstein et al. that these probe theories could be harnessed as numerous checks for the correspondence between gauge theory and near horizon geometry.

Into various aspects of these probes theories we shall delve throughout the writing and attention will be paid to two classes of algebraic singularities, namely orbifolds and toric singularities,

With the wealth of dualities in String Theory it is perhaps of no surprise that the three methods introduced above are equivalent by a sequence of T-duality (mirror) transformations. Though we shall make extensive usage of the techniques of all three throughout this writing, focus will be on the latter two, especially the last. We shall elucidate these three main ideas: geometrical engineering, Hanany-Witten brane configurations and D-branes transversely probing algebraic singularities, respectively in Chapters 6, 7 and 8 of Book II.

The abovementioned, of tremendous interest to the physicist, is only half the story. In the course of this study of compactification on Ricci-flat manifolds, beautiful and unexpected mathematics were born. Indeed, our very understanding of classical geometry underwent modifications and the notions of "stringy" or "quantum" geometry emerged. Properties of algebro-differential geometry of the target space-time manifested as the supersymmetric conformal field theory on the world-sheet. Such delicate calculations as counting of holomorphic curves and intersection of homological cycles mapped elegantly to computations of world-sheet instantons and Yukawa couplings.

The mirror principle, initiated by Candelas et al. in the early 90's, greatly simplified the aforementioned computations. Such unforeseen behaviour as pairs of Calabi-Yau manifolds whose Hodge diamonds were mirror reflections of each other naturally arose as spectral flow in the associated world-sheet conformal field theory. Though we shall too make usage of versions of mirror symmetry, viz., the **local mirror**, this writing will not venture too much into the elegant inter-relation between the mathematics and physics of string theory through mirror geometry.

What we shall delve into, is the local model of Calabi-Yau manifolds. These are the *algebraic singularities* of which we speak. In particular we concentrate on **canonical Gorenstein** singularities that admit crepant resolutions to smooth Calabi-Yau varieties. In particular, attention will be paid to orbifolds, i.e., quotients of flat space by finite groups, as well as toric singularities, i.e., local behaviour of toric varieties near the singular point.

As early as the mid 80's, the string partition function of Dixon-Harvey-Vafa-Witten (DHVW) proposed a resolution of orbifolds then unknown to the mathematician and made elegant predictions on the Euler characteristic of orbifolds. These gave new directions to such remarkable observations as the **McKay Correspondence** and its generalisations to beyond dimension 2 and beyond du Val-Klein singularities. Recent work by Bridgeland, King, and Reid on the generalised McKay from the derived category of coherent sheafs also tied deeply with similar structures arising in D-brane technologies as advocated by Aspinwall, Douglas et al. Stringy orbifolds thus became a topic of pursuit by such noted mathematicians as Batyrev, Kontsevich and Reid.

Intimately tied thereto, were applications of the construction of certain hyper-Kähler quotients, which are themselves moduli spaces of certain gauge theories, as gravitational instantons. The works by Kronheimer-Nakajima placed the McKay Correspondence under the light of representation theory of *quivers*. Douglas-Moore's construction mentioned above for the orbifold gauge theories thus brought these quivers into a string theoretic arena.

With the technology of D-branes to probe sub-stringy distance scales, Aspinwall-Greene-Douglas-Morrison-Plesser made space-time a derived concept as moduli space of world-volume theories. Consequently, novel perspectives arose, in the understanding of the field known as Geometric Invariant Theory (GIT), in the light of gauge invariant operators in the gauge theories on the D-brane. Of great significance, was the realisation that the Landau-Ginzberg/Calabi-Yau correspondence in the linear sigma model of Witten, could be used to translate between the gauge theory as a world-volume theory and the moduli space as a **GIT quotient**.

In the case of toric varieties, the sigma-model fields corresponded nicely to generators of the homogeneous coördinate ring in the language of Cox. This provided us with a alternative and computationally feasible view from the traditional approaches to toric varieties. We shall take advantage of this fact when we deal with toric duality later on.

This work will focus on how the above construction of gauge theories leads to various intricacies in algebraic geometry, representation theory and finite graphs, and vice versa, how we could borrow techniques from the latter to address the physics of the former. In order to refresh the reader's mind on the requisite mathematics, Book I is devoted to a review on the relevant topics. Chapter 2 will be an overview of the geometry, especially algebraic singularities and Picard-Lefschetz theory. Also included will be a discussion on symplectic quotients as well as the special case of toric varieties. Chapter 3 then prepares the reader for the orbifolds, by reviewing the pertinent concepts from representation theory of finite groups. Finally in Chapter 4, a unified outlook is taken by studying quivers as well as the constructions of Kronheimer and Nakajima.

Thus prepared with the review of the mathematics in Book I and the physics in II, we shall then take the reader to Books III and IV, consisting of some of the author's work in the last four years at the Centre for Theoretical Physics at MIT.

We begin with the D-brane probe picture. In Chapters **??** and **??** we classify and study the singularities of the orbifold type by discrete subgroups of $SU(3)$ and $SU(4)$ [**?**, **?**]. The resulting physics consists of catalogues of finite four dimensional Yang-Mills theories with 1 or 0 supersymmetry. These theories are nicely encoded by certain finite graphs known as **quiver diagrams**. This generalises the work of Douglas and Moore for abelian ALE spaces and subsequent work by Johnson-Meyers for all ALE spaces as orbifolds of $SU(2)$. Indeed McKay's Correspondence facilitates the ALE case; moreover the ubiquitous ADE meta-pattern, emerging in so many seemingly unrelated fields of mathematics and physics greatly aids our understanding.

In our work, as we move from two-dimensional quotients to three and four dimensions, interesting observations were made in relation to generalised McKay's Correspondences. Connections to Wess-Zumino-Witten models that are conformal field theories on the world-sheet, especially the remarkable resemblance of the McKay graphs from the former and fusion graphs from the latter were conjectured in [**?**]. Subsequently, a series of activities were initiated in [**?**, **?**, **?**] to attempt to address why weaker versions of the complex of dualities which exists in dimension two may persist in higher dimensions. Diverse subject matters such as symmetries of the modular invariant partition functions, graph algebras of the conformal field theory, matter content of the probe gauge theory and crepant resolution of quotient singularities all contribute to an intricate web of inter-relations. Axiomatic approaches such as the quiver and ribbon categories were also attempted. We will discuss these issues in Chapters **??**, **??** and **??**.

Next we proceed to address the T-dual versions of these D-brane probe theories in terms of Hanany-Witten configurations. As mentioned earlier, understanding these would greatly enlighten the understanding of how these gauge theories embed into string theory. With the help of orientifold planes, we construct the first examples of non-Abelian configurations for $\mathbb{C}^3$ orbifolds [**?**, **?**]. These are direct generalisations of the well-known elliptic models and brane box models, which are a widely studied class of conformal theories. These constructions will be the theme for Chapters **??** and **??**.

Furthermore, we discuss the steps towards a general method [**?**], which we dubbed as "stepwise projection," of finding Hanany-Witten setups for arbitrary orbifolds in Chapter **??**. With the help of Frøbenius' induced representation theory, the stepwise procedure of systematically obtaining non-Abelian gauge theories from the Abelian theories, stands as a non-trivial step towards solving the general problem of T-dualising pure geometry into Hanany-Witten setups.

Ever since Seiberg and Witten's realisation that the NS-NS B-field of string theory, turned on along world-volumes of D-branes, leads to non-commutative field theories, a host of activity ensued. In our context, Vafa generalised the DHVW closed sector orbifold partition function to include phases associated with the B-field. Subsequently, Douglas and Fiol found that the open sector analogue lead to projective representation of the orbifold group.

This inclusion of the background B-field has come to be known as turning on **discrete torsion**. Indeed a corollary of a theorem due to Schur tells us that orbifolds of dimension two, i.e., the ALE spaces do not admit such turning on. This is in perfect congruence with the rigidity of the $\mathcal{N} = 2$ superpotential. For $\mathcal{N} = 0, 1$ theories however, we can deform the superpotential consistently and arrive at yet another wide class of field theories.

With the aid of such elegant mathematics as the Schur multiplier, covering groups and the Cartan-Leray spectral sequence, we systematically study how and when it is possible to arrive at these theories with discrete torsion by studying the projective representations of orbifold groups [?, ?] in Chapters **??** and **??**.

Of course orbifolds, the next best objects to flat (complex-dimensional) space, are but one class of local Calabi-Yau singularities. Another intensively studied class of algebraic varieties are the so-called toric varieties. As finite group representation theory is key to the former, combinatorial geometry of convex bodies is key to the latter. It is pleasing to have such powerful interplay between such esoteric mathematics and our gauge theories.

We address the problem of constructing gauge theories of a D-brane probe on toric singularities [?] in Chapter **??**. Using the technique of partial resolutions pioneered by Douglas, Greene and Morrison, we formalise a so-called "Inverse Algorithm" to Witten's gauged linear sigma model approach and carefully investigate the type of theories which arise given the type of toric singularity.

Harnessing the degree of freedom in the toric data in the above method, we will encounter a surprising phenomenon which we call **Toric Duality**. [?]. This in fact gives us an algorithmic technique to engineer gauge theories which flow to the same fixed point in the infra-red moduli space. The manifestation of this duality as Seiberg Duality for $\mathcal{N} = 1$ [?] came as an additional bonus. Using a combination of field theory calculations, Hanany-Witten-type of brane configurations and the intersection theory of the mirror geometry [?], we check that all the cases produced by our algorithm do indeed give Seiberg duals and conjecture the validity in general [?]. These topics will constitute Chapters **??** and **??**.

All these intricately tied and inter-dependent themes of D-brane dynamics, construction of four-dimensional gauge theories, algebraic singularities and quiver graphs, will be the subject of this present writing.

I

LIBER PRIMUS: Invocatio Mathematicæ

Chapter 2

Algebraic and Differential Geometry

Nomenclature

Unless otherwise stated, we shall adhere to the following notations throughout the writing:

X	Complex analytic variety
T_pX, T_p^*X	Tangent and cotangent bundles (sheafs) of X at point p
$\mathcal{O}(X)$	Sheaf of analytic functions on X
$\mathcal{O}^*(X)$	Sheaf of non-zero analytic functions on X
$\Gamma(X, \mathcal{O})$	Sections of the sheaf (bundle) $\mathcal{O}$ over X
$\Omega^{p,q}(X)$	Dolbeault (p, q)-forms on X
ω_X	The canonical sheaf of X
$f : \tilde{X} \to X$	Resolution of the singularity X
$\mathfrak{g} = Lie(G)$	The Lie Algebra of the Lie group G
$\widetilde{\mathfrak{g}}$	The Affine extension of $\mathfrak{g}$
$\mu : M \to Lie(G)^*$	Moment map associated with the group G
$\mu^{-1}(c)//G$	Symplectic quotient associated with the moment map μ
$\lvert G \rvert$	The order of the finite group G
$\chi_\gamma^{(i)}(G)$	Character for the i-th irrep in the γ-th conjugacy class of G

As the subject matter of this work is on algebraic singularities and their applications to string theory, what better place to commence our mathematical invocations indeed, than a brief review on some rudiments of the vast field of singularities in algebraic varieties. The material contained herein shall be a collage from such canonical texts as [?, ?, ?, ?], to which the reader is highly recommended to refer.

2.1 Singularities on Algebraic Varieties

Let M be an m-dimensional complex algebraic variety; we shall usually deal with projective varieties and shall take M to be $\mathbb{P}^m$, the complex projective m-space, with projective coördinates $(z_1, \ldots, z_m) = [Z_0 : Z_2 : \ldots : Z_m] \in \mathbb{C}^{m+1}$. In general, by Chow's Theorem, any analytic subvariety X of M can be locally given as the zeores of a finite collection of holomorphic functions $g_i(z_1, \ldots, z_m)$. Our protagonist shall then be the variety $X := \{z | g_i(z_1, \ldots, z_m) = 0 \;\; \forall \; i = 1, \ldots, k\}$, especially the singular points thereof. The following definition shall distinguish such points for us:

DEFINITION 2.1.1 *A point $p \in X$ is called a smooth point of X if X is a submanifold of M near p, i.e., the Jacobian $\mathcal{J}(X) := \left(\frac{\partial g_i}{\partial z_j}\right)_p$ has maximal rank, namely k.*

Denoting the locus of smooth points as X^*, then if $X = X^*$, X is called a smooth variety. Otherwise, a point $s \in V \setminus V^*$ is called a **singular point**.

Given such a singularity s on a X, the first exercise one could perform is of course its **resolution**, defined to be a birational morphism $f : \tilde{X} \to X$ from a nonsingular variety $\tilde{X}$. The preimage $f^{-1}(s) \subset \tilde{X}$ of the singular point is called the **exceptional divisor** in $\tilde{X}$. Indeed if X is a projective variety, then if we require the resolution f to be projective (i.e., it can be composed as $\tilde{X} \to X \times \mathbb{P}^N \to X$), then $\tilde{X}$ is a projective variety.

The singular variety X, of (complex) dimension n, is called *normal* if the structure sheafs obey $\mathcal{O}_X = f^*\mathcal{O}_{\tilde{X}}$. We henceforth restrict our attention to normal varieties. The point is that as a topological space the normal variety X is simply the quotient

$$X = \tilde{X} / \sim,$$

where $\sim$ is the equivalence which collapses the exceptional divisor to a point[1], the so-called process of blowing down. Indeed the reverse, where we replace the singularity s by a set of directions (i.e., a projective space), is called blowing up . As we shall mostly concern ourselves with Calabi-Yau manifolds (CY) of dimensions 2 and 3, of the uttermost importance will be exceptional divisors of dimension 1, to these we usually refer as **$\mathbb{P}^1$-blowups**.

Now consider the **canonical divisors** of $\tilde{X}$ and X. We recall that the canonical divisor K_X of X is any divisor in the linear equivalence (differing by principal divisors) class as the canonical sheaf ω_X, the n-th (hence maximal) exterior power of the sheaf of differentials. Indeed for X Calabi-Yau, K_X is trivial. In general the canonical sheaf of the singular variety and that of its resolution $\tilde{X}$ are not so naïvely related but differ by a term depending on the exceptional divisors E_i:

$$K_{\tilde{X}} = f^*(K_X) + \sum_i a_i E_i.$$

The term $\sum\limits_i a_i E_i$ is a formal sum over the exceptional divisors and is called the *discrepancy* of the resolution and the values of the numbers a_i categorise some commonly encountered subtypes of singularities characterising X, which we tabulate below:

$a_i \geq 0$	canonical	$a_i > 0$	terminal
$a_i \geq -1$	log canonical	$a_i > -1$	log terminal

The type which shall be pervasive throughout this work will be the canonical singularities. In the particular case when all $a_i = 0$, and the discrepancy term vanishes, we have what is known as a **crepant resolution**. In this case the canonical sheaf of the resolution is simply the pullback of that of the singularity, when the latter is trivial, as in the cases of orbifolds which we shall soon see, the former remains trivial and hence Calabi-Yau. Indeed crepant resolutions always exists for dimensions 2 and 3, the situations of our interest, and are related by flops. Although in dimension

[1]And so X has the structure sheaf $f^*\mathcal{O}_{\tilde{X}}$, the set of regular functions on $\tilde{X}$ which are constant on $f^{-1}(s)$.

3, the resolution may not be unique (q.v. e.g. [**?**]). On the other hand, for terminal singularities, any resolution will change the canonical sheaf and such singular Calabi-Yau's will no longer have resolutions to Calabi-Yau manifolds.

In this vein of discussion on Calabi-Yau's, of the greatest relevance to us are the so-called[2] **Gorenstein** singularities , which admit a nowhere vanishing global holomorphic n-form on $X \setminus s$; these are then precisely those singularities whose resolutions have the canonical sheaf as a trivial line bundle, or in other words, these are the local Calabi-Yau singularities.

Gorenstein canonical singularities which admit crepant resolutions to smooth Calabi-Yau varieties are therefore the subject matter of this work.

2.1.1 Picard-Lefschetz Theory

We have discussed blowups of singularities in the above, in particular $\mathbb{P}^1$-blowups. A most useful study is when we consider the vanishing behaviour of these S^2-cycles. Upon this we now focus. Much of the following is based on [**?**]; The reader is also encouraged to consult e.g. [**?**, **?**] for aspects of Picard-Lefschetz monodromy in string theory.

Let X be an n-fold, and $f : X \to U \subset \mathbb{C}$ a holomorphic function thereupon. For our purposes, we take f to be the embedding equation of X as a complex algebraic variety (for simplicity we here study a hypersurface rather than complete intersections). The singularities of the variety are then, in accordance with Definition 2.1.1, $\{\vec{x}|f'(\vec{x}) = 0\}$ with $\vec{x} = (x_1, ..., x_n) \in M$. f evaluated at these critical points $\vec{x}$ is called a critical value of f.

We have level sets $F_z := f^{-1}(z)$ for complex numbers z; these are $n-1$ dimensional varieties. For any non-critical value z_0 one can construct a loop γ beginning and ending at z_0 and encircling no critical value. The map $h_\gamma : F_{z_0} \to F_{z_0}$, which generates

[2]The definition more familiar to algebraists is that a singularity is Gorenstein if the local ring is a Gorenstein ring, i.e., a local Artinian ring with maximal ideal m such that the annihilator of m has dimension 1 over A/m. Another commonly encountered terminology is the $\mathbb{Q}$-Gorenstein singularity; these have $\Gamma(X \setminus p, K_X^{\otimes n})$ a free $\mathcal{O}(X)$-module for some finite n and are cyclic quotients of Gorenstein singularities.

the monodromy as one cycles the loop, the main theme of Picard-Lefschetz Theory. In particular, we are concerned with the induced action $h_{\gamma *}$ on the homology cycles of F_{z_0}.

When f is Morse[3], in the neighbourhood of each critical point p_i , f affords the Taylor series $f(x_1, \ldots, x_n) = z_i + \sum_{j=1}^{n}(x_j - p_j)^2$ in some coordinate system. Now adjoin a critical value $z_i = f(p_i)$ with a non-critical value z_0 by a path $u(t) : t \in [0, 1]$ which does not pass through any other critical value. Then in the level set $F_{u(t)}$ we fix sphere $S(t) = \sqrt{u(t) - z_i}S^{n-1}$ (with S^{n-1} the standard $(n-1)$-sphere $\{(x_1, \ldots, x_n) : |x|^2 = 1, \mathrm{Im} x_i = 0\}$. In particular $S(0)$ is precisely the critical point p_i. Under these premises, we call the homology class $\Delta \in H_{n-1}(F_{z_0})$ in the non-singular level set F_{z_0} represented by the sphere $S(1)$ the **Picard-Lefschetz vanishing cycle**.

Fixing z_0, we have a set of such cycles, one from each of the critical values z_i. Let us consider what are known as *simple loops*. These are elements of $\pi_1 (U \backslash \{z_i\}, z_0)$, the fundamental group of loops based at z_0 and going around the critical values. For these simple loops τ_i we have the corresponding *Picard-Lefschetz monodromy operator*

$$h_i = h_{\tau_i *} : H_\bullet(F_{z_0}) \to H_\bullet(F_{z_0}).$$

On the other hand if $\pi_1 (U \backslash \{z_i\}, z_0)$ is a free group then the cycles $\{\Delta_i\}$ are *weakly distinguished*.

The *point d'appui* is the Picard-Lefschetz Theorem which determines the monodromy of f under the above setup:

THEOREM 2.1.1 *The monodromy group of the singularity is generated by the Picard-Lefschetz operators h_i, corresponding to a weakly distinguished basis $\{\Delta_i\} \subset H_{n-1}$ of the non-singular level set of f near a critical point. In particular for any cycle $a \in H_{n-1}$ (no summation in i)*

$$h_i(a) = a + (-1)^{\frac{n(n+1)}{2}}(a \circ \Delta_i)\Delta_i.$$

[3]That is to say, at all critical points x_i, the Hessian $\frac{\partial f}{\partial x_i \partial x_j}$ has non-zero determinant and all critical values $z_i = f(x_i)$ are distinct.

2.2 Symplectic Quotients and Moment Maps

We have thus far introduced canonical algebraic singularities and monodromy actions on exceptional $\mathbb{P}^1$-cycles. The spaces we shall be concerned are Kähler (Calabi-Yau) manifolds and therefore naturally we have more structure. Of uttermost importance, especially when we encounter moduli spaces of certain gauge theories, is the symplectic structure.

DEFINITION 2.2.2 *Let M be a complex algebraic variety, a* symplectic form ω *on M is a holomorphic 2-form, i.e. $\omega \in \Omega^2(M) = \Gamma(M, \bigwedge^2 T^*M)$, such that*

- *ω is closed: $d\omega = 0$;*
- *ω is non-degenerate: $\omega(X,Y) = 0$ for any $Y \in T_pM \Rightarrow X = 0$.*

Therefore on the *symplectic manifold* (M, ω) (which by the above definition is locally a complex symplectic vector space, implying that $\dim_{\mathbb{C}} M$ is even) ω induces an isomorphism between the tangent and cotangent bundles by taking $X \in TM$ to $i_X(\omega) := \omega(X, \cdot) \in \Omega^1(M)$. Indeed for any global analytic function $f \in \mathcal{O}(M)$ we can obtain its differential $df \in \Omega^1(M)$. However by the (inverse map of the) above isomorphism, we can define a vector field X_f, which we shall call the *Hamiltonian* vector field associated to f (a scalar called the Hamiltonian). In the language of classical mechanics, this vector field is the generator of infinitesimal canonical transformations[4]. In fact, $[X_f, X_g]$, the commutator between two Hamiltonian vector fields is simply $X_{\{f,g\}}$, where $\{f, g\}$ is the familiar Poisson bracket.

The vector field X_f is actually symplectic in the sense that

$$L_{X_f}\omega = 0,$$

where L_X is the Lie derivative with respect to the vector field X. This is so since $L_{X_f}\omega = (d \circ i_{X_f} + i_{X_f} \circ d)\omega = d^2 f + i_{X_f} d\omega = 0$. Let $H(M)$ be the Lie subalgebra

[4] If we were to write local coördinates (p_i, q_i) for M, then $\omega = \sum_i dq_i \wedge dq_i$ and the Hamiltonian vector field is $X_f = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - (p_i \leftrightarrow q_i)$ and our familiar Hamilton's Equations of motion are $i_{X_f}(\omega) = \omega(X_f, \cdot) = df$.

of Hamiltonian vector fields (of the tangent space at the identity), then we have an obvious exact sequence of Lie algebras (essentially since energy is defined up to a constant),

$$0 \to \mathbb{C} \to \mathcal{O}(M) \to H(M) \to 0,$$

where the Lie bracket in $\mathcal{O}(M)$ is the Poisson bracket.

Having presented some basic properties of symplectic manifolds, we proceed to consider quotients of such spaces by certain equivariant actions. We let G be some algebraic group which acts symplectically on M. In other words, for the action g^* on $\Omega^2(M)$, induced from the action $m \to gm$ on the manifold for $g \in G$, we have $g^*\omega = \omega$ and so the symplectic structure is preserved. The infinitesimal action of G is prescribed by its Lie algebra, acting as symplectic vector fields; this gives homomorphisms $k : Lie(G) \to H(M)$ and $\tilde{k} : Lie(G) \to \mathcal{O}(M)$. The action of G on M is called Hamiltonian if the following modification to the above exact sequence commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}(M) \arrow[r] & H(M) \arrow[r] & 0 \\ & & & Lie(G) \arrow[ul, "\tilde{k}"] \arrow[u, "k"'] & \end{tikzcd}$$

DEFINITION 2.2.3 *Any such Hamiltonian G-action on M gives rise to a G-equivariant* **Moment Map** $\mu : M \to Lie(G)^*$ *which corresponds*[5] *to the map $\tilde{k}$ and satisfies*

$$k(A) = X_{A\circ\mu} \quad \textit{for any} \quad A \in Lie(G),$$

i.e., $d(A \circ \mu) = i_{k(A)}\omega$.

Such a definition is clearly inspired by the Hamilton equations of motion as presented in Footnote **??**. We shall not delve into many of the beautiful properties of the moment map, such as when G is translation in Euclidean space, it is nothing more than momentum, or when G is rotation, it is simply angular momentum; for what we shall interest ourselves in the forthcoming, we are concerned with a crucial property of the moment map, namely the ability to form certain smooth quotients.

[5] *Because* $\hom(Lie(G), \hom(M,\mathbb{C})) = \hom(M, Lie(G)^*)$.

Let $\mu : M \to Lie(G)^*$ be a moment map and $c \in [Lie(G)^*]^G$ be the G-invariant subalgebra of $Lie(G)^*$ (in other words the co-centre), then the equivariance of μ says that G acts on the fibre $\mu^{-1}(c)$ and we can form the quotient of the fibre by the group action. This procedure is called the *symplectic quotient* and the subsequent space is denoted $\mu^{-1}(c)//G$. The following theorem guarantees that the result still lies in the category of algebraic varieties.

THEOREM 2.2.2 *Assume that G acts freely on $\mu^{-1}(c)$, then the* **symplectic quotient** *$\mu^{-1}(c)//G$ is a symplectic manifold, with a unique symplectic form $\bar{\omega}$, which is the pullback of the restriction of the symplectic form on M $\omega|_{\mu^{-1}(c)}$; i.e., $\omega|_{\mu^{-1}(c)} = q^*\bar{\omega}$ if $q : \mu^{-1}(c) \to \mu^{-1}(c)//G$ is the quotient map.*

A most important class of symplectic quotient varieties are the so-called toric varieties. These shall be the subject matter of the next section.

2.3 Toric Varieties

The types of algebraic singularities with which we are most concerned in the ensuing chapters in Physics are quotient and toric singularities. The former are the next best thing to flat spaces and will constitute the topic of the Chapter on finite groups. For now, having prepared ourselves with symplectic quotients from the above section, we give a lightening review on the vast subject matter of toric varieties, which are the next best thing to tori. The reader is encouraged to consult [?, ?, ?, ?, ?] as canonical mathematical texts as well as [?, ?, ?] for nice discussions in the context of string theory.

As a holomorphic quotient, a toric variety is simply a generalisation of the complex projective space $\mathbb{P}^d := (\mathbb{C}^{d+1}\ \ \{0\})/\mathbb{C}^*$ with the $\mathbb{C}^*$-action being the identification $x \sim \lambda x$. A toric variety of complex dimension d is then the quotient

$$(\mathbb{C}^n \setminus F)/\mathbb{C}^{*(n-d)}.$$

Here the $\mathbb{C}^{*(n-d)}$-action is given by $x_i \sim \lambda_a^{Q_i^a} x_i$ $(i = 1, \ldots, n; a = 1, \ldots, n-d)$ for some integer matrix (of charges) Q_i^a. Moreover, $F \in \mathbb{C}^n \setminus \mathbb{C}^{*n}$ is a closed set of points one must remove to make the quotient well-defined (Hausdorff).

In the language of symplectic quotients, we can reduce the geometry of such varieties to the combinatorics of certain convex sets.

2.3.1 The Classical Construction

Before discussing the quotient, let us first outline the standard construction of a toric variety. What we shall describe is the classical construction of a toric variety from its defining fan, due originally to MacPherson. Let $N \simeq \mathbb{Z}^n$ be an integer lattice and let $M = \hom_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$ be its dual. Moreover let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ (and similarly for $M_{\mathbb{R}}$). Then

DEFINITION 2.3.4 *A (strongly convex) polyhedral cone σ is the positive hull of a finitely many vectors $v_1, \ldots, v_k$ in N, namely*

$$\sigma = \mathrm{pos}\{v_{i=1,\ldots,k}\} := \sum_{i=1}^{k} \mathbb{R}_{\geq 0} v_i.$$

From σ we can compute its **dual cone** $\sigma^\vee$ as

$$\sigma^\vee := \{u \in M_{\mathbb{R}} | u \cdot v \geq 0 \forall v \in \sigma\}.$$

Subsequently we have a finitely generated monoid

$$S_\sigma := \sigma^\vee \cap M = \{u \in M | u \cdot \sigma \geq 0\}.$$

We can finally associate maximal ideals of the monoid algebra of the polynomial ring adjoint S_σ to points in an algebraic (variety) scheme. This is the affine toric variety X_σ associated with the cone σ:

$$X_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

To go beyond affine toric varieties, we simply paste together, as coördinate patches, various X_{σ_i} for a collection of cones σ_i; such a collection is called a **fan** $\Sigma = \bigsqcup_i \sigma_i$ and we finally arrive at the general toric variety X_Σ.

As we are concerned with the singular behaviour of our varieties, the following definition and theorem shall serve us greatly.

DEFINITION 2.3.5 *A cone* $\sigma = \text{pos}\{v_i\}$ *is* **simplicial** *is all the vectors* v_i *are linearly independent; it is* **regular** *if* $\{v_i\}$ *is a* $\mathbb{Z}$*-basis for* N*. The fan* Σ *is* **complete** *if its cones span the entirety of* $\mathbb{R}^n$ *and it is regular if all its cones are regular and simplicial.*

Subsequently, we have

THEOREM 2.3.3 X_Σ *is compact iff* Σ *is complete; it is* **non-singular** *iff* Σ *is regular.*

Finally we are concerned with Calabi-Yau toric varieties, these are associated with what is know (recalling Section 1.1 regarding Gorenstein resolutions) as **Gorenstein cones**. It turns out that an n-dimensional toric variety satisfies the Ricci-flatness condition if all the endpoints of the vectors of its cones lie on a single $n-1$-dimensional hypersurface, in other words,

THEOREM 2.3.4 *The cone* σ *is called* Gorenstein *if there exists a vector* $w \in N$ *such that* $\langle v_i, w \rangle = 1$ *for all the generators* v_i *of* σ*. Such cones give rise to toric Calabi-Yau varieties.*

We refer the reader to [**?**] for conditions when Gorenstein cones admit crepant resolutions.

The name *toric* may not be clear from the above construction but we shall see now that it is crucial. Consider each point t the algebraic torus $T^n := (\mathbb{C}^*)^n \simeq N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \hom(M, \mathbb{C}^*) \simeq \text{spec}(\mathbb{C}[M])$ as a group homomorphism $t : M \to \mathbb{C}^*$ and each point $x \in X_\sigma$ as a monoid homomorphism $x : S_\sigma \to \mathbb{C}$. Then we see that there is a *natural torus action* on the toric variety by the algebraic torus T^n as $x \to t \cdot x$ such that $(t \cdot x)(u) := t(u)x(u)$ for $u \in S_\sigma$. For $\sigma = \{0\}$, this action is nothing other than the group multiplication in $T^n = X_{\sigma=\{0\}}$.

2.3.2 The Delzant Polytope and Moment Map

How does the above tie in together with what we have discussed on symplectic quotients? We shall elucidate here. It turns out such a construction is canonically done for compact toric varieties embedded into projective spaces, so we shall deal more with *polytopes* rather than *polyhedral cones*. The former is simply a compact version of the latter and is a bounded set of points instead of extending as a cone. The argument below can be easily extended for fans and non-compact (affine) toric varieties. For now our toric variety X_Δ is encoded in a polytope Δ.

Let (X, ω) be a symplectic manifold of real dimension $2n$. Let $\tau : T^n \to \mathrm{Diff}(X, \omega)$ be a Hamiltonian action from the n-torus to vector fields on X. This immediately gives us a moment map $\mu : X \to \mathbb{R}^n$, where $\mathbb{R}^n$ is the dual of the Lie algebra for T^n considered as the Lie group $U(1)^n$. The image of μ is a polytope Δ, called a moment or **Delzant Polytope**. The inverse image, up to equivalence of the T^n-action, is then nothing but our toric variety X_Δ. But this is precisely the statement that

$$X_\Delta := \mu^{-1}(\Delta)//T^n$$

and the toric variety is thus naturally a symplectic quotient.

In general, given a convex polytope, Delzant's theorem guarantees that if the following conditions are satisfied, then the polytope is Delzant and can be used to construct a toric variety:

THEOREM 2.3.5 (Delzant) *A convex polytope $\Delta \subset \mathbb{R}^n$ is Delzant if:*

1. *There are n edges meeting at each vertex p_i;*

2. *Each edge is of the form $p_i + \mathbb{R}_{\geq 0} v_i$ with $v_{i=1,\ldots,n}$ a basis of $\mathbb{Z}^n$.*

We shall see in Liber II and III, that the moduli space of certain gauge theories arise as toric singularities. In Chapter 5, we shall in fact see a third, physically motivated construction for the toric variety. For now, let us introduce another class of Gorenstein singularities.

Chapter 3

Representation Theory of Finite Groups

A wide class of Gorenstein canonical singularities are of course quotients of flat spaces by appropriate discrete groups. When the groups are chosen to be discrete subgroups of special unitary groups, i.e., the holonomy groups of Calabi-Yau's, and when crepant resolutions are admissible, these quotients are singular limits of CY's and provide excellent local models thereof. Such quotients of flat spaces by discrete finite subgroups of certain Lie actions, are called **orbifolds** (or V-manifolds, in their original guise in [**?**]). It is therefore a natural *point de départ* for us to go from algebraic geometry to a brief discussion on finite group representations (q.v. e.g. [**?**] for more details of which much of the following is a condensation).

3.1 Preliminaries

We recall that a **representation** of a finite group G on a finite dimensional (complex) vector space V is a homomorphism $\rho : G \to GL(V)$ to the group of automorphisms $GL(V)$ of V. Of great importance to us is the **regular** representation, where V is the vector space with basis $\{e_g | g \in G\}$ and G acts on V as $h \cdot \sum a_g e_g = \sum a_g e_{hg}$ for $h \in G$.

Certainly the corner-stone of representation theory is Schur's Lemma:

THEOREM 3.1.6 (Schur's Lemma) *If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then (a) either ϕ is an isomorphism or $\phi = 0$. If $V = W$, then ϕ is a homothety (i.e., a multiple of the identity).*

The lemma allows us to uniquely decompose any representation R into irreducibles $\{R_i\}$ as $R = R_1^{\oplus a_1} \oplus \ldots \oplus R_n^{\oplus a_n}$. The three concepts of regular representations, Schur's lemma and unique decomposition we shall extensively use later in Liber III. Another crucial technique is that of character theory into which we now delve.

3.2 Characters

If V is a representation of G, we define its character χ_V to be the $\mathbb{C}$-function on $g \in G$:

$$\chi_V(g) = \mathrm{Tr}(V(g)).$$

Indeed the character is a **class function**, constant on each conjugacy class of G; this is due to the cyclicity of the trace: $\chi_V(hgh^{-1}) = \chi_V(g)$. Moreover χ is a homomorphism from vector spaces to $\mathbb{C}$ as

$$\chi_{V \oplus W} = \chi_V + \chi_W \qquad \chi_{V \otimes W} = \chi_V \chi_W.$$

From the following theorem

THEOREM 3.2.7 *There are precisely the same number of conjugacy classes are there are irreducible representations of a finite group G,*

and the above fact that χ is a class function, we can construct a square matrix, the so-called **character table**, whose entries are the characters $\chi_\gamma^{(i)} := \mathrm{Tr}(R_i(\gamma))$, as i goes through the irreducibles R_i and γ, through the conjugacy classes. This table will be of tremendous computational use for us in Liber III.

The most important important properties of the character table are its two *orthogonality conditions*, the first of which is for the rows, where we sum over conjugacy classes:

$$\sum_{g\in G} \chi_g^{(i)*}\chi_g^{(j)*} = \sum_{\gamma=1}^{n} r_\gamma \chi_\gamma^{(i)*}\chi_\gamma^{(j)*} = |G|\delta_{ij},$$

where n is the number of conjugacy classes (and hence irreps) and r_γ the size of the γ-th conjugacy class. The other orthogonality is for the columns, where we sum over irreps:

$$\sum_{i=1}^{n} \chi_k^{(i)*}\chi_l^{(i)*} = \frac{|G|}{r_k}\delta_{kl}.$$

We summarise these relations as

THEOREM 3.2.8 *With respect to the inner product* $(\alpha,\beta) := \frac{1}{|G|}\sum\limits_{g\in G}\alpha^*(g)\beta(g) = \frac{1}{|G|}\sum\limits_{\gamma=1}^{n} r_\gamma\alpha^*(\gamma)\beta(\gamma)$, *the characters of the irreducible representations (i.e. the character table) are orthonormal.*

Many interesting corollaries follow. Of the most useful are the following. Any representation R is irreducible iff $(\chi_R,\chi_R) = 1$ and if not, then (χ_R,χ_{R_i}) gives the multiplicity of the decomposition of R into the i-th irrep.

For the *regular representation* R_r, the character is simply $\chi(g) = 0$ if $g \neq \mathbb{I}$ and it is $|G|$ when $g = \mathbb{I}$ (this is simply because any group element h other than the identity will permute $g \in G$ and in the vector basis e_g correspond to a non-diagonal element and hence do not contribute to the trace). Therefore if we were to decompose the R_r in to irreducibles, the i-th would receive a multiplicity of $(R_r, R_i) = \frac{1}{|G|}\chi_{R_i}(\mathbb{I})|G| = \dim R_i$. Therefore any irrep R_i appears in the regular representation precisely $\dim R_i$ times.

3.2.1 Computation of the Character Table

There are some standard techniques for computing the character table given a finite group G; the reader is referred to [**?**, **?**, **?**] for details.

For the j-th conjugacy class c_j, define a *class operator* $C_j := \sum\limits_{g\in c_j} g$, as a formal sum of group elements in the conjugacy class. This gives us a **class multiplication**:

$$C_j C_k = \sum_{g\in c_j; h\in c_k} gh = \sum_k c_{jkl} C_l,$$

where c_{jkl} are "fusion coefficients" for the class multiplication and can be determined from the multiplication table of the group G. Subsequently one has, by taking characters,

$$r_j r_k \chi_j^{(i)} \chi_k^{(i)} = \dim R_i \sum_{l=1}^{n} c_{jkl} r_l \chi_k^{(l)}.$$

These are n^2 equations in $n^2 + n$ variables $\{\chi_j^{(i)}; \dim R_i\}$. We have another n equations from the orthonormality $\frac{1}{|G|} \sum_{j=1}^{n} r_j |\chi_j^{(i)}|^2 = 1$; these then suffice to determine the characters and the dimensions of the irreps.

3.3 Classification of Lie Algebras

In Book the Third we shall encounter other aspects of representation theory such as induced and projective representation; we shall deal therewith accordingly. For now let us turn to the representation of Lie Algebras. It may indeed seem to the reader rather discontinuous to include a discussion on the the classification of Lie Algebras in a chapter touching upon finite groups. However the reader's patience shall soon be rewarded in Chapter 4 as well as Liber III when we learn that certain classifications of finite groups are intimately related, by what has become known as McKay's Correspondence, to that of Lie Algebras. Without further ado then let us simply present, for the sake of refreshing the reader's memory, the classification of complex Lie algebras.

Given a complex Lie algebra $\mathfrak{g}$, it has the *Levi Decomposition*

$$\mathfrak{g} = \mathrm{Rad}(\mathfrak{g}) \oplus \tilde{\mathfrak{g}} = \mathrm{Rad}(\mathfrak{g}) \oplus \bigoplus_i \mathfrak{g}_i,$$

where $\mathrm{Rad}(\mathfrak{g})$ is the radical, or the maximal solvable ideal, of $\mathfrak{g}$. The representation of such solvable algebras is trivial and can always be brought to $n \times n$ upper-triangular matrices by a basis change. On the other hand $\tilde{\mathfrak{g}}$ is semisimple and contains no nonzero solvable ideals. We can decompose $\tilde{\mathfrak{g}}$ further into a direct sum of **simple** Lie algebras $\mathfrak{g}_i$ which contain no nontrivial ideals. The $\mathfrak{g}_i$'s are then the nontrivial pieces of $\mathfrak{g}$.

The great theorem is then the complete classification of the complex simple Lie algebras due to Cartan, Dynkin and Weyl. These are the

- **Classical Algebras:** $A_n := \mathfrak{sl}_{n+1}(\mathbb{C})$, $B_n := \mathfrak{so}_{2n+1}(\mathbb{C})$, $C_n := \mathfrak{sp}_{2n}(\mathbb{C})$ and $D_n := \mathfrak{so}_{2n}(\mathbb{C})$ for $n = 1, 2, 3 \ldots$;

- **Exceptional Algebras:** $E_{6,7,8}$, F_4 and G_2.

The Dynkin diagrams for these are given in Figure **??**. The nodes are marked with the so-called comarks $a_i^\vee$ which we recall to be the expansion coefficients of the highest root θ into the simple coroots $\alpha_i^\vee := 2\alpha_i/|\alpha_i|^2$ (α_i are the simple roots)

$$\theta = \sum_i^r a_i^\vee \alpha_i^\vee,$$

where r is the rank of the algebra (or the number of nodes).

The dual Coxeter numbers are defined to be

$$c := \sum_i^r a_i^\vee + 1$$

and the **Cartan Matrix** is

$$C_{ij} := (\alpha_i, \alpha_j^\vee).$$

We are actually concerned more with **Affine** counterparts of the above simple algebras. These are central extensions of the above in the sense that if the commutation relation in the simple $\mathfrak{g}$ is $[T^a, T^b] = f_c^{ab} T^c$, then that in the affine $\widehat{\mathfrak{g}}$ is $[T_m^a, T_m^b] = f_c^{ab} T_{m+n}^c + kn\delta_{ab}\delta_{m,-n}$. The generators T^a of $\mathfrak{g}$ are seen to be generalised to $T_m^a := T^a \otimes t^m$ of $\widehat{\mathfrak{g}}$ by Laurent polynomials in t. The above concepts of roots etc. are directly generalised with the inclusion of the affine root. The Dynkin diagrams are as in Figure **??** but augmented with an extra affine node.

We shall see in Liber III that the comarks and the dual Coxeter numbers will actually show up in the dimensions of the irreducible representations of certain fi-

Figure 3-1: The Dynkin diagrams of the simple complex Lie Algebras; the nodes are labelled with the comarks.

nite groups. Moreover, the Cartan matrices will correspond to certain graphs constructable from the latter.

Chapter 4

Finite Graphs, Quivers, and Resolution of Singularities

We have addressed algebraic singularities, symplectic quotients and orbifolds in relation to finite group representations. It is now time to embark on a journey which would ultimately give a unified outlook. To do so we must involve ourselves with yet another field of mathematics, namely the theory of graphs.

4.1 Some Rudiments on Graphs and Quivers

As we shall be dealing extensively with algorithms on finite graphs in our later work on toric singularities, let us first begin with the fundamental concepts in graph theory. The reader is encouraged to consult such classic texts as [**?**, **?**].

DEFINITION 4.1.6 *A finite graph is a triple* (V, E, I) *such that* V, E *are disjoint finite sets (respectively the set of vertices and edges) with members of* E *joining those of* V *according to the incidence relations* I.

The graph is *undirected* if for each edge e joining vertex i to j there is another edge e' joining j to i; it is directed otherwise. The graph is *simple* if there exists no *loops* (i.e., edges joining a vertex to itself). The graph is connected if any two vertices can be linked a series of edges, a so-called *walk*. Two more commonly encountered concepts are the *Euler* and *Hamilton* cycles, the first of which is walk returning to the beginning vertex which traverses each edge only once and each vertex at least once, while the latter, the vertices only once. Finally we call two graphs isomorphic if they are topologically homeomorphic; we emphasise the unfortunate fact that the graph isomorphism problem (of determining whether two graphs are isomorphic) is thus far unsolved; it is believed to be neither P nor NP-complete. This will place certain restrictions on our computations later.

We can represent a graph with n vertices and m edges by an $n \times n$ matrix, the so-named **adjacency matrix** a_{ij} whose ij-th entry is the number of edges from i to j. If the graph is simple, then we can also represent the graph by an **incidence matrix**, an $n \times m$ matrix d_{ia} in whose a-th columns there is a -1 (resp. 1) in row i (resp. row j) if there an a-th edge going from i to j. We emphasise that the graph must be *simple* for the incidence matrix to fully encapture its information. Later on in Liber III we will see this is a shortcoming when we are concerned with gauged linear sigma models.

4.1.1 Quivers

Now let us move onto a specific type of directed graphs, which we shall call a **quiver**. To any such a quiver (V, E, I) is associated the abelian category $\text{Rep}(V, E, I)$, of its representations (over say, $\mathbb{C}$). A (complex) *representation* of a quiver associates to every vertex $i \in V$ a vector space V_i and to any edge $i \stackrel{a}{\to} j$ a linear map $f_a : V_i \to V_j$. The vector $\vec{d} = (d_i := \dim_{\mathbb{C}} V_i)$ is called the dimension of the representation.

Together with its representation dimension, we can identify a quiver as a *labelled graph* (i.e., a graph with its nodes associate to integers) $(V, E, I; \vec{d})$. Finally, as we shall encounter in the case of gauge theories, one could attribute certain algebraic meaning to the arrows by letting them be formal variables which satisfy certain sets algebraic relations R; now we have to identify the quiver as a quintuple $(V, E, I; \vec{d}, R)$. These labelled directed finite quivers with relations are what concern string theorist the most.

In Liber III we shall delve further into the representation theory of quivers in relation to gauge theories, for now let us introduce two more preliminary concepts. We say a representation with dimension $\vec{d'}$ is a sub-representation of that with $\vec{d}$ if $(V, E, I; \vec{d'}) \hookrightarrow (V, E, I; \vec{d})$ is an injective morphism. In this case given a vector θ such that $\theta \cdot d = 0$, we call a representation with dimension d **θ-semistable** if for any subrepresentation with dimension d', $\theta \cdot d' \geq 0$; we call it θ-stable for the strict inequality. King's beautiful work [**?**] has shown that θ-stability essentially implies existence of solutions to certain BPS equations in supersymmetric gauge theories, the so-called F-D flatness conditions. But pray be patient as this discussion would have to wait until Liber II.

4.2 du Val-Kleinian Singularities

Having digressed some elements of graph and quiver theories, let us return to algebraic geometry. We shall see below a beautiful link between the theory of quivers and that of orbifold of $\mathbb{C}^2$.

First let us remind the reader of the classification of the quotient singularities of $\mathbb{C}^2$, these date as far back as F. Klein [**?**]. The affine equations of these so-called **ALE** (Asymptotically Locally Euclidean) singularities can be written in $\mathbb{C}[x, y, z]$ as

$$\boxed{\begin{array}{c} A_n : xy + z^n = 0 \\ D_n : x^2 + y^2 z + z^{n-1} = 0 \\ E_6 : x^2 + y^3 + z^4 = 0 \\ E_7 : x^2 + y^3 + yz^3 = 0 \\ E_8 : x^2 + y^3 + z^5 = 0. \end{array}}$$

We have not named these ADE by coincidence. The resolutions of such singularities were studied extensively by [**?**] and one sees in fact that the $\mathbb{P}^1$-blowups intersect precisely in the fashion of the Dynkin diagrams of the simply-laced Lie algebras ADE. For a illustrative review upon this elegant subject, the reader is referred to [**?**].

4.2.1 McKay's Correspondence

Perhaps it is a good point here to introduce the famous McKay correspondence, which will be a major part of Liber III. We shall be brief now, promising to expound upon the matter later.

Due to the remarkable observation of McKay in [**?**], there is yet another justification of naming the classification of the discrete finite subgroups Γ of $SU(2)$ as ADE. Take the defining representation R of Γ, and consider its tensor product with all the irreducible representations R_i:

$$R \otimes R_i = \bigoplus_j a_{ij} R_j.$$

Now consider a_{ij} as an adjacency matrix of a finite quiver with labelling the dimensions of the irreps. Then McKay's Theorem states that a_{ij} of the ADE finite group is precisely the Dynkin diagram of the *affine* ADE Lie algebra and the dimensions correspond to the comarks of the algebra. Of course for any finite group we can perform such a procedure, and we shall call the quiver so-obtained the **McKay Quiver**.

4.3 ALE Instantons, hyper-Kähler Quotients and McKay Quivers

It is the unique perspective of Kronheimer's work [**?**] which uses the methods of certain symplectic quotients in conjunction with quivers to study the resolution of the $\mathbb{C}^2$ orbifolds. We must digress one last time, to introduce instanton constructions.

4.3.1 The ADHM Construction for the E^4 Instanton

For the Yang-Mills equation $D^a F_{ab} := \nabla^a F_{ab} + [A^a, F_{ab}] = 0$ obtained from the action $L_{\text{YM}} = -\frac{1}{4} F_{ab} F^{ab}$ with connexion A_a and field strength $F_{ab} := \nabla_{[a} A_{b]} + [A_a, A_b]$, we seek *finite action* solutions. These are known as **instantons**. A theorem due to Uhlenbeck [**?**] ensures that finding such an instanton solution in Euclidean space E^4

amounts to investigating G-bundles over S^4 since finite action requires the gauge field to be well-behaved at infinity and hence the one-point compactification of E^4 to S^4.

Such G-bundles, at least for simple G, are classified by integers, viz., the second Chern number of the bundle E, $c_2(E) := \frac{1}{8\pi}\int_{S^4} \mathrm{Tr}(F \wedge F)$; this is known as the *instanton number* of the gauge field. In finding the saddle points, so as to enable the evaluation of the Feynman path integral for L_{YM}, one can easily show that only the self-dual and self-anti-dual solutions $F_{ab} = \pm F_{ab}$ give rise to absolute minima in each topological class (i.e., for fixed instanton number). Therefore we shall focus in particular on the self-dual instantons. We note that self-duality implies solution to the Yang-Mills equation due to the Bianchi identity. Hence we turn our attention to self-dual gauge fields. There is a convenient theorem (see e.g. [?]) which translates the duality condition into the language of holomorphic bundles:

THEOREM 4.3.9 (Atiyah et al.) *There is a natural 1-1 correspondence between*

- *Self-dual SU(n) gauge fields*[1] *on U, an open set in S^4, and*
- *Holomorphic rank n vector bundles E over $\hat{U}$, an open set*[2] *in $\mathbb{P}^3$, such that (a) $E|_{\hat{x}}$ is trivial $\forall x \in U$; (b) $\det E$ is trivial; (c) E admits a positive real form.*

Therefore the problem of constructing self-dual instantons amounts to constructing a holomorphic vector bundle over $\mathbb{P}^3$. The key technique is due to the monad concept of Horrocks [?] where a sequence of vector bundles $F \xrightarrow{A} G \xrightarrow{B} H$ is used to produce the bundle E as a quotient $E = \ker B/\mathrm{Im} A$. Atiyah, Hitchin, Drinfeld and Manin then utilised this idea in their celebrated paper [?] to reduce the self-dual Yang-Mills instanton problem from partial differential equations to matrix equations; this is now known as the **ADHM construction**. Let V and W be complex vector spaces of dimensions $2k + n$ and k respectively and $A(Z)$ a linear map

$$A(Z) : W \to V$$

[1] *Other classical groups have also been done, but here we shall exemplify with the unitary groups.*
[2] *There is a canonical mapping from $x \in U$ to $\hat{x} \in \hat{U}$ into which we shall not delve.*

depending linearly on coordinates $\{Z^{a=0,1,2,3}\}$ of $\mathbb{P}^3$ as $A(Z) := A_a Z^a$ with A_a constant linear maps from W to V. For any subspace $U \subset V$, we define

$$U^0 := \{v \in V | (u,v) = 0 \ \ \forall u \in U\}$$

with respect to the symplectic (nondegenerate skew bilinear) form $(\ ,\ )$. Moreover we introduce antilinear maps $\sigma : W \to W$ with $\sigma^2 = 1$ and $\sigma : V \to V$ with $\sigma^2 = -1$ and impose the conditions

$$\begin{array}{ll} (1) & \forall Z^a \neq 0, U_Z := A(Z)W \text{ has dimension } k \text{ and is isotropic } (U_Z \subset U_Z^0); \\ (2) & \forall w \in W, \sigma A(Z) w = A(\sigma Z) \sigma w. \end{array} \tag{4.3.1}$$

Then the quotient space $E_Z := U_Z^0 / U_Z$ of dimension $(2k+n-k)-k = n$ is precisely the rank n $SU(n)$-bundle E over $\mathbb{P}^3$ which we seek. One can further check that E satisfies the 3 conditions in theorem **??**, whereby giving us the required self-dual instanton. Therefore we see that the complicated task of solving the non-linear partial differential equations for the self-dual instantons has been reduced to finding $(2k+n) \times k$ matrices $A(Z)$ satisfying condition (**??**), the second of which is usually known - though perhaps here not presented in the standard way - as the ADHM equation.

4.3.2 Moment Maps and Hyper-Kähler Quotients

The other ingredient we need is a generalisation of the symplectic quotient discussed in Section 1.2, the so-called Hyper-Kähler Quotients of Kronheimer [**?**] (see also the elucidation in [**?**]). A Riemannian manifold X with three covariantly constant complex structures $i := I, J, K$ satisfying the quaternionic algebra is called **Hyper-Kähler**[3]. From these structures we can define closed (hyper-)Kähler 2-forms:

$$\omega_i(V, W) := g(V, iW) \qquad \text{for} \quad i = I, J, K$$

[3]In dimension 4, simply-connectedness and self-duality of the Ricci tensor suffice to guarantee hyper-Kählerity.

mapping tangent vectors $V, W \in T(X)$ to $\mathbb{R}$ with g the metric tensor.

On a hyper-Kähler manifold with Killing vectors V (i.e., $\mathcal{L}_V g = 0$) we can impose **triholomorphicity**: $\mathcal{L}_V \omega_i = V^\nu (d\omega_i)_\nu + d(V^\nu(\omega_i)_\nu) = 0$ which together with closedness $d\omega_i = 0$ of the hyper-Kähler forms imply the existence of potentials μ_i, such that $d\mu_i = V^\nu(\omega_i)_\nu$. Since the dual of the Lie algebra $\mathfrak{g}$ of the group of symmetries G generated by the Killing vectors V is canonically identifiable with left-invariant forms, we have an induced map of such potentials:

$$\mu_i : X \to \mu_i^a \in \mathbb{R}^3 \otimes \mathfrak{g}^* \qquad i = 1, 2, 3; \quad a = 1, ..., \dim(G)$$

These maps are the (hyper-Kähler) **moment maps** and usually grouped as $\mu_{\mathbb{R}} = \mu_3$ and $\mu_{\mathbb{C}} = \mu_1 + i\mu_2$

Thus equipped, for any hyper-Kähler manifold Ξ of dimension $4n$ admitting k freely acting triholomorphic symmetries, we can construct another, X_ζ, of dimension $4n - 4k$ by the following two steps:

1. We have $3k$ moment maps and can thus define a level set of dimension $4n - 3k$:

$$P_\zeta := \{\xi \in \Xi | \mu_i^a(\xi) = \zeta_i^a\};$$

2. When $\zeta \in \mathbb{R}^3 \otimes \text{Centre}(\mathfrak{g}^*)$, P_ζ turns out to be a principal G-bundle over a new hyper-Kähler manifold

$$X_\zeta := P_\zeta / G \cong \{\xi \in \Xi | \mu_{\mathbb{C}}^a(\xi) = \zeta_{\mathbb{C}}^a\} / G^{\mathbb{C}}.$$

This above construction , where in fact the natural connection on the bundle $P_\zeta \to X_\zeta$ is self-dual, is the celebrated **hyper-Kähler quotient** construction [**?**].

Now we present a remarkable fact which connects these moment maps to the previous section. If we write (**??**) for $SU(n)$ groups into a (perhaps more standard) component form, we have the ADHM data

$$M := \{A, B; s, t^\dagger | A, B \in \text{End}(V); s, t^\dagger \in \text{Hom}(V, W)\},$$

with the ADHM equations

$$\begin{aligned}[A,B]+ts &= 0;\\ ([A,A^\dagger]+[B,B^\dagger])-ss^\dagger+tt^\dagger &= 0.\end{aligned}$$

Comparing with the hyper-Kähler forms $\omega_{\mathbb{C}} = \mathrm{Tr}(dA\wedge dB)+\mathrm{Tr}(dt\wedge ds)$ and $\omega_{\mathbb{R}} = \mathrm{Tr}(dA\wedge dA^\dagger + dB\wedge dB^\dagger) - \mathrm{Tr}(ds^\dagger\wedge ds - dt\wedge dt^\dagger)$ which are invariant under the action by $A, B, s, t^\dagger$, we immediately arrive at the following fact:

PROPOSITION 4.3.1 *The moment maps for the triholomorphic $SU(n)$ isometries precisely encode the ADHM equation for the $SU(n)$ self-dual instanton construction.*

4.3.3 ALE as a Hyper-Kähler Quotient

Kronheimer subsequently used the above construction for the case of X being the ALE space, i.e. the orbifolds $\mathbb{C}^2/(\Gamma \in SU(2))$. Let us first clarify some notations: $\Gamma \subset SU(2) :=$ Finite discrete subgroup of $SU(2)$, i.e., A_n, D_n, or $E_{6,7,8}$; $Q :=$ The defining $\mathbb{C}^2$-representation; $R :=$ The regular $|\Gamma|$-dimensional complex representation; $R_{i=0,..,r} :=$ irreps(Γ) of dimension n_i with 0 corresponding to the affine node (the trivial irrep); $(\ )_\Gamma :=$ The Γ-invariant part; $a_{ij} :=$ The McKay quiver matrix for Γ, i.e., $Q\otimes R_i = \bigoplus\limits_j a_{ij}R_j$; $T :=$ A one dimensional quaternion vector space $= \{x_0+x_1 i+x_2 j+x_3 k | x_i \in \mathbb{R}\}$; $\Lambda^+T^* :=$ The self-dual part of the second exterior power of the dual space = span{hyper-Kähler forms $\omega_{i=I,J,K}$}; $[y\wedge y] := (T^*\wedge T^*)\otimes [\mathrm{End}(V),\mathrm{End}(V)]$, for $y\in T^*\otimes \mathrm{End}(V)$; Endskew$(R) :=$ The anti-Hermitian endomorphisms of R; $Z :=$ Trace free part of Centre(Endskew$_\Gamma(R)$); $G := \prod\limits_{i=1}^r U(n_i) =$ The group of unitary automorphisms of R commuting with the action of Γ, modded out by $U(1)$ scaling[4] $X_\zeta := \{y \in (T^*\otimes_{\mathbb{R}} \mathrm{Endskew}(R))_\Gamma | [y\wedge y]^+ = \zeta\}/G$ for generic$\zeta \in \Lambda^+T^*\otimes Z$; $\mathcal{R} :=$ The natural bundle over X_ζ, viz., $Y_\zeta \times_G R$, with $Y_\zeta := \{y | [y\wedge y]^+ = \zeta\}$; and finally $\xi :=$ A tautological vector-bundle endormorphism as an element in $T^*\otimes_{\mathbb{R}}$ Endskew$(\mathcal{R})$.

[4]This is in the sense that the group $U(|\Gamma|)$ is broken down, by Γ-invariance, to $\prod\limits_{i=0}^r U(n_i)$, and then further reduced to G by the modding out.

We now apply the hyper-Kähler construction in the previous subsection to the ALE manifold

$$\boxed{\begin{array}{rl} \Xi := & (Q \otimes \mathrm{End}(R))_\Gamma = \{\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\} \\ = & \bigoplus\limits_{ij} a_{ij}\,\mathrm{hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \\ \cong & (T^* \otimes_{\mathbb{R}} \mathrm{Endskew}(R))_\Gamma = \{\xi = \begin{pmatrix} \alpha & -\beta^\dagger \\ \beta & \alpha^\dagger \end{pmatrix}\} \end{array}}$$

where α and β are $|\Gamma| \times |\Gamma|$ matrices satisfying $\begin{pmatrix} R_\gamma \alpha R_{\gamma^{-1}} \\ R_\gamma \beta R_{\gamma^{-1}} \end{pmatrix} = Q_\gamma \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $\gamma \in \Gamma$. Of course this is simply the Γ-invariance condition; or in a physical context, the projection of the matter content on orbifolds. In the second line we have directly used the definition of the McKay matrices[5] a_{ij} and in the third, the canonical isomorphism between $\mathbb{C}^4$ and the quaternions.

The hyper-Kähler forms are $\omega_{\mathbb{R}} = \mathrm{Tr}(d\alpha \wedge d\alpha^\dagger) + \mathrm{Tr}(d\beta \wedge d\beta^\dagger)$ and $\omega_{\mathbb{C}} = \mathrm{Tr}(d\alpha \wedge d\beta)$, the moment maps, $\mu_{\mathbb{R}} = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]$ and $\mu_{\mathbb{C}} = [\alpha, \beta]$. Moreover, the group of triholomorphic isometries is $G = \prod\limits_{i=1}^{r} U(n_i)$ with a trivial $U(n_0) = U(1)$ modded out. It is then the celebrated theorem of Kronheimer [**?**] that

THEOREM 4.3.10 (Kronheimer) *The space*

$$X_\zeta := \{\xi \in \Xi | \mu_i^a(\xi) = \zeta_i^a\}/G$$

is a smooth hyper-Kähler manifold of dimension[6] *four diffeomorphic to the resolution of the ALE orbifold* $\mathbb{C}^2/\Gamma$*. And conversely all ALE hyper-Kähler four-folds are obtained by such a resolution.*

We remark that in the metric, $\zeta_{\mathbb{C}}$ corresponds to the complex deformation while

[5]The steps are as follows: $(Q \otimes \mathrm{End}(R))_\Gamma = (Q \otimes \mathrm{Hom}(\bigoplus\limits_i R_i \otimes \mathbb{C}^{n_i}, \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})))_\Gamma = (\bigoplus\limits_{ijk} a_{ik}\mathrm{Hom}(R_k, R_j))_\Gamma \otimes \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) = \bigoplus\limits_{ij} a_{ij}\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$ by Schur's Lemma.

[6]*Since* $\dim(X_\zeta) = \dim(\Xi) - 4\dim(G) = 2\sum\limits_{ij} a_{ij} n_i n_j - 4(|\Gamma| - 1) = 4|\Gamma| - 4|\Gamma| + 1 = 4.$

$\zeta_{\rm I\!R} = 0$ corresponds to the singular limit $\mathbb{C}^2/\Gamma$.

4.3.4 Self-Dual Instantons on the ALE

Kronheimer and Nakajima [?] subsequently applied the ADHM construction on the ALE quotient constructed in the previous section. In analogy to the usual ADHM construction, we begin with the data $(V, W, \mathcal{A}, \Psi)$ such that

$$
\begin{array}{rl}
V, W := & \text{A pair of unitary } \Gamma\text{-modules of complex dimensions} \\
 & k \text{ and } n \text{ respectively;} \\
A, B := & \Gamma\text{-equivariant endomorphisms of } V; \\
\mathcal{A} := & \left(\begin{array}{cc} A & -B^\dagger \\ B & A^\dagger \end{array}\right) \in (T^* \otimes_{\rm I\!R} \text{Endskew}(R))_\Gamma = \bigoplus\limits_{ij} a_{ij} \text{Hom}(V_i, V_j); \\
s, t^\dagger := & \text{homomorphisms from } V \text{ to } W; \\
\Psi := & (s, t^\dagger) \in \text{Hom}(S \otimes V, W)_\Gamma.
\end{array}
$$

Let us explain the terminology above. By Γ-module we simply mean that V and W admit decompositions into the irreps of Γ in the canonical way: $V = \bigoplus\limits_i V_i \otimes R_i$ with $V_i \cong \mathbb{C}^{v_i}$ such that $k = \dim(V) = \sum\limits_i v_i n_i$ and similarly for W. By Γ-equivariance we mean the operators as matrices can be block-decomposed (into $n_i \times n_j$) according to the decomposition of the modules V and W. In the definition of $\mathcal{A}$ we have used the McKay matrices in the reduction of $(T^* \otimes_{\rm I\!R} \text{Endskew}(R))_\Gamma$ in precisely the same fashion as was in the definition of Ξ. For Φ, we use something analogous to the standard spin-bundle decomposition of tangent bundles $T^* \otimes \mathbb{C} = S \otimes \bar{S}$, to positive and (dual) negative spinors S and $\bar{S}$. We here should thus identify S as the right-handed spinors and Q, the left-handed.

Finally we have an additional structure on X_ζ. Now since X_ζ is constructed as a quotient, with P_ζ as a principal G-bundle, we have an induced natural bundle $\mathcal{R} := P_\zeta \times_G R$ with trivial R fibre. From this we have a **tautological bundle** $\mathcal{T}$ whose endomorphisms are furnished by $\xi \in T^* \otimes_{\rm I\!R} \text{Endskew}(\mathcal{R})$. This is tautological in the sense that $\xi \in \Xi$ and the points of the base X_ζ are precisely the endomorphisms of the fibre R.

On X_ζ we define operators $\mathcal{A}\otimes \mathrm{Id}_{\mathcal{T}}, \mathrm{Id}_V \otimes \xi$ and $\Psi \otimes \mathrm{Id}_{\mathcal{T}} : S\otimes V \otimes \mathcal{T} \to W \otimes \mathcal{T}$. Finally we define the operator (which is a $(2k+n)|\Gamma| \times 2k|\Gamma|$ matrix because S and Q are of complex dimension 2, V, of dimension k and $\mathcal{R}$ and $\mathcal{T}$, of dimension $|\Gamma|$)

$$\mathcal{D} := (\mathcal{A}\otimes \mathrm{Id} - \mathrm{Id}\otimes \xi) \oplus \Psi \otimes \mathrm{Id}$$

mapping $S\otimes V\otimes \mathcal{R} \to Q \otimes V \otimes \mathcal{T} \oplus W \otimes \mathcal{R}$. We can restrict this operator to the Γ-invariant part, viz., $\mathcal{D}_\Gamma$, which is now a $(2k+n)\times 2k$ matrix. The adjoint is given by

$$\mathcal{D}_\Gamma^\dagger : \left(\bar{Q}\otimes \bar{V} \otimes \mathcal{T}\right)_\Gamma \oplus \left(\bar{W}\otimes \mathcal{T}\right)_\Gamma \to S \otimes \left(\bar{V}\otimes \mathcal{T}\right)_\Gamma,$$

where $\bar{V}, \bar{W}$ and $\bar{Q}$ denote the trivial (Cartesian product) bundle over X_ζ with fibres V, W and Q.

Now as with the $\mathbb{R}^4$ case, the moment maps encode the ADHM equations, except that instead of the right hand side being zero, we now have the deformation parametres ζ. In other words, we have $[\mathcal{A}\wedge\mathcal{A}]^+ + \{\Psi^\dagger, \Psi\} = -\zeta_V$, where $\{\Psi^\dagger,\Psi\} \in \Lambda^+ T^* \otimes \mathrm{Endskew}(V)$ is the symmetrisation in the S indices and contracting in the W indices of $\Psi^\dagger \otimes \Psi$, and ζ_V is such that $\zeta_V \otimes \mathrm{Id} \in \Lambda^+ T^* \otimes \mathrm{End}((V\otimes R)^\Gamma)$. In component form this reads

$$\begin{aligned} [A,B] + ts &= -\zeta_{\mathbb{C}}; \\ ([A,A^\dagger] + [B,B^\dagger]) - ss^\dagger + tt^\dagger &= \zeta_{\mathbb{R}}, \end{aligned} \tag{4.3.2}$$

where as before $\zeta = \bigoplus_{i=1}^{r} \zeta_i \mathrm{Id}_{v_i} \in \mathbb{R}^3 \otimes Z$.

Thus equipped, the anti-self-dual[7] instantons can be constructed by the following theorem:

THEOREM 4.3.11 (Kronheimer-Nakajima) *For $\mathcal{A}$ and Ψ satisfying injectivity of $\mathcal{D}_\Gamma$ and (??), all anti-self-dual $U(n)$ connections of instanton number k, on ALE can be obtained as the induced connection on the bundle $E = \mathrm{Coker}(\mathcal{D}_\Gamma)$.*

[7]The self-dual ones are obtained by reversing the orientation of the bundle.

More explicitly, we take an orthonormal frame U of sections of $\mathrm{Ker}(\mathcal{D}_\Gamma^\dagger)$, i.e., a $(2k+n)\times n$ complex matrix such that $\mathcal{D}_\Gamma^\dagger U = 0$ and $U^\dagger U = \mathrm{Id}$. Then the required connection (gauge field) is given by

$$A_\mu = U^\dagger \nabla_\mu U.$$

4.3.5 Quiver Varieties

We can finally take a unified perspective, combining what we have explained concerning the construction of ALE-instantons as Hyper-Kähler quotients and the quivers for th orbifolds of $\mathbb{C}^2$. Given an $SU(2)$ quiver (i.e., a McKay quiver constructed out of Γ, a finite discrete subgroup of $SU(2)$) Q with edges $H=\{h\}$, vertices $\{1,2,...,r\}$, and beginning (resp. ends) of h as $\alpha(h)$ (resp. $\beta(h)$), we study the representation by associating vector spaces as follows: to each vertex q, we associate a pair of hermitian vector spaces V_q and W_q. We then define the complex vector space:

$$\begin{aligned} M(v,w) &:= \left(\bigoplus_{h\in H}\mathrm{Hom}(V_{\alpha(h)},V_{\beta(h)})\right)\oplus\left(\bigoplus_{q=1}^{r}\mathrm{Hom}(W_q,V_q)\oplus\mathrm{Hom}(V_q,W_q)\right)\\ &:= \bigoplus_{h,q}\{B_h,i_q,j_q\}\end{aligned}$$

with $v:=(\dim_{\mathbb{C}}V_1,\ \ldots\ ,\dim_{\mathbb{C}}V_n)$ and $w:=(\dim_{\mathbb{C}}W_1,\ \ldots\ ,\dim_{\mathbb{C}}W_n)$ being vectors of dimensions of the spaces associated with the nodes.

Upon $M(v,w)$ we can introduce the action by a group

$$G := \prod_q U(V_q) : \{B_h, i_q, j_q\} \to \left\{g_{\alpha(h)}B_h g^{-1}_{\beta(h)}, g_q i_q, j_q g_q^{-1}\right\}$$

with each factor acting as the unitary group $U(V_q)$. We shall be more concerned with $G' := G/U(1)$ where the trivial scalar action by an overall factor of $U(1)$ has been modded out.

In Q we can choose an orientation Ω and hence a signature for each (directed) edge h, viz., $\epsilon(h)=1$ if $h\in\Omega$ and $\epsilon(h)=-1$ if $h\in\bar{\Omega}$. Hyper-Kähler moment maps are subsequently given by:

$$\begin{aligned}
\mu_{\mathbb{R}}(B,i,j) &:= \tfrac{i}{2}\left(\sum_{h\in H, q=\alpha(h)} B_h B_h^\dagger - B_{\bar h}^\dagger B_{\bar h} + i_q i_q^\dagger - j_q^\dagger j_q\right) \in \bigoplus_q \mathfrak{u}(V_q) := \mathfrak{g}, \\
\mu_{\mathbb{C}}(B,i,j) &:= \left(\sum_{h\in H, q=\alpha(h)} \epsilon(h) B_h B_{\bar h} + i_q j_q\right) \in \bigoplus_q \mathfrak{gl}(V_q) := \mathfrak{g}\otimes\mathbb{C}.
\end{aligned} \tag{4.3.3}$$

These maps (**??**) we recognise as precisely the ADHM equations in a different guise. Moreover, the center Z of $\mathfrak{g}$, being a set of scalar $r\times r$ matrices, can be identified with $\mathbb{R}^n$. For Dynkin graphs[8] we can then define R_+, the set of positive roots, $R_+(v)$, the positive roots bounded by v and D_θ, the wall defined by the root θ.

We rephrase Kronheimer's theorem as [**?**]:

THEOREM 4.3.12 *For the discrete subgroup* $\Gamma \in SU(2)$*, let* $v = (n_0, n_1, ..., n_n)$*, the vector of Dynkin labels of the Affine Dynkin graph associated with* Γ *and let* $w = 0$*, then for*[9] $\zeta := (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \left\{\mathbb{R}^3 \otimes Z\right\} \setminus \bigcup_{\theta\in\mathbb{R}_+\setminus\{n\}} \mathbb{R}^3 \otimes D_\theta$*, the manifold*

$$X_\zeta := \{B \in M(v,0) | \mu(B) = \zeta\} // G'$$

is the smooth resolution of $\mathbb{C}^2/\Gamma$ *with corresponding ALE metric.*

For our purposes this construction induces a natural bundle which will give us the required instanton. In fact, we can identify $G' = \prod_{q\neq 0} U(V_q)$ as the gauge group over the non-Affine nodes and consider the bundle

$$\mathcal{R}_l = \mu^{-1}(\zeta) \times_{G'} \mathbb{C}^{n_l}$$

for $l = 1, ..., r$ indexing the non-Affine nodes where $\mathbb{C}^{n_l}$ is the space acted upon by the irreps of Γ (whose dimensions, by the McKay Correspondence, are precisely the Dynkin labels) such that $U(V_q)$ acts trivially (by Schur's Lemma) unless $q = l$. For the

[8] In general they are defined as $R_+ := \{\theta \in \mathbb{Z}^n_{\geq 0} | \theta^t \cdot C \cdot \theta \leq 2\}\setminus\{0\}$ for generalised Cartan matrix $C := 2I - A$ with A the adjacency matrix of the graph; $R_+(v) := \{\theta \in R_+ | \theta_q \leq v_q = \dim_{\mathbb{C}} V_q \ \forall q\}$ and $D_\theta := \{x \in \mathbb{R}^n | x \cdot \theta = 0\}$.

[9] *Z is the trace-free part of the centre and $\mu(B) = \zeta$ means, component-wise $\mu_{\mathbb{R}} = \zeta_{\mathbb{R}}$ and $\mu_{\mathbb{C}} = \zeta_{\mathbb{C}}$.*

affine node, we define $\mathcal{R}_0$ to be the trivial bundle (inspired by the fact that this node corresponds to the trivial principal 1-dimensional irrep of Γ). There is an obvious tautological bundle endomorphism:

$$\xi := (\xi_h) \in \bigoplus_{h \in H} \mathrm{Hom}(\mathcal{R}_{\alpha(h)}, \mathcal{R}_{\beta(h)}).$$

We now re-phrase the Kronheimer-Nakajima theorem above as

THEOREM 4.3.13 *The following sequence of bundle endomorphisms*

$$\bigoplus_q V_q \otimes \mathcal{R}_q \xrightarrow{\sigma} \left(\bigoplus_{h \in H} V_{\alpha(h)} \otimes \mathcal{R}_{\beta(h)} \right) \oplus \left(\bigoplus_q W_q \otimes \mathcal{R}_q \right) \xrightarrow{\tau} \bigoplus_q V_q \otimes \mathcal{R}_q,$$

where

$$\begin{array}{rcl} \sigma & := & \left(B_{\bar{h}} \otimes \mathrm{Id}_{\mathcal{R}_{\beta(h)}} + \epsilon(h) \mathrm{Id}_{V_{\alpha(h)}} \otimes \xi_h \right) \oplus \left(j_q \otimes \mathrm{Id}_{\mathcal{R}_q} \right) \\ \tau & := & \left(\epsilon(h) B_{\bar{h}} \otimes \mathrm{Id}_{\mathcal{R}_{\beta(h)}} - \mathrm{Id}_{V_{\alpha(h)}} \otimes \xi_{\bar{h}}, i_q \otimes \mathrm{Id}_{V_q} \right) \end{array}$$

is a complex (since the ADHM equation $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$ implies $\tau\sigma = 0$) and the induced connection A on the bundle

$$E := \mathrm{Coker}(\sigma, \tau^\dagger) \subset \left(\bigoplus_{h \in H} V_{\alpha(h)} \otimes \mathcal{R}_{\beta(h)} \right) \oplus \left(\bigoplus_q W_q \otimes \mathcal{R}_q \right)$$

is anti-self-dual. And conversely all such connections are thus obtained.

We here illustrate the discussions above via explicit quiver diagrams; though we shall use the $\widehat{A_2}$ as our diagrammatic example, the generic structure should be captured. The quiver is represented in Figure **??** and the concepts introduced in the previous sections are elucidated therein. In the figure, the vector space V of dimension k is decomposed into $V_0 \oplus V_1 \oplus ... \oplus V_r$, each of dimension v_i and associated with the i-th node of Dynkin label $n_i = \dim(R_i)$ in the affine Dynkin diagram of rank r. This is simply the usual McKay quiver for $\Gamma \subset SU(2)$. Therefore we have $k = \sum\limits_i n_i v_i$.

To this we add the vector space W of dimension n decomposing similarly as $W = W_0 \oplus W_1 \oplus ... \oplus W_r$, each of dimension w_i and $n = \sum\limits_i n_i w_i$. Now we have the McKay quiver with extra legs. Between each pair of nodes V_{q_1} and V_{q_2} we have the

Figure 4-1: The Kronheimer-Nakajima quiver for $\mathbb{C}^2/A_n$, extending the McKay quiver to also encapture the information for the construction of the ALE instanton.

map B_h with h the edge between these two nodes. We note of course that due to McKay h is undirected and single-valence for $SU(2)$ thus making specifying merely one map between two nodes sufficient. Between each pair V_q and W_q we have the maps $i_q : W_q \to V_q$ and j_q, in the other direction. The group $U(k)$ is broken down to $(\prod\limits_{q=0}^{r} U(v_q))/U(1)$. This is the group of Γ-compatible symplectic diffeomorphisms. This latter gauge group is our required rank $n = \dim(W)$ unitary bundle with anti-self-dual connection, i.e., an $U(n)$ instanton with instanton number $k = \dim(V)$.

Epilogue

Thus we conclude Liber I, our preparatory journey into the requisite mathematics. We have introduced canonical Gorenstein singularities and monodromies thereon. Thereafter we have studied symplectic structures one could impose, especially in the context of symplectic quotients and moment maps. As a powerful example of such quotients we have reviewed toric varieties.

We then digressed to the representation of finite groups, in preparation of studying a wide class of Gorenstein singularities: the orbifolds. We shall see in Liber III how all of the Abelian orbifolds actually afford toric descriptions. Subsequently we digressed again to the theory of finite graphs and quiver, another key constituent of this writing.

A unified outlook was finally performed in the last sections of Chapter 4 where symplectic quotients in conjunction with quivers were used to address orbifolds of $\mathbb{C}^2$, the so-called ALE spaces. With all these tools in hand, let us now proceed to string theory.

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