Symbolic Computations in Hybrid Systems Verification

Why symbolic computations are required for hybrid systems analysis

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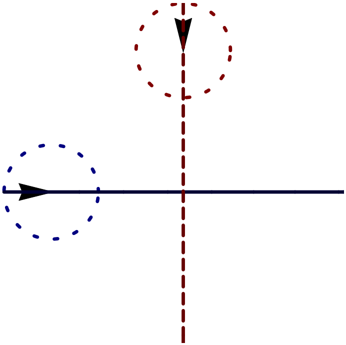
Arlington, VA 2008

Outline

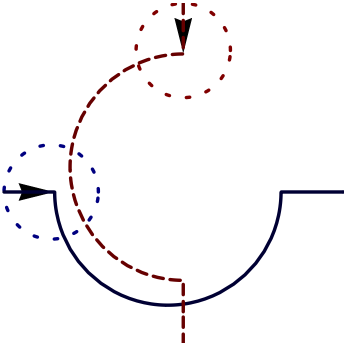
1. Motivation
   - Air Traffic Control
   - The Importance of Verification
   - Image Computation in Hybrid Systems
2. Approximation in Model Checking
   - Approximation Refinement Model Checking
   - Exact Image Computation: Polynomials and Beyond
   - Image Approximation
3. Flow Approximation
   - Bounded Flow Approximation
   - Continuous Image Computation
   - Probabilistic Model Checking
   - Differential Flow Approximation
4. Symbolic Differential Invariants
5. Case Studies
6. Conclusions and Future Work

Outline

Air Traffic Control



Air Traffic Control



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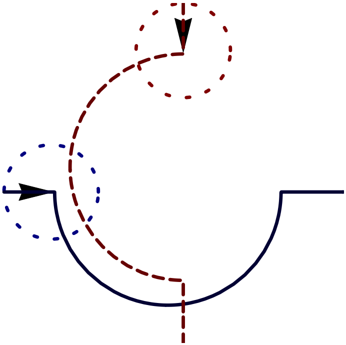
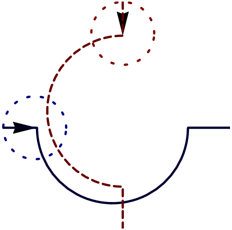


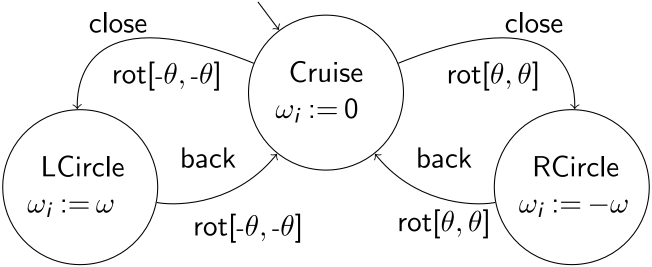
Hybrid Systems

continuous evolution along differential equations + discrete change

ATC: Roundabout Maneuver Automaton

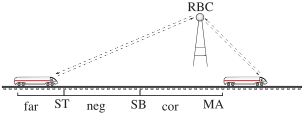


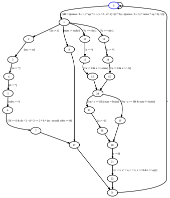
$$\begin{bmatrix} \dot{x} & = -v_1 & +v_2 \cos\phi & +\omega_1 y \\ \dot{y} & = & v_2 \sin\phi & -\omega_1 x \\ \dot{\phi} & = & \omega_2 & -\omega_1 \end{bmatrix}$$

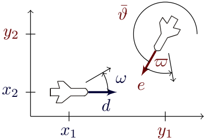


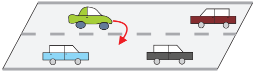
Details

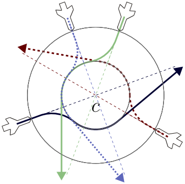
Hybrid Systems Verification is Important for . . .











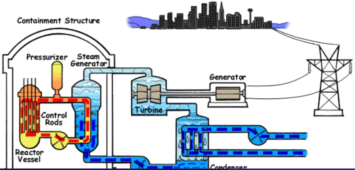
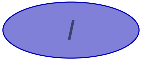


Image Computation in Hybrid Systems



- Analyse image computation problem in hybrid systems
- Approximation refinement techniques and their limits

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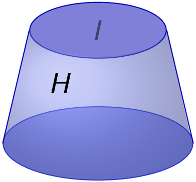

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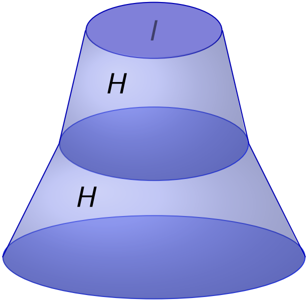

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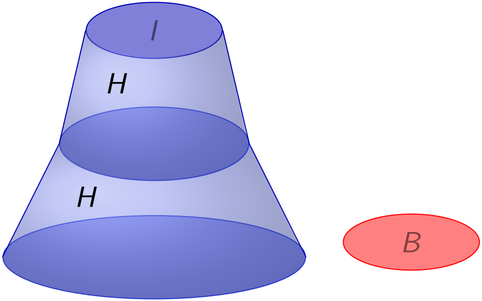

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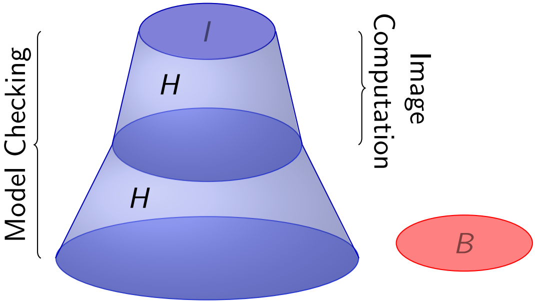

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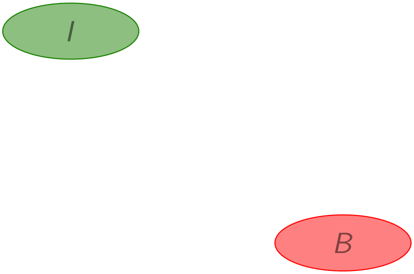
Outline

AMC: Approximation Refinement Model Checking

AMC(B reachable from I in H):

1. $A :=$ approx(H) uniformly
2. blur by uniform approximation error $+\epsilon$
3. check(B reachable from I in $A + \epsilon$)
4. B not reachable $\Rightarrow$ H safe



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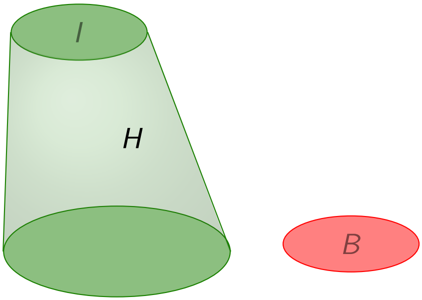

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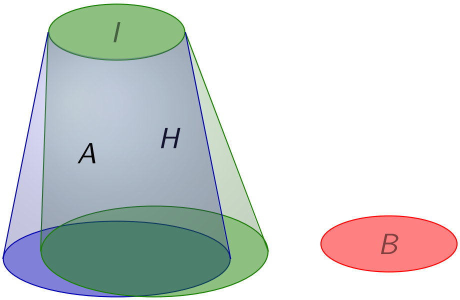

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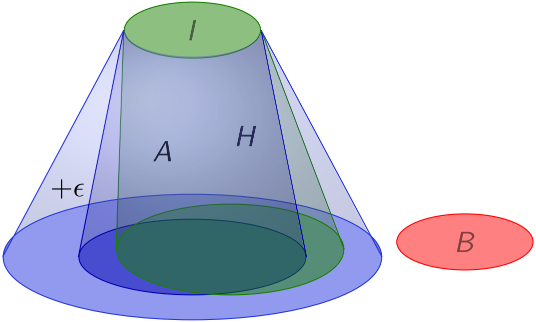

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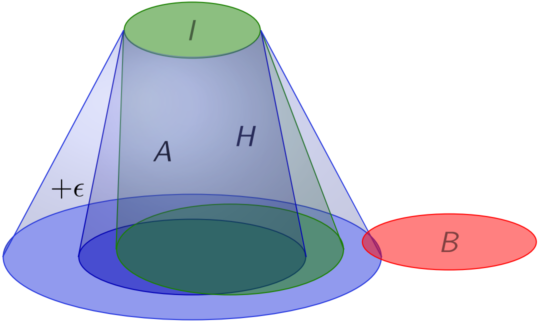

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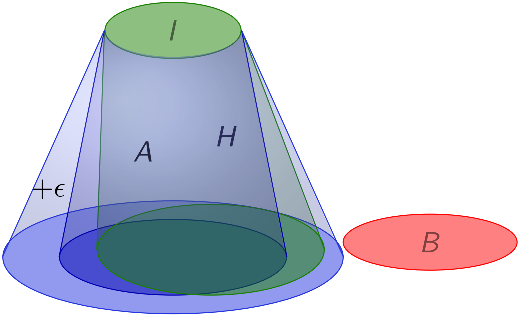

AMC: Exact Image Computation

AMC(B reachable from I in H):

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Proposition

check and blur can be implemented for

- *I and B semialgebraic*
- *A with polynomial flows over* $\mathbf{R}$
- *+Piecewise definitions*
- *+Rational extensions (e.g. multivariate rational splines)*

AMC: Image Approximation

AMC(B reachable from I in H):

1. $A :=$ approx(H) uniformly
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Proposition

approx exists for all uniform errors $\epsilon > 0$ when

- *using polynomials to build A*
- *Flows $\varphi \in C(D, \mathbf{R}^n)$ of H*
- *$D \subset \mathbf{R} \times \mathbf{R}^n$ compact closure of an open set*

Outline

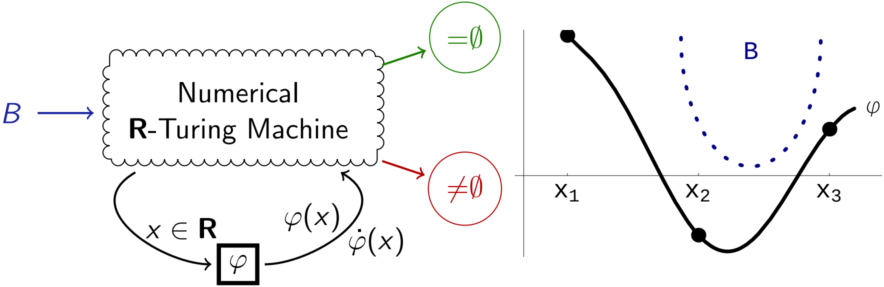
Bounded Flow Approximation

Proposition (Effective Weierstraß approximation)

- *Flows* $\varphi \in C^1(D, \mathbf{R}^n)$
- *Bounds* $b := \max_{x \in D} \|\dot{\varphi}(x)\|$

$\Rightarrow$ *approx computable, hence image computation decidable*

Continuous Image Computation



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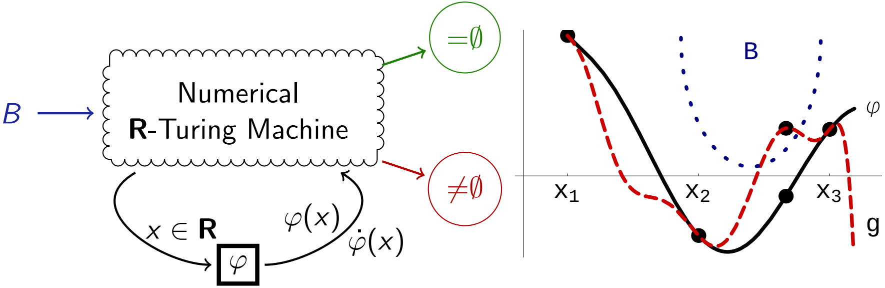

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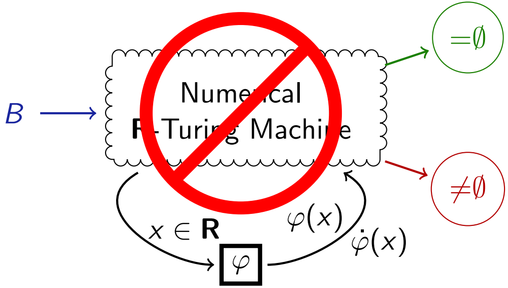
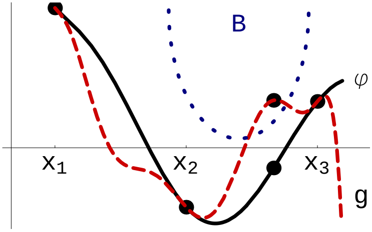
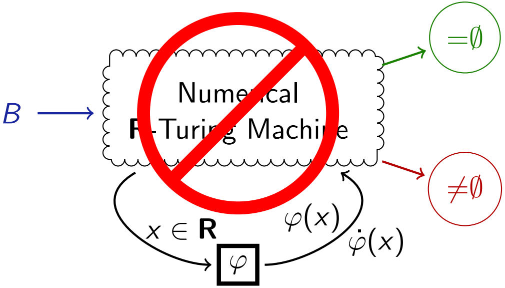


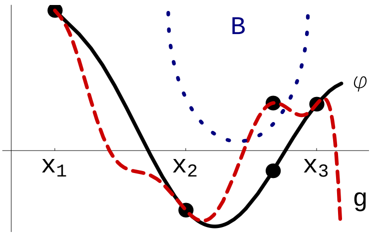


Proposition (Image computation undecidable for. . .)

- *arbitrarily effective flow* $\varphi \in C^k(D \subseteq \mathbf{R}^n, \mathbf{R}^m)$*; D, B effective*
- *tolerate error* $\epsilon > 0$ *in decisions*

Continuous Image Computation

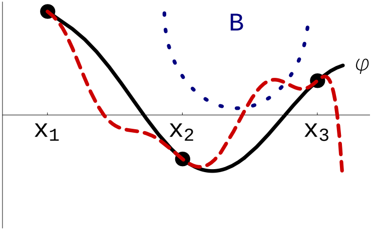




Proposition (Image computation undecidable for. . .)

- *arbitrarily effective flow* $\varphi \in C^k(D \subseteq \mathbf{R}^n, \mathbf{R}^m)$*; D, B effective*
- *tolerate error* $\epsilon > 0$ *in decisions*
- φ *smooth polynomial function with* $\mathbf{Q}$*-coefficients*

Probabilistic Model Checking



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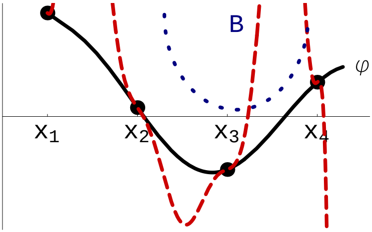

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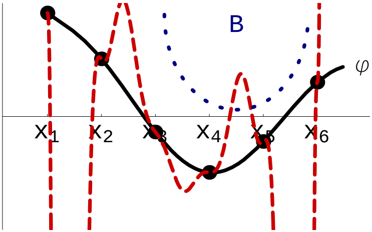

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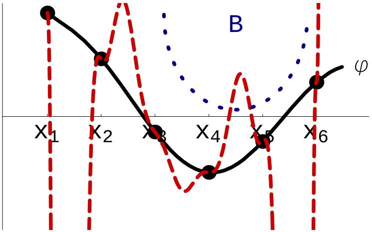
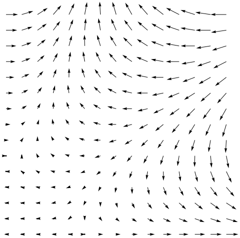


Proposition

- $P(\|\dot{\varphi}\|_\infty > b) \to 0$ *as* $b \to \infty$
- φ *evaluated on finite subset* $X = \{x_i\}$ *of open or compact* D

$\Rightarrow$ $P(\textit{decision correct}) \to 1$ *as* $\|d(\cdot, X)\|_\infty \to 0$

Differential Flow Approximation



φ solves
$\dot{x}(t) = f(t, x)$

Proposition

- *Flow* φ *is solution of* $\dot{x}(t) = f(t, x)$
- $f \in C([a, b] \times \mathbf{R}^n, \mathbf{R}^n)$
- ℓ*-Lipschitz-continuous:* $\|f(t, x_1) - f(t, x_2)\| \leq \ell \|x_1 - x_2\|$

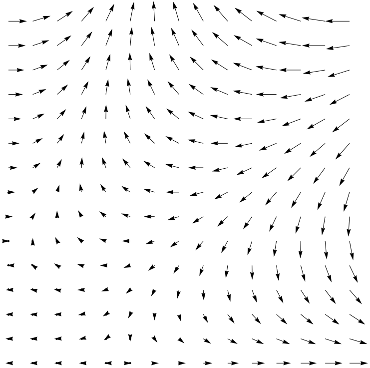
$\Rightarrow$ *Continuous image computation decidable*

Outline

Symbolic: Exploit Vector Field of Differential Equations

"Definition" (Differential Invariant)

"Property that remains true in the direction of the dynamics"



Symbolic: Exploit Vector Field of Differential Equations

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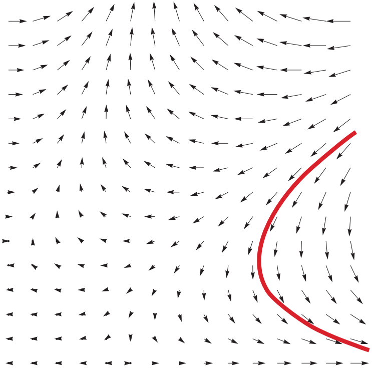

Symbolic: Exploit Vector Field of Differential Equations

"Definition" (Differential Invariant)

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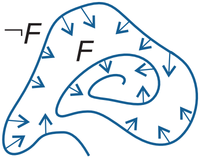
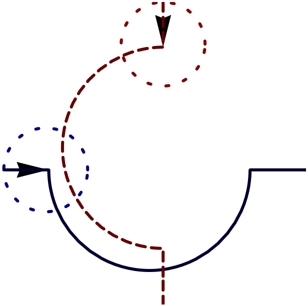
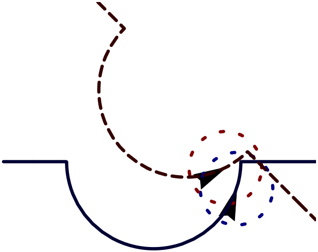
$$\nabla_{\dot{x}_1 = f_1(x), \ldots, \dot{x}_n = f_n(x)} F \quad \text{is} \quad \bigwedge_{(b \geq c) \in F} \left(\sum_{i=1}^{n} \frac{\partial b}{\partial x_i} f_i(x) \geq \sum_{i=1}^{n} \frac{\partial c}{\partial x_i} f_i(x) \right)$$

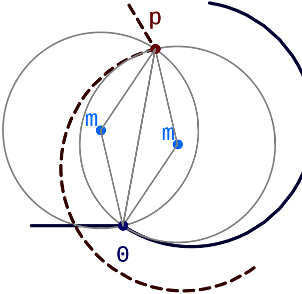


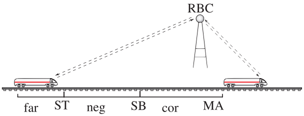
Outline

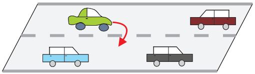
Case Studies

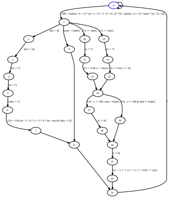












Outline

Conclusions

- Image computation in hybrid systems model checking
  1. approx uniformly
  2. blur by uniform error
  3. check for B

flows	approx / image computation
continuous	uniform approx exists, but...
smooth	undecidable numerically
bounded by b	decidable
bound probabilities	probabilistically decidable
ODE ℓ-Lipschitz	decidable

- Differential invariants:
  1. based on symbolic computations
  2. sound!
  3. scalable

Challenges

- Symbolic computation techniques for hybrid systems are strictly required!
- Scaling symbolic computation to nontrivial dynamics
- Combine numerical algorithms with symbolic analysis
- Joint symbolic stochastic analysis

A. Platzer and E. M. Clarke.
The image computation problem in hybrid systems model checking.
In A. Bemporad, A. Bicchi, and G. Buttazzo, editor, *HSCC*, 2007.

A. Platzer and E. M. Clarke.
Computing differential invariants of hybrid systems as fixedpoints.
In A. Gupta and S.Malik, editor, *CAV*, 2008.