NETS 412: Algorithmic Game Theory March 2, 2017

Lecture 13

Lecturer: Aaron Roth *Scribe: Aaron Roth*

Walrasian Equilibrium

In this lecture, we bring money into the picture, while still thinking about a "matching" like problem.

Suppose we have:

1. We have m *goods* G for sale
2. n *buyers* i who each have *valuation functions* over bundles, $v_i : 2^G \to [0, 1]$.

Buyers have *quasi-linear* utility functions, which means they can (linearly) trade off their value for goods and their value for money. If each good $j \in G$ has a price p_j, then a buyer i gets the following utility for buying a bundle $S \subseteq G$:

$$u_i(S) = v_i(S) - \sum_{j \in S} p_j$$

The question we want to ask in this class is how we should *price* and *allocate* goods so that everyone is happy with their allocation. Is this even possible? If it is, can we do so and *also* achieve a high welfare allocation? We will see that in some settings, the answer is surprisingly *yes*.

First we need to talk about feasible allocations:

Definition 1 *An allocation $S_1, \ldots, S_n \subseteq G$ is* feasible *if for all $i \neq j$, $S_i \cap S_j = \emptyset$ We write OPT to denote the socially optimal feasible allocation:*

$$OPT = \max_{S_1,\ldots,S_n \text{ feasible}} \sum_i v_i(S)$$

We now introduce a notion of pricing equilibrium, that corresponds to everyone being happy with the chosen allocation.

Definition 2 *A set of prices p together with an allocation $S_1, \ldots, S_n$ form an (ϵ-approximate)* Walrasian equilibrium *if:*

1. *$S_1, \ldots, S_n$ is feasible, and*
2. *For all i, buyer i is receiving his (ϵ) most preferred bundle given the prices:*

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq \max_{S^* \subseteq G} \left(v_i(S^*) - \sum_{j \in S^*} p_j \right) - \epsilon$$

and,

3. *All unallocated items have zero price: for all $j \notin S_1 \cup \ldots \cup S_n$, $p_j = 0$.*

At Walrasian equilibrium, no buyer wants to buy a different bundle, and the seller does not want to lower any of the prices – the only things that aren't selling *can't* sell (they already have price 0). We might naturally have two questions about Walrasian equilibria: first, do they exist, in general? Second, when they do exist, are they compatible with achieving welfare close to OPT?

We answer the second question first.

Theorem 3 *If $S_1, \ldots, S_n$ form an ϵ-Walrasian equilibrium allocation, then they achieve nearly optimal welfare. In particular:*

$$\sum_i v_i(S_i) \geq OPT - \epsilon n$$

Proof Let p be the corresponding Walrasian equilibrium prices, and let $S'_1, \ldots, S'_n$ be any other feasible allocation. We know from the 2nd Walrasian equilibrium condition that for every player i, we have:

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(S'_i) - \sum_{j \in S'_i} p_j - \epsilon$$

(Since in particular, $v_i(S_i)$ is within ϵ of the value of player i's *most preferred* bundle.) Since this is true for every player i, we can sum over all n of these inequalities to get:

$$\sum_i \left(v_i(S_i) - \sum_{j \in S_i} p_j \right) \geq \sum_i \left(v_i(S'_i) - \sum_{j \in S'_i} p_j \right) - \epsilon n$$

In particular, by simply re-ordering the sums, we have equivalently:

$$\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n$$

Noting that for any $j \notin S_1 \cup \ldots \cup S_n$, we must have $p_j = 0$ (the third Walrasian equilibrium condition), we can conclude that on the left hand side:

$$\sum_{j \in S_1 \cup \ldots \cup S_n} p_j = \sum_j p_j$$

and hence we can rewrite the inequality to conclude:

$$\sum_i v_i(S_i) \geq \sum_i v_i(S'_i) + (\sum_j p_j - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j) - \epsilon n \geq \sum_i v_i(S'_i) - \epsilon n$$

Finally, taking $S'_1, \ldots, S'_n$ to be the optimal allocation gives the theorem. ■

Ok – so Walrasian equilibria are great when we have them. Not only is everyone getting their most preferred bundle, but the allocation is also always globally optimal! Of course, this is only interesting if they ever exist. Do they?

Lets start by considering a simple case: when buyers have so-called *unit-demand* valuations, meaning they want to buy a bundle of only 1 item. Formally, for each buyer i:

$$v_i(S) = \max_{j \in S} v_i(\{j\})$$

We can think about such a valuation function as being determined by just m numbers, one for each good:

$$v_{i,j} \equiv v_i(\{j\}).$$

Note that finding the welfare maximizing allocation for unit demand bidders is just the problem of finding the maximum weight matching in a bipartite graph between buyers and goods, where the edge weight between buyer i and good j is $v_{i,j}$.

We will show:

Theorem 4 *For any set of unit demand buyers, a Walrasian equilibrium always exists.*

Proof We will prove this theorem constructively, by considering a natural ascending price auction that works much like the deferred acceptance algorithm. Initially, every bidder will be unmatched, and all goods will have price 0. Then, in turns, unmatched bidders will bid on their most preferred item at the current prices. Their bid will make them the new high bidder on the good they bid on, and we will increment its price. At all times, we will consider the bidder i tentatively matched to good j to be the current high bidder on good j, and we will halt when all bidders are matched.

Lets analyze the ascending price auction. First, does it even ever halt and output anything? (It does)

Algorithm 1 The Ascending Price Auction with increment ϵ.

For all $j \in G$, set $p_j = 0$, $\mu(j) = \emptyset$.
while There exist any unmatched bidders **do**
  for Each unmatched bidder i **do**
    i "bids" on $j^* = \arg\max_j(v_{i,j} - p_j)$ if $v_{i,j^*} - p_{j^*} > 0$. Otherwise, bidder i drops out. (and is "matched" to nothing):
    $\mu(j^*)$ is now unmatched. Set $\mu(j^*) \leftarrow i$
    $p_{j^*} \leftarrow p_{j^*} + \epsilon$
  end for
end while
Output (p, μ).

Lemma 5 *The ascending price auction halts after at most $\frac{n}{\epsilon}$ bids.*

Proof We first claim that at any point during the algorithm, we must have:

$$\sum_j p_j \leq n$$

This is because once a good becomes matched, it stays matched for the rest of the algorithm (even though bidders might become unmatched). Hence, all *unmatched* goods must have price $p_j = 0$. We also have that for any fixed good j, $p_j \leq 1$. This is because no bidder ever bids on any good j such that $v_{i,j} - p_j \leq 0$, and by assumption $v_{i,j} \leq 1$ for all i, j. Finally, since there are at most n agents, at most n goods are ever matched, and so at most n goods can have positive price.

To complete the proof, simply note that each bid increases $\sum_j p_j$ by ϵ. Since $\sum_j p_j$ begins at 0 and never exceeds n, there can be no more than $\frac{n}{\epsilon}$ bids. ■

So we know the algorithm halts and outputs some pricing and matching.

Lemma 6 *The output (p, μ) of the ascending price auction is an ϵ-approximate Walrasian equilibrium.*

Proof We verify the equilibrium conditions. First, observe that by construction, the auction only returns feasible allocations.

Next observe that if good j is unallocated, it must never have received a bid in the auction (once a good is allocated, it stays allocated), and hence $p_j = 0$.

Finally, note that $v_{i,\mu(i)} - p_{\mu(i)} \geq \max_j(v_{i,j} - p_j) - \epsilon$. To see this, note that by construction, at the time bidder i was matched to good $\mu(i)$, we must have had:

$$\mu(i) \in \arg\max_j(v_{i,j} - p_j)$$

Since that time, p_j increased by only ϵ, and other prices have only increased (not decreased). That completes the proof. ■

Hence, equilibrium prices always exist in matching markets. ■

What about for more general valuation functions? We will see on the homework that Walrasian equilibrium need not exist for *all* valuation functions.

Lets think about what we need in order to generalize the ascending price auction and the proof that it constructs a Walrasian equilibrium. We can certainly generalize the auction to work for more general valuation functions. Rather than having each bidder i bid on their most preferred item, we have them bid on their most preferred bundle. We say a bidder is unsatisfied if she is not currently matched to her ϵ-most preferred bundle:

For each unsatisfied bidder i:

1. i bids on every item she is not the high bidder on in a set $S^* \in \arg\max_{S \subseteq G}(v_i(S) - \sum_{j \in S} p_j)$
2. For all $j \in S^*$, $\mu(j) \leftarrow i$, $p_j \leftarrow p_j + \epsilon/m$.

Certainly now, after a bidder gets to bid, she is matched to her ϵ-most preferred bundle, and she remains so if she is not out-bid on any of her items (since other prices only rise). However, for our analysis, we also needed the fact that once a good became matched, it stayed matched (so that we could argue that unmatched goods have price 0). Hence, we do not want that when a bidder i bids, she abandons any of the goods she is currently matched to. That is, we always want bidder i's most preferred bundle to include as a subset all of the goods she is currently matched to. If this is so, our proof that the ascending price auction always terminates at a Walrasian equilibrium will go through.

Lets formalize this condition.

For price vectors p, p', write $p \preceq p'$ to mean that $p_j \leq p'_j$ for all j. Let $w_i(p) = \arg\max_{S \subseteq G}(v_i(S) - \sum_{j \in S} p_j)$ be player i's demand set at prices p.

Definition 7 *Valuation function v_i satisfies the* gross substitutes *property if for every $p \preceq p'$ and for every $S \in w_i(p)$, if $S' = \{j \in S : p_j = p'_j\}$, then there exits $S^* \in w_i(p')$ such that $S' \subseteq S^*$.*

In other words, "Raising the prices on goods $j \neq i$ doesn't decrease a bidder's demand for good j".

Operationally, Gross substitutes valuations satisfy the condition we want. Any good for which bidder i has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder i's demand set.

Hence, we have:

Theorem 8 *In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.*

In fact, it is known that this is the *largest* set of valuation functions for which Walrasian equilibria are guaranteed to exist. For any valuation function v_i that does not satisfy the gross substitutes conditions, there exist a set of valuation functions v_{-i} for the other players, each satisfying gross substitutes, for which there are no Walrasian equilibrium prices!