Algorithmic Game Theory: Problem Set 5
Due on Tuesday, April 28
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Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment.

Generalized Groves Mechanism (25 pts)

In class, we saw that the Groves mechanism with allocation rule:

\[ x(v) = \arg \max_{a \in A} \sum_{i=1}^{n} v_i(a) \]

and payment rule

\[ p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*) \]

where \( a^* = x(v) \) and \( h_i \) is an arbitrary function independent of \( v_i \) is truthful. Suppose instead we have a generalized allocation rule:

\[ x(v) = \arg \max_{a \in A} \left( \sum_{i=1}^{n} \alpha_i v_i(a) + \beta_a \right) \]

where \( \alpha_i > 0 \) are positive weights corresponding to individuals and \( \beta_a \) are weights corresponding to alternatives \( a \in A \). Show that this allocation rule can be paired with the payment rule such that the resulting mechanism is truthful. Hint: As with the Groves mechanism from class, you want to design the payment rule so that a player’s utility is maximized exactly when the mechanism’s objective function is maximized.

Public Projects Auction (30 pts)

Recall the public projects auction, in which a city is deciding whether or not to build a bridge at cost \( C > 0 \). In this case, \( A = \{yes, no\} \). This is a single parameter domain, in which \( w(\text{yes}) = 1 \) and \( w(\text{no}) = 0 \) and an individual with valuation \( v_i \in \mathbb{R}^+ \) has value \( v_i(a) = v_i \cdot w(a) \) for outcome \( a \). Consider the (generalized) VCG mechanism which has allocation rule

\[ x(v) = \arg \max_{a \in A} \sum_{i=1}^{n} v_i(a) - w(a) \cdot C \]

(i.e. \( \beta_{yes} = -C \)) and payment rule

\[ p_i(v) = \left( \sum_{j \neq i} v_j(a^*_{-i}) - w(a^*_{-i})C \right) - \left( \sum_{j \neq i} v_j(a^*) - w(a^*)C \right) \]

where \( a^* = x(v) \) and \( a^*_{-i} = x(v_{-i}) \). This VCG mechanism builds the bridge if and only if \( \sum_{i=1}^{n} v_i \geq C \).

1. Show that if the bridge is built, the total payments are enough to cover the cost of the bridge (i.e. \( \sum_{i=1}^{n} p_i(v) \geq C \)) if and only if \( \sum_{i=1}^{n} v_i = C \). (15 pts)

2. Show how you can use a profit extractor to truthfully determine whether or not to build the bridge in a way in which the cost of the bridge will always be covered by the payments that are collected. The profit extractor will not be able to always maximize social welfare: give an example of a collection of values \( v_i \) such that the VCG mechanism would build the bridge but the profit extractor would not. (15 pts).