Algorithmic Game Theory: Problem Set 1
Due as hard copy on Tuesday, January 27
Aaron Roth

Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment. Ask questions on Piazza.

Problem 1) Games with Infinite Action Sets (20 points)
John Nash proved that every game with finitely many players and finitely many actions has a Nash equilibrium in mixed strategies. These conditions are important!

(a) Give an example of a 2 player game in which each player has infinitely many actions and your game has a Nash equilibrium. Briefly and precisely describe the equilibrium. (10 points)

(b) Give an example of a 2 player game in which each player has infinitely many actions, and prove that your game does not have any Nash equilibrium. (10 points)
       Hint: Don’t forget about mixed strategies!

Problem 2) Iterated Elimination (45 points)
Recall that in class we considered one way of solving a game: by iterated elimination of weakly dominated strategies. We can also consider iterated elimination of strictly dominated strategies. An action \(a_i \in A_i\) is strictly dominated if \(u_i(a_i, a_{-i}) < u_i(a'_i, a_{-i})\) for some \(a'_i \in A\) and for all \(a_{-i} \in A_{-i}\). (i.e. the inequality is always strict.)

We can write this method as an algorithm, which takes as input a set of \(n\) action sets \(A_1, \ldots, A_n\) and a set of \(n\) utility functions \(u_1, \ldots, u_n\), where each \(u_i\) is a function \(u_i : A_1 \times \ldots \times A_n \rightarrow \mathbb{R}\).

Part 1 (15 points)
Consider the following 2 player game.

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<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>2,0</td>
<td>1,1</td>
<td>4,2</td>
</tr>
<tr>
<td>M</td>
<td>3,4</td>
<td>1,2</td>
<td>2,3</td>
</tr>
<tr>
<td>B</td>
<td>1,3</td>
<td>0,2</td>
<td>3,0</td>
</tr>
</tbody>
</table>

(a) Which strategies survive iterated elimination of strictly dominated strategies? (5 points)

(b) What are the pure strategy Nash equilibria of the game? (5 points)

(c) Find a non-trivial (i.e. someone should be randomizing and not just playing a pure strategy) mixed-strategy Nash equilibrium of the game. (5 points)
Algorithm 1 Iterated Elimination of Strictly Dominated Strategies

IteratedElim$(A_1, \ldots, A_n, u_1, \ldots, u_n)$.

- Initialize a counter $t = 0$
- For each $i$, Let $B^t_i = A_i$
- while TRUE do
  - For each $i$ let:
    - $\text{Dom}^t_i = \{a_i \in B^t_i \mid \text{there exists } a_i' \in B^t_i \text{ such that for all } s \in B^t_1 \times \ldots \times B^t_n, u_i(a_i', s) > u_i(a_i, s)\}$
    - if There exists an $i$ such that $\text{Dom}^t_i \neq \emptyset$ then
      - Let $B^{t+1}_i = B^t_i \setminus \text{Dom}^t_i$
      - Update $t = t + 1$
    - else
      - Break
    - end if
  - end while
- Return $B^t_1, \ldots, B^t_n$.

Part 2 (30 points)

(a) Prove that if only a single strategy profile $s$ survives iterated elimination of weakly dominated strategies (i.e. if at the end for all $i$, $|B^t_i| = 1$ and $s_i \in B^t_i$ is the surviving action of player $i$) then $s$ is a pure strategy Nash equilibrium of the game. (10 points)

(b) Prove that if only a single strategy profile $s$ survives iterated elimination of strictly dominated strategies, then it is the unique pure strategy Nash equilibrium of the game. (10 points)

(c) Give an example of a game that has two pure strategy Nash equilibria, and depending on the order in which actions are chosen for elimination, either of them can be selected as the single surviving strategy profile when we apply iterated elimination of weakly dominated strategies. (10 points)