Programming Up-to-Congruence

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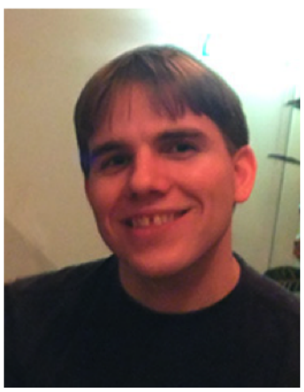
January 16, 2015

POPL 2015

Zombie

A functional programming language with a dependent type system intended for "lightweight" verification

With:



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The Zombie programming language

Goal: FP++

- Functional programming enhanced by reasoning in constructive logic
- Full-spectrum dependent types (for uniformity)
- Erasable arguments (for efficient compilation)
- Simple semantics for indexed types and dependently-typed pattern matching

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- Simple semantics for indexed types and dependently-typed pattern matching
- **Proof automation based on congruence closure**

Zombie: A language, in two parts

1. Programmatic fragment: nontermination allowed (similar to ML and Haskell)

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log add : Nat → Nat → Nat
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Uniformity: Both fragments use the same syntax, have the same (call-by-value) operational semantics.

Dependent types in Zombie

The logical fragment can reason about the programmatic fragment.

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    div62 = join
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Type checking `join` is undecidable, so includes an overridable timeout—the programmer is in control.

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In other words, β-conversion is only available for *propositional* equality.

```
f (x |> [Vec Bool ~div62])
```

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ind npluszero n =
  case n [eq] of
   Zero → (join : 0 + 0 = 0)
              |> [~eq + 0 = ~eq]       -- explicit type coercion
                                       -- eq : 0 = n
   Suc m →
     let ih = npluszero m [ord eq] in
       (join : (Suc m) + 0 = Suc (m + 0))
         |> [(Suc m) + 0 = Suc ~ih]    -- ih : m + 0 = m
         |> [~eq + 0 = ~eq]            -- eq : Suc m = n
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But we can do better.

Better

What if the type checker could determine those coercions automatically?

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Capture this idea with a relation:

$$\texttt{eq: n = 0} \vdash \big(\texttt{0 + 0 = 0}\big) = \big(\texttt{n + 0 = n}\big)$$

Opportunity: Congruence Closure

The relation that we need is the *congruence closure* of equations in the context.

$$\frac{x : a = b \in \Gamma}{\Gamma \vdash a = b} \qquad \frac{\Gamma \vdash a = b}{\Gamma \vdash \{a/x\}\, c = \{b/x\}\, c}$$

$$\frac{}{\Gamma \vdash a = a} \qquad \frac{\Gamma \vdash a = b}{\Gamma \vdash b = a} \qquad \frac{\Gamma \vdash a = b \quad \Gamma \vdash b = c}{\Gamma \vdash a = c}$$

Efficient algorithms for deciding this relation exist [Nieuwenhuis and Oliveras, 2007].
Note, extending this relation with β-conversion makes it undecidable.

What we have done

Designed and implemented a concise **surface language** for ZOMBIE programmers

- Specification via bidirectional type system

$$\Gamma \vdash a \Rightarrow A \quad \text{and} \quad \Gamma \vdash a \Leftarrow A$$

- Type checking is up-to Congruence Closure

$$\frac{\Gamma \vdash a \Rightarrow A \quad \Gamma \vDash A = B}{\Gamma \vdash a \Rightarrow B} \qquad \frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vDash A = B}{\Gamma \vdash a \Leftarrow B}$$

- Elaborates to explicitly-typed **core language**, previously proven sound
  [POPL '14][MSFP'12]

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5. and generates proof terms in the core language

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Properties of Elaboration

- **Elaboration is sound**
  If elaboration succeeds, it produces a well-typed core language term.
- **Elaboration is complete**
  If a term type checks according to the surface language specification, then elaboration will succeed.
- **Elaboration doesn't change the semantics**
  If elaboration succeeds, it produces a core language term that differs from the source term only in irrelevant information (type annotations, type coercions, erasable arguments).

Extensions

Proof inference

Congruence closure can also supply proofs of equality

```
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
   case n [eq] of
      Zero →
        let _ = (join : 0 + 0 = 0) in _
      Suc m →
        let _ = npluszero m [ord eq] in
        let _ = (join : (Suc m) + 0 = Suc (m + 0)) in _
```

Extension: Unfold

Common to reduce terms as much as possible

```
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
   case n [eq] of
      Zero  → unfold (0 + 0) in _
      Suc m →
         let _ = npluszero m [ord eq] in
         unfold ((Suc m) + 0) in _
```

The expression `unfold a in b` expands to

```
let _ = (join : a = a1)   in
let _ = (join : a1 = ...) in
...
let _ = (join : ... = an) in
  b
```

when `a` ⇝ `a1` ⇝ `...` ⇝ `an`

Extension: Reduction Modulo

The type checker makes use of congruence closure when reducing terms with `unfold`.

```
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
   case n [eq] of
      Zero → unfold (n + 0) in _
      Suc m →
        let ih = npluszero m [ord eq] in
        unfold (n + 0) in _
```

E.g., if we have $h : n = \mathsf{0}$ in the context, allow the step

$$n + 0 \leadsto_{\mathsf{cbv}} 0$$

Extension: Smartjoin

Use unfold (and reduction modulo) on both sides of an equality when type checking `join`.

```
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
   case n [eq] of
      Zero  → smartjoin
      Suc m → let ih = npluszero m [ord eq] in
               smartjoin
```

Conclusions

- Dependently-typed languages should allow nonterminating programs, but compile-time reduction is tricky
- Restricting β-reduction allows alternative forms of automatic reasoning, specifically congruence closure
- Congruence closure powers smart case, a simple specification of dependently-typed pattern matching
- Proof automation is an important part of the design of dependently-typed languages, and should be backed up by specifications

Implementation and examples available:

`https://code.google.com/p/trellys/source/browse/trunk/zombie-trellys/`

or Google: zombie trellys

Thanks!