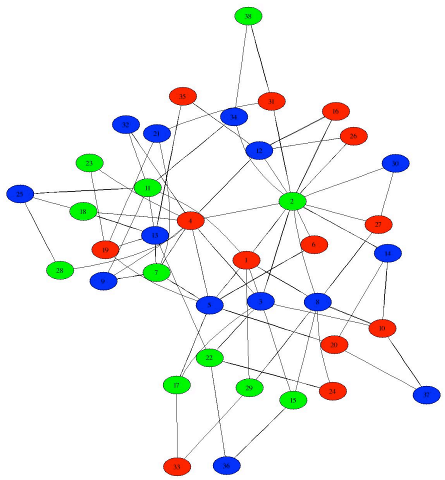
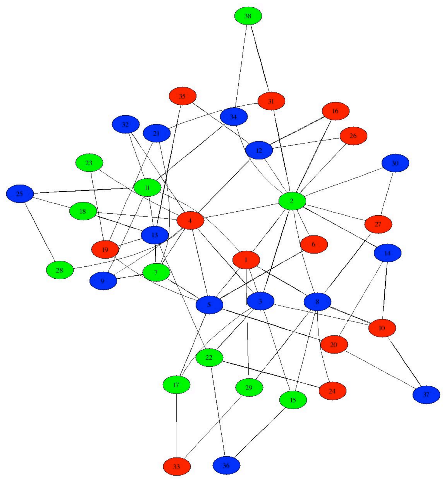
How Do “Real” Networks Look?

Networked Life
NETS 112
Fall 2018
Prof. Michael Kearns

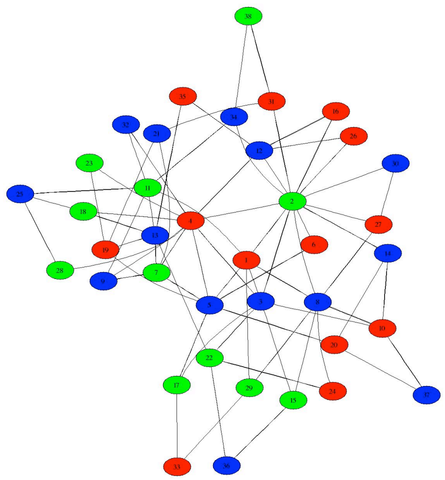


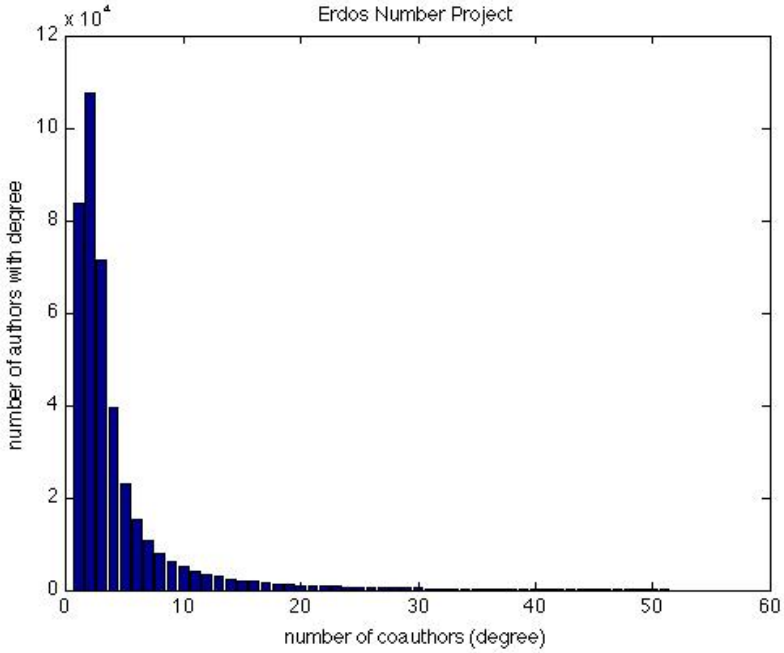
Roadmap

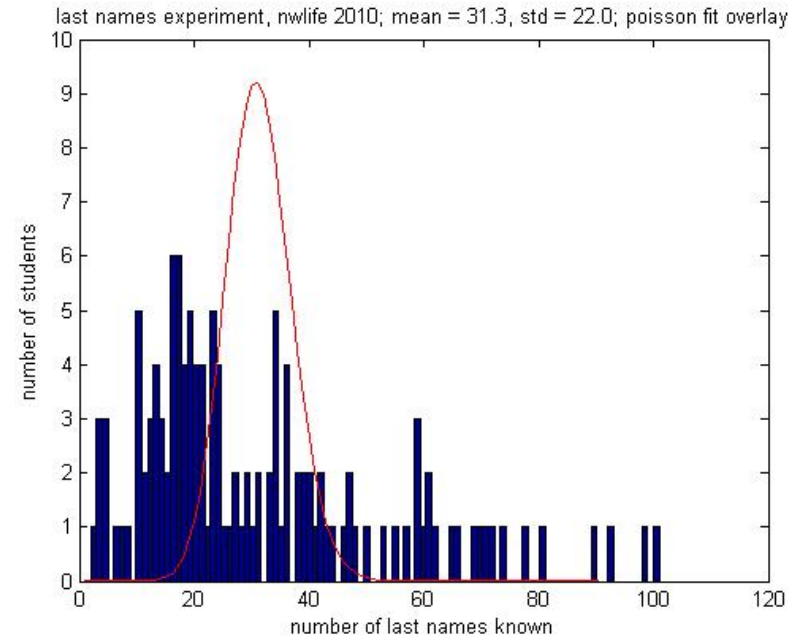
- Next several lectures: “universal” structural properties of networks
- Each large-scale network is unique microscopically, but with appropriate definitions, striking macroscopic commonalities emerge
- Main claim: “typical” large-scale network exhibits:
  - heavy-tailed degree distributions → “hubs” or “connectors”
  - existence of giant component: vast majority of vertices in same component
  - small diameter (of giant component) : generalization of the “six degrees of separation”
  - high clustering of connectivity: friends of friends are friends
- For each property:
  - define more precisely; say what “heavy”, “small” and “high” mean
  - look at empirical support for the claims
- First up: heavy-tailed degree distributions

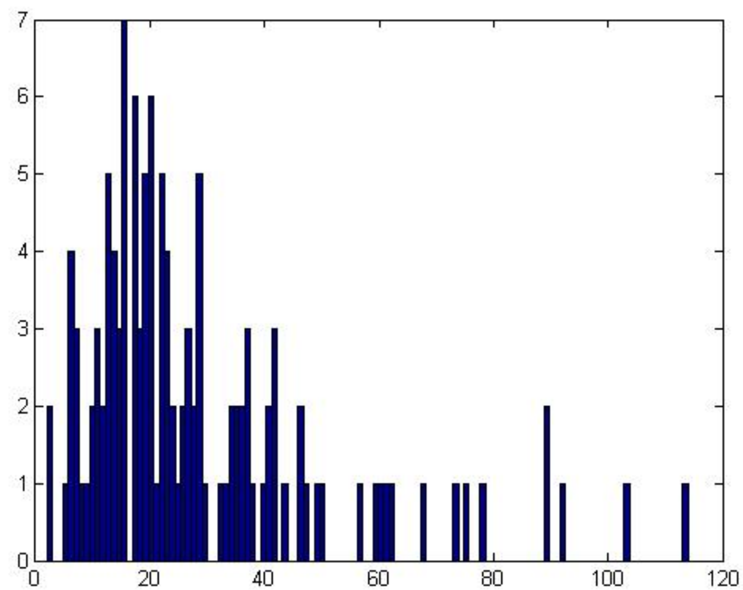


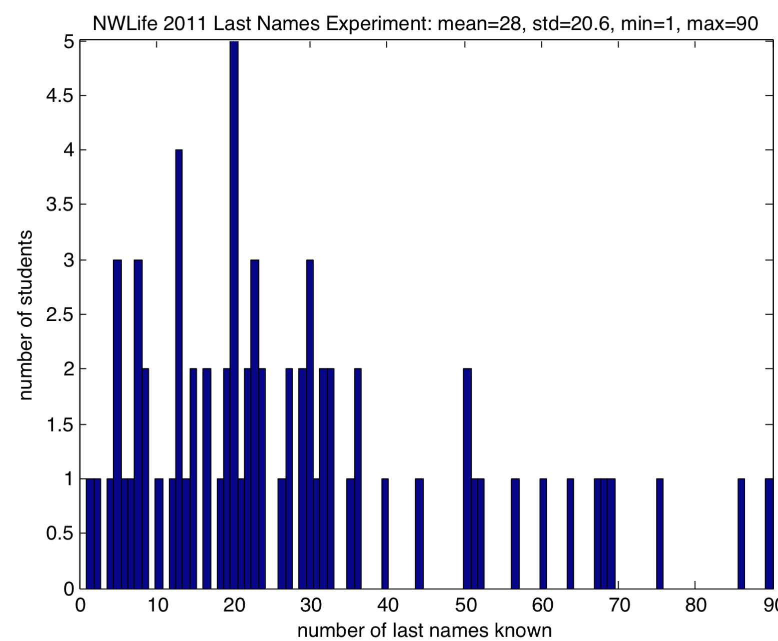
How Do “Real” Networks Look?
I. Heavy-Tailed Degree Distributions




Erdos Number Project
x 10^4
number of authors with degree
number of coauthors (degree)


last names experiment, nwlife 2010; mean = 31.3, std = 22.0; poisson fit overlay
number of students
number of last names known




NWLife 2011 Last Names Experiment: mean=28, std=20.6, min=1, max=90
number of students
number of last names known

What Do We Mean By *Not* "Heavy-Tailed"?

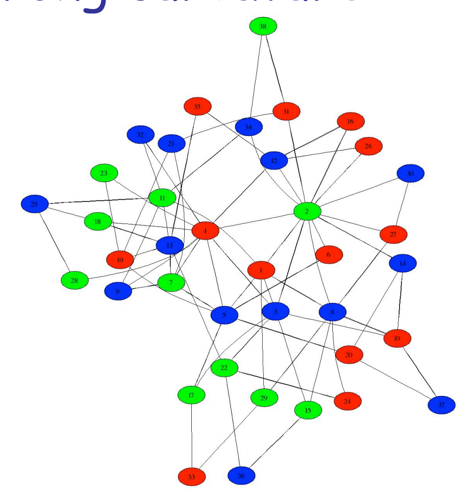
- Mathematical model of a typical "bell-shaped" distribution:
  - the Normal or Gaussian distribution over some quantity x
  - Good for modeling many real-world quantities… but not degree distributions
  - if mean/average is μ then probability of value x is:

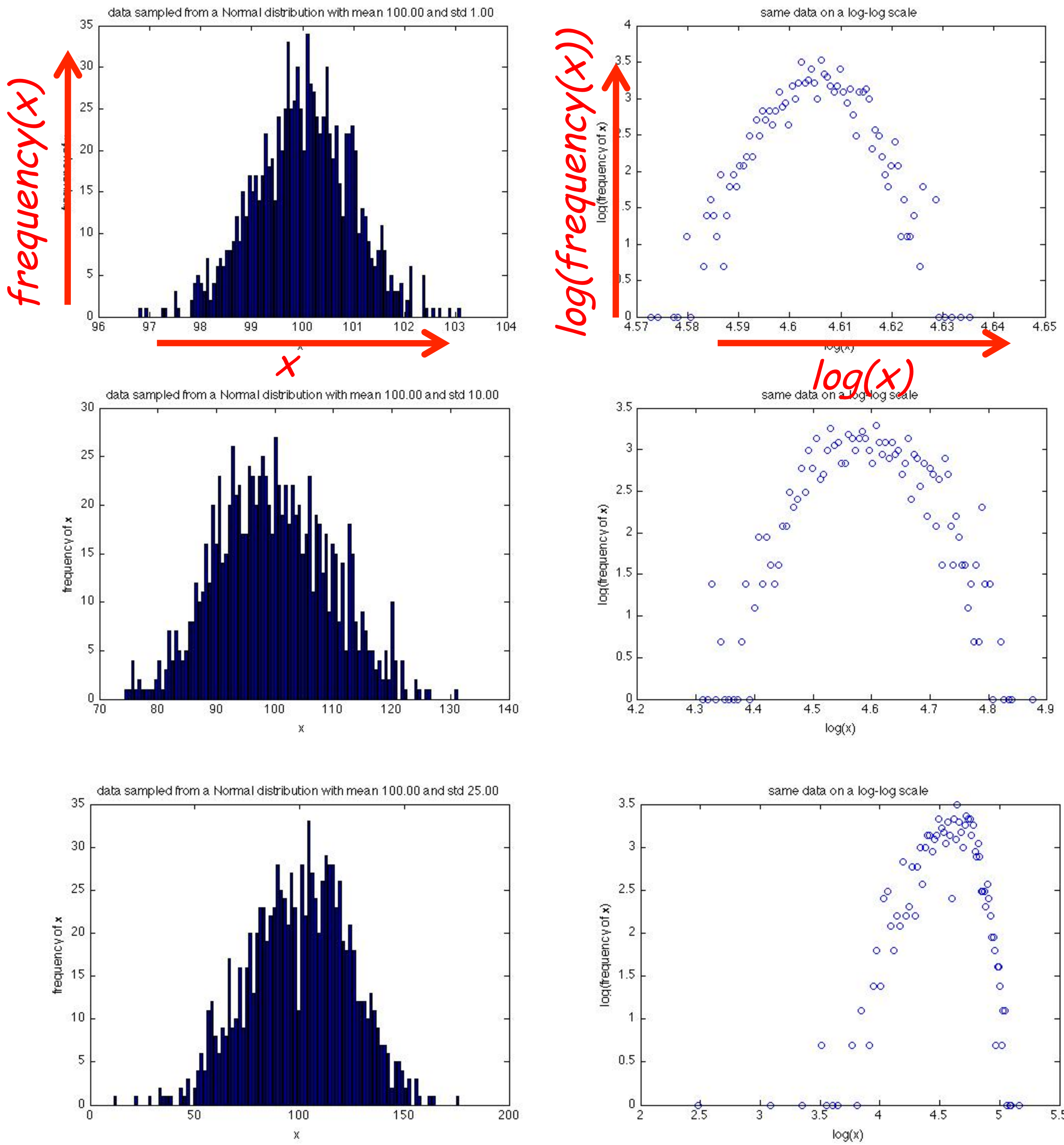
$$probability(x) \propto e^{-(x-\mu)^2}$$

  - main point: exponentially fast decay as x moves away from μ
  - if we take the logarithm:

$$\log(probability(x)) \propto -(x-\mu)^2$$

- Claim: if we plot log(x) vs log(probability(x)), will get strong curvature
- Let's look at some (artificial) sample data…
  - (Poisson better than Normal for degrees, but same story holds)




data sampled from a Normal distribution with mean 100.00 and std 1.00
frequency(x)
x
same data on a log-log scale
log(frequency(x))
log(x)
data sampled from a Normal distribution with mean 100.00 and std 10.00
frequency of x
x
same data on a log-log scale
log(frequency of x)
log(x)
data sampled from a Normal distribution with mean 100.00 and std 25.00
frequency of x
x
same data on a log-log scale
log(frequency of x)
log(x)

What *Do* We Mean By "Heavy-Tailed"?

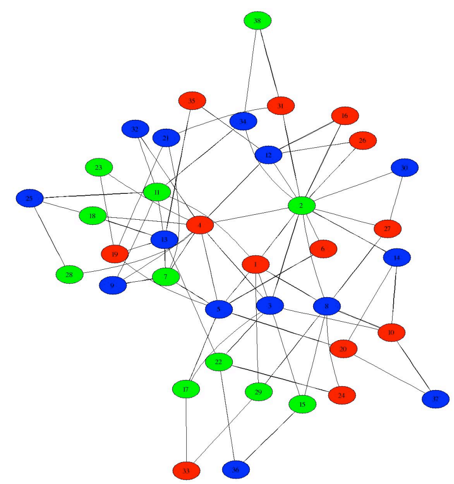
- One mathematical model of a typical "heavy-tailed" distribution:
  - the Power Law distribution with exponent β

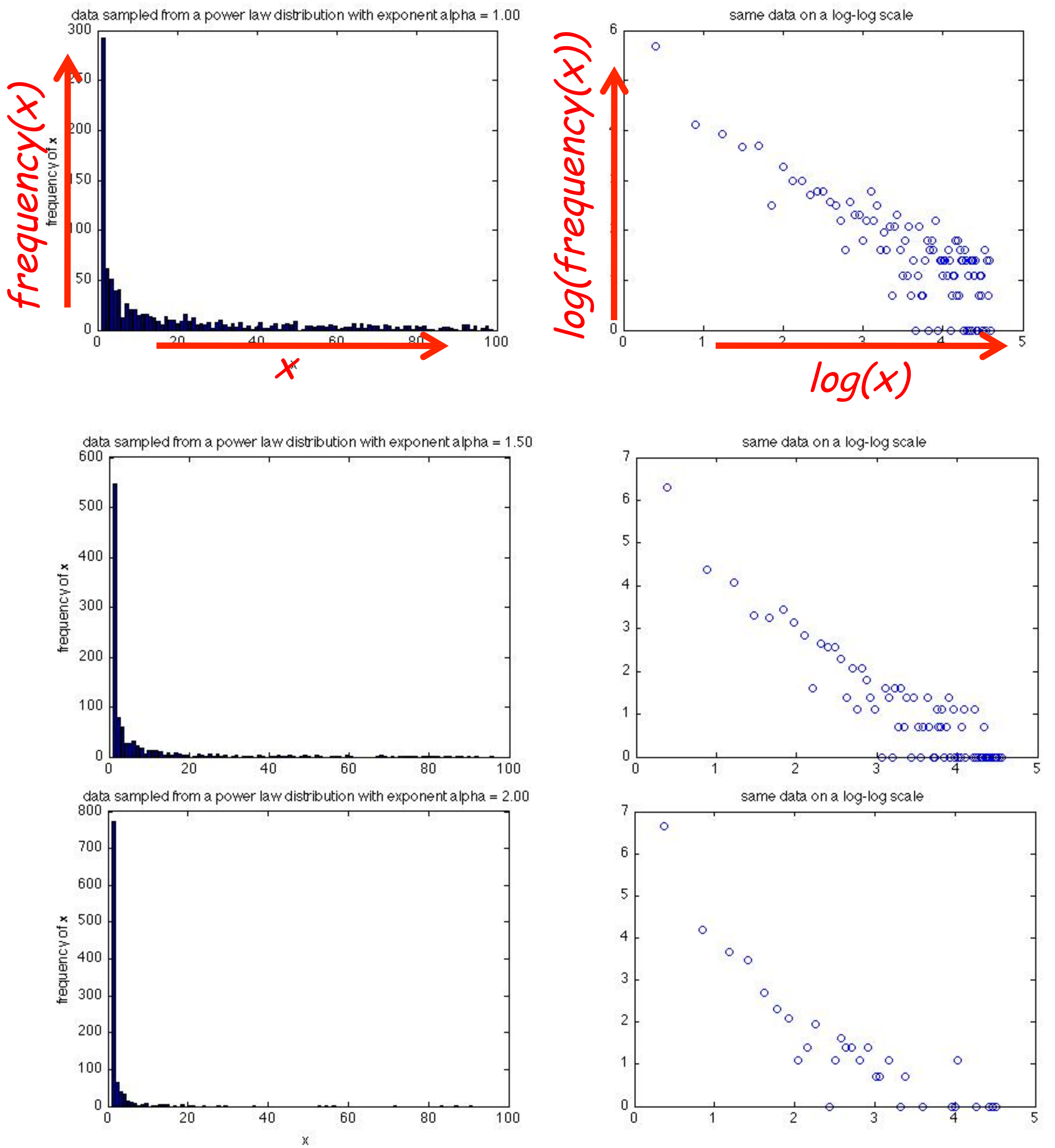
$$probability(x) \propto 1/x^{\beta}$$

  - main point: inverse polynomial decay as x increases
  - if we take the logarithm:

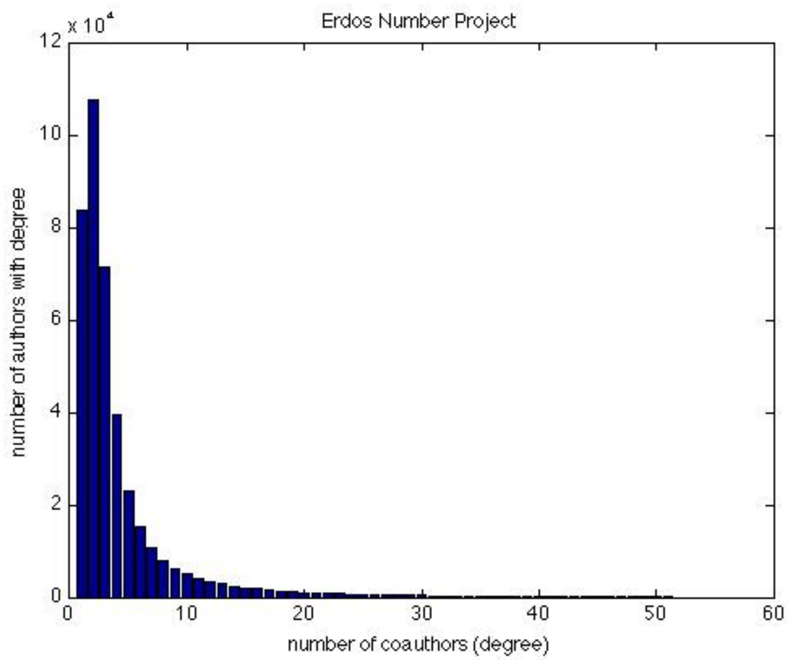
$$\log(probability(x)) \propto -\beta \log(x)$$

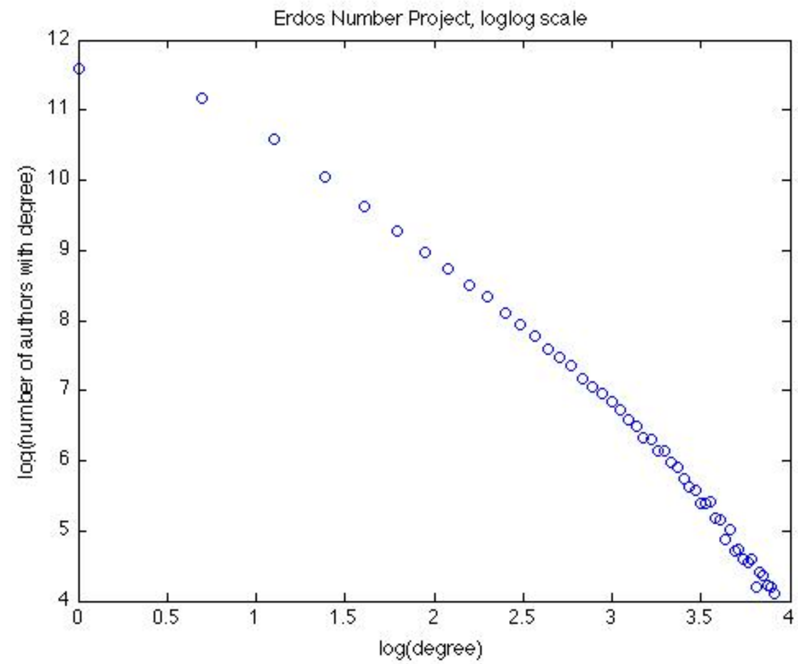
- Claim: if we plot log(x) vs log(probability(x)), will get a straight line!
- Let's look at (artificial) some sample data…



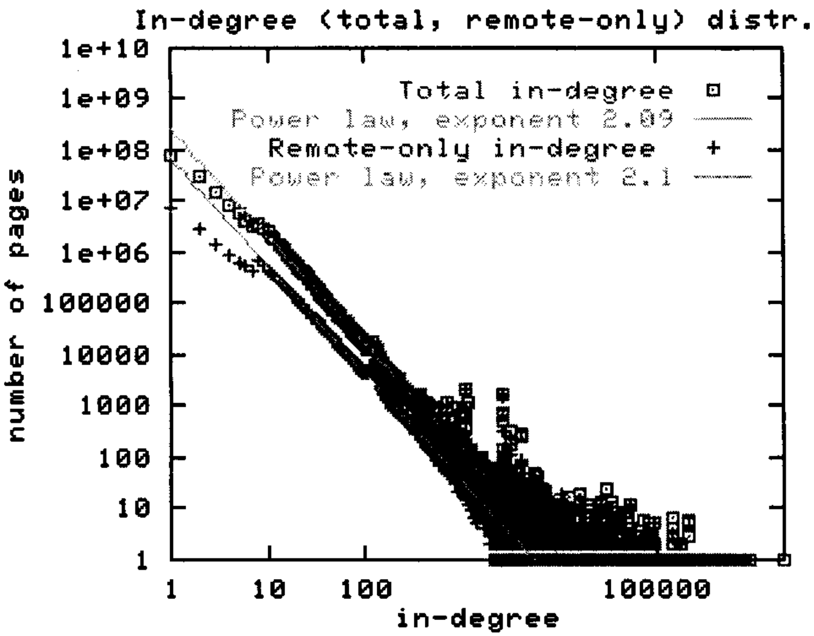


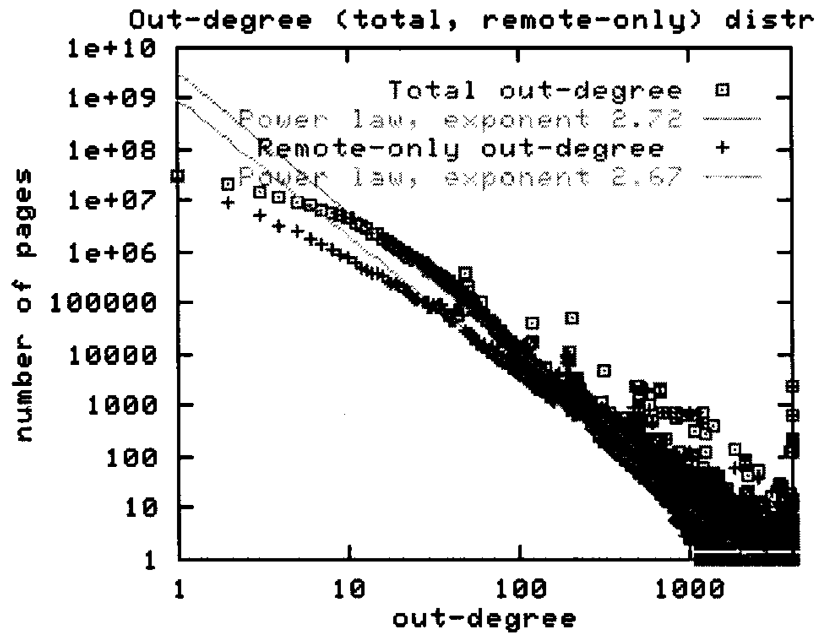
data sampled from a power law distribution with exponent alpha = 1.00
frequency of x
frequency(x)
x
same data on a log-log scale
log(frequency(x))
log(x)
data sampled from a power law distribution with exponent alpha = 1.50
frequency of x
same data on a log-log scale
data sampled from a power law distribution with exponent alpha = 2.00
frequency of x
x
same data on a log-log scale





Erdos Number Project Revisited





Figures 1 and 2: In-degree and out-degree distributions subscribe to the power law. The law also holds if only off-site (or "remote-only") edges are considered.

Degree Distribution of the Web Graph [Broder et al.]

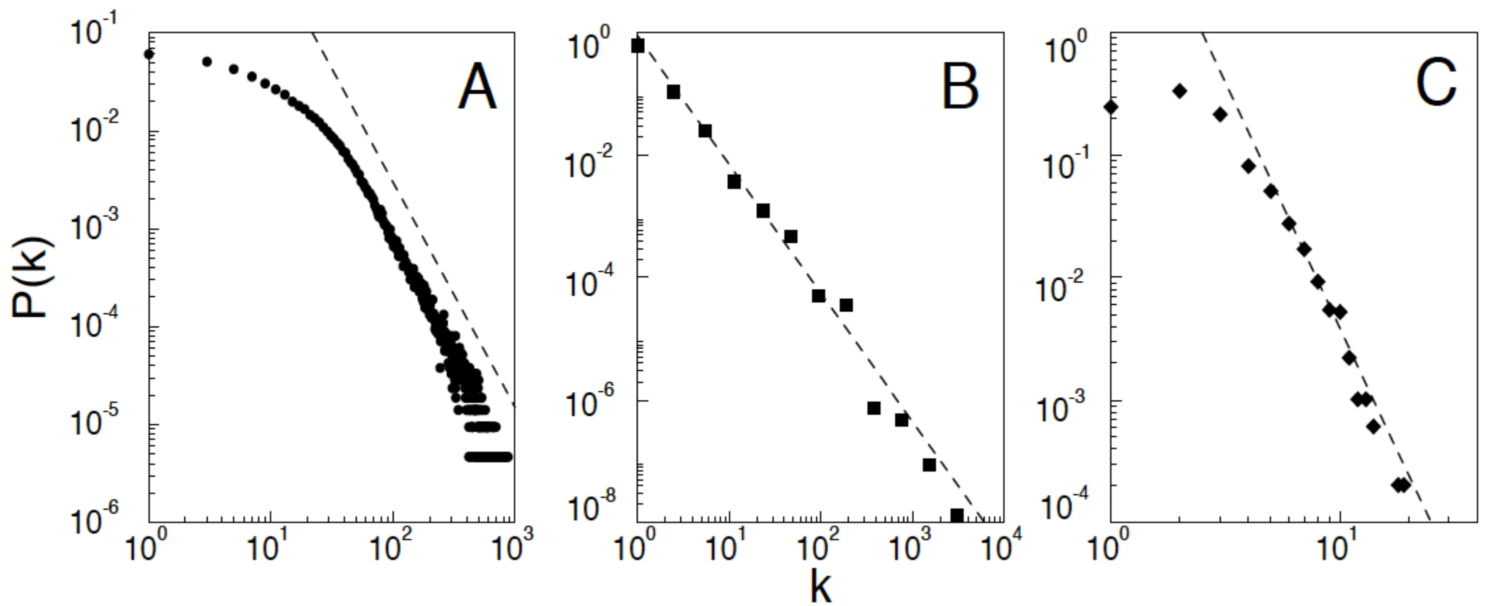


FIG. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$; (**B**) World wide web, $N = 325,729$, $\langle k \rangle = 5.46$ (6); (**C**) Powergrid data, $N = 4,941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (**A**) $\gamma_{actor} = 2.3$, (**B**) $\gamma_{www} = 2.1$ and (**C**) $\gamma_{power} = 4$.

Actor Collaborations; Web; Power Grid [Barabasi and Albert]

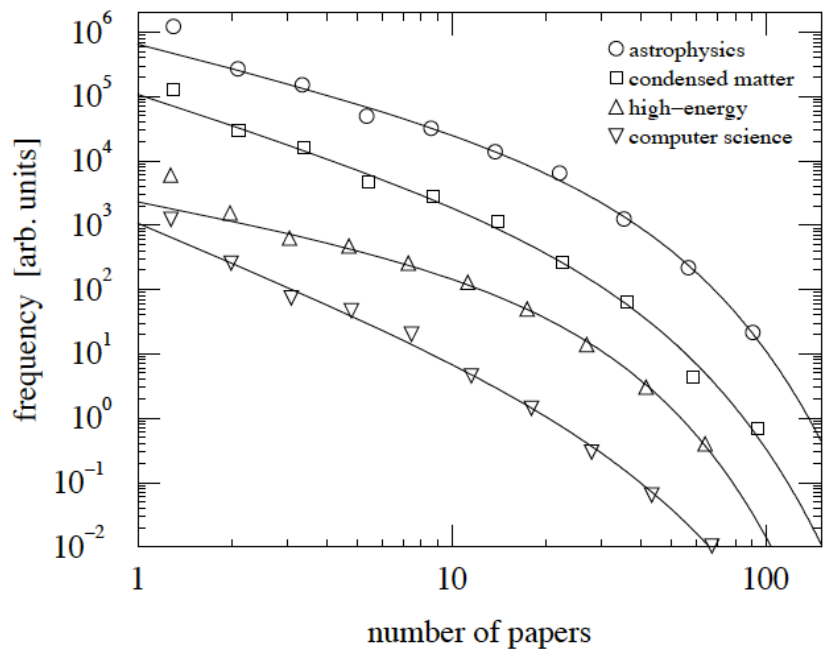
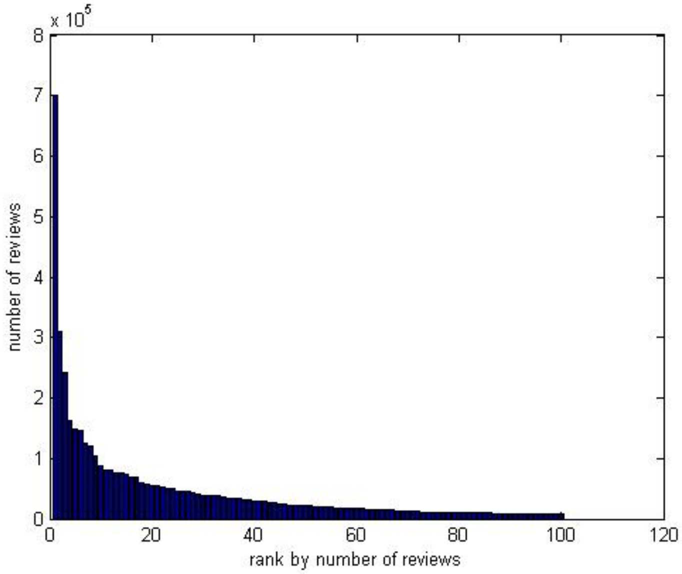


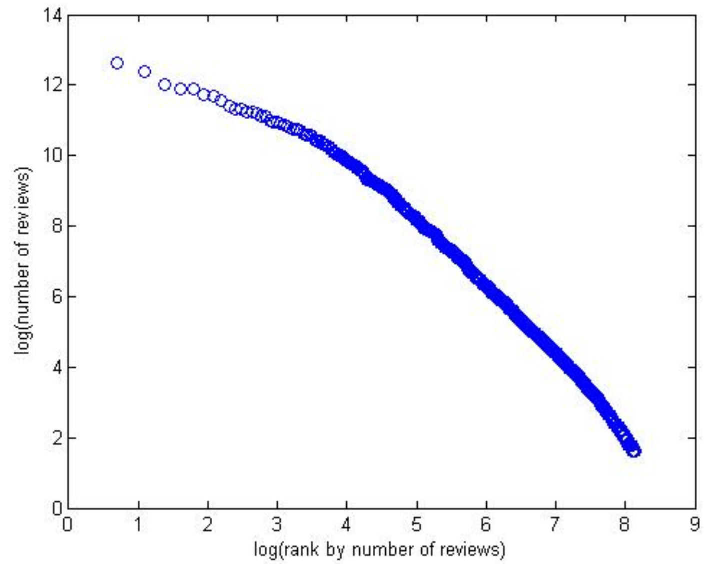
FIG. 2. Histograms of the number of papers written by scientists in four of the databases. As with Fig. 1, the solid lines are least-squares fits to Eq. (1).

Scientific Productivity (Newman)

Zipf's Law

- Look at the frequency of English words:
  - "the" is the most common, followed by "of", "to", etc.
  - claim: frequency of the n-th most common ~ 1/n (power law, a ~ 1)
- General theme:
  - *rank* events by their *frequency of occurrence*
  - resulting distribution often is a power law!
- Other examples:
  - North America city sizes
  - personal income
  - file sizes
  - genus sizes (number of species)
  - the "long tail of search" (on which more later…)
  - let's look at log-log plots of these
- People seem to dither over exact form of these distributions
  - e.g. value of a
  - but not over heavy tails



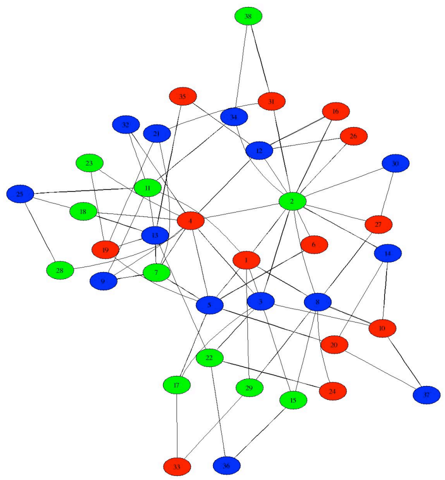


iPhone App Popularity

Summary

- Power law distribution is a good mathematical model for heavy tails; Normal/bell-shaped is not
- Statistical signature of power law and heavy tails: linear on a log-log scale
- Many social and other networks exhibit this signature
- Next "universal": small diameter

How Do "Real" Networks Look?
II. Small Diameter

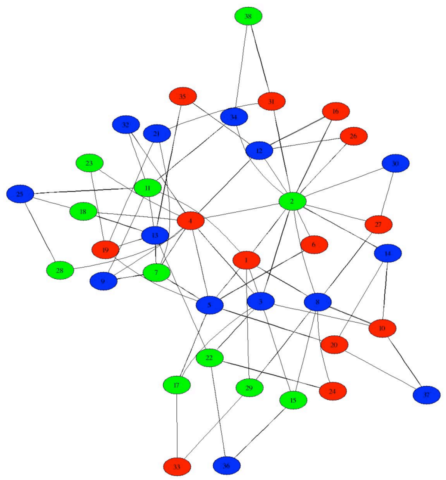


What Do We Mean By “Small Diameter”?

- First let’s recall the definition of diameter:
  - assumes network has a single connected component (or examine “giant” component)
  - for every pair of vertices u and v, compute shortest-path distance d(u,v)
  - then (average-case) diameter of entire network or graph G with N vertices is

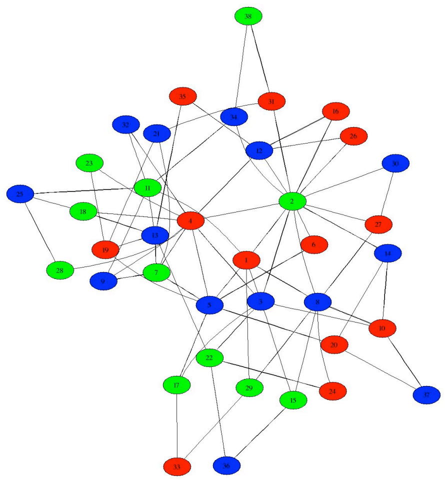
$$diameter(G) = 2/(N(N-1))\sum_{u,v} d(u,v)$$

  - equivalent: pick a random pair of vertices (u,v); what do we expect d(u,v) to be?
- What’s the smallest/largest diameter(G) could be?
  - smallest: 1 (complete network, all N(N-1)/2 edges present); independent of N
  - largest: linear in N (chain or line network)
- “Small” diameter:
  - no precise definition, but certainly << N
  - Travers and Milgram: ~5; any fixed network has fixed diameter
  - may want to allow diameter to grow slowly with N (?)
  - e.g. log(N) or log(log(N))



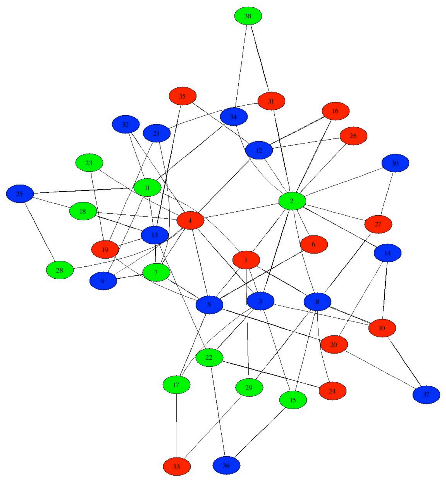
Empirical Support

- Travers and Milgram, 1969:
  - diameter ~ 5-6, N ~ 200M
- Columbia Small Worlds, 2003:
  - diameter ~4-7, N ~ web population?
- Lescovec and Horvitz, 2008:
  - Microsoft Messenger network
  - Diameter ~6.5, N ~ 180M
- Backstrom et al., 2012:
  - Facebook social graph
  - diameter ~5, N ~ 721M

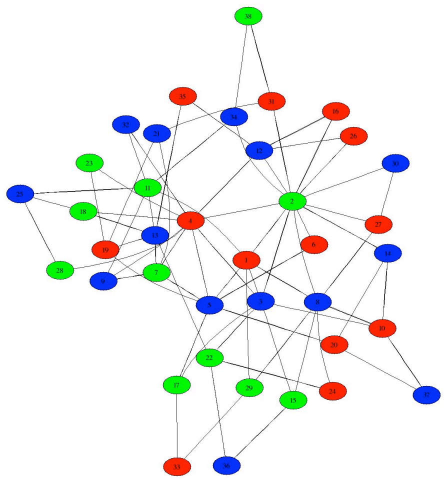


Summary

- So far: naturally occuring, large-scale networks exhibit:
  - heavy-tailed degree distributions
  - small diameter
- Next up: clustering of connectivity



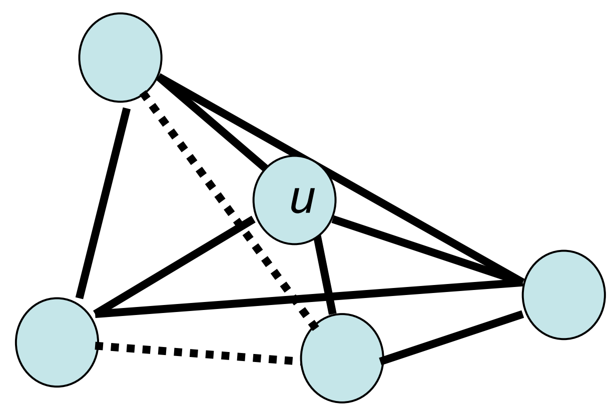
How Do “Real” Networks Look?
III. Clustering of Connectivity



The Clustering Coefficient of a Network

- Intuition: a measure of how "bunched up" edges are
- The clustering coefficient of vertex u:
  - let k = degree of u = number of neighbors of u
  - k(k-1)/2 = *max possible # of edges* between neighbors of u
  - c(u) = (*actual* # of edges between neighbors of u)/[k(k-1)/2]
  - fraction of pairs of friends that are also friends
  - 0 <= c(u) <= 1; measure of *cliquishness* of u's neighborhood
- Clustering coefficient of a graph G:
  - CC(G) = average of c(u) over all vertices u in G

k = 4
k(k-1)/2 = 6
c(u) = 4/6 = 0.666…



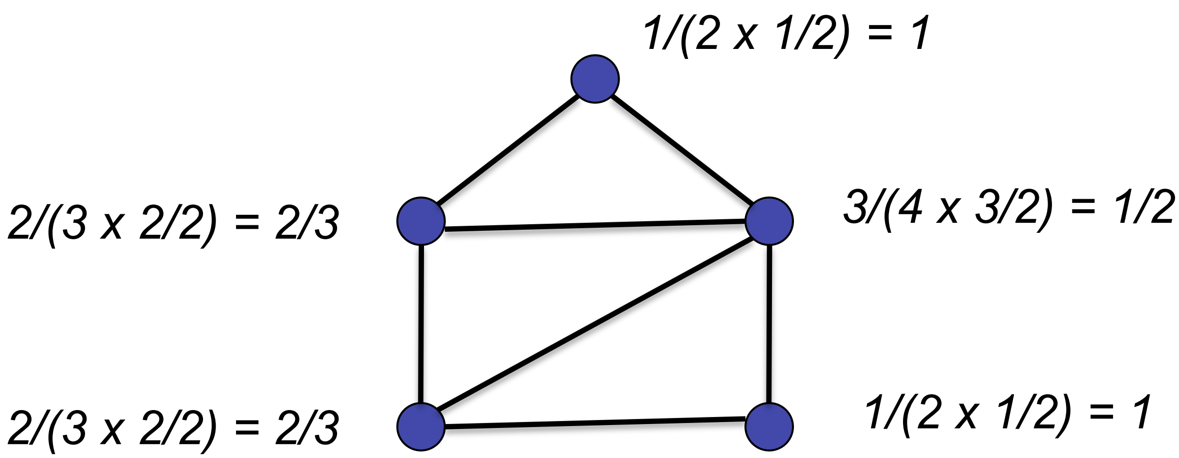
What Do We Mean By "High" Clustering?

- CC(G) measures how likely vertices with a common neighbor are to be neighbors themselves
- Should be compared to how likely *random* pairs of vertices are to be neighbors
- Let p be the edge density of network/graph G:

$$p = E/(N(N-1)/2)$$

- Here E = total number of edges in G
- If we picked a pair of vertices at random in G, probability they are connected is exactly p
- So we will say clustering is high if CC(G) >> p

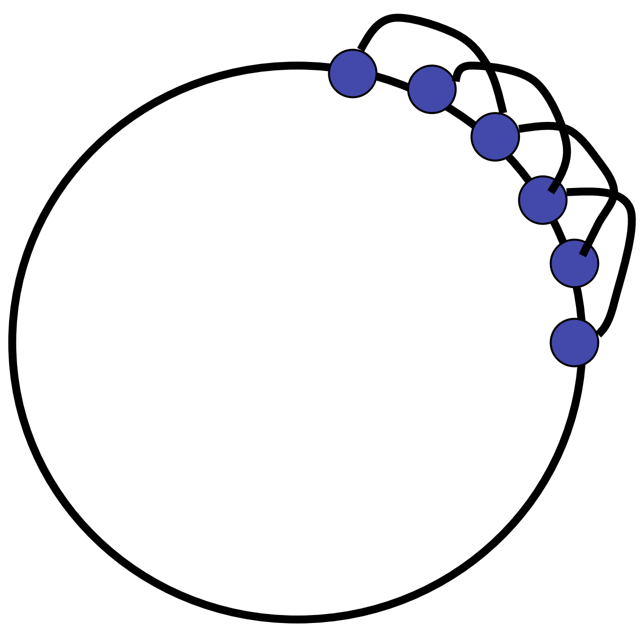
Clustering Coefficient Example 1



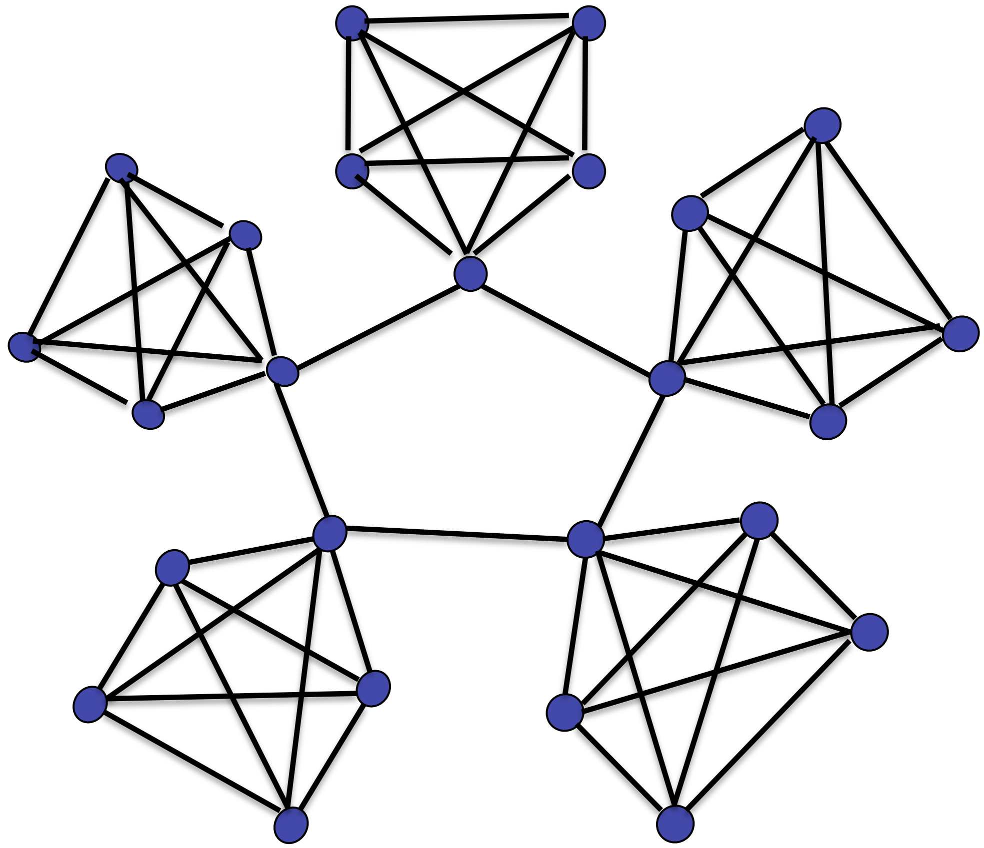
C.C. = (1 + ½ + 1 + 2/3 + 2/3)/5 = 0.7666…
p = 7/(5 x 4/2) = 0.7
Not highly clustered

Clustering Coefficient Example 2

- Network: simple cycle + edges to vertices 2 hops away on cycle
- By symmetry, all vertices have the same clustering coefficient
- Clustering coefficient of a vertex v:
  - Degree of v is 4, so the number of *possible* edges between pairs of neighbors of v is 4 x 3/2 = 6
  - How many pairs of v's neighbors actually *are* connected? 3 --- the two clockwise neighbors, the two counterclockwise, and the immediate cycle neighbors
  - So the c.c. of v is 3/6 = ½
- Compare to overall edge density:
  - Total number of edges = 2N
  - Edge density p = 2N/(N(N-1)/2) ~ 4/N
  - As N becomes large, ½ >> 4/N
  - So this cyclical network is highly clustered



Clustering Coefficient Example 3



Divide N vertices into sqrt(N) groups of size sqrt(N) (here N = 25)
Add all connections within each group (cliques), connect "leaders" in a cycle
N – sqrt(N) non-leaders have C.C. = 1, so network C.C. → 1 as N becomes large
Edge density is p ~ 1/sqrt(N)

TABLE 3.2 STATISTICS OF SMALL WORLD NETWORKS

	L_{ACTUAL}	L_{RANDOM}	C_{ACTUAL}	C_{RANDOM}
MOVIE ACTORS	3.65	2.99	0.79	0.00027
POWER GRID	18.7	12.4	0.080	0.005
C. ELEGANS	2.65	2.25	0.28	0.05

L=Path Length; C=Clustering Coefficient.