Algebraic Geometry

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Preface

This manuscript is based on lectures given by Steve Shatz for the course *Math 624/625–Algebraic Geometry*, during Fall 2001 and Spring 2002. The process for producing this manuscript was the following: I (Jean Gallier) took notes and transcribed them in LaTeX at the end of every week. A week later or so, Steve reviewed these notes and made changes and corrections. After the course was over, Steve wrote up additional material that I transcribed into LaTeX. We also met numerous times from 2002 until 2005 to make corrections and additions.

The following manuscript is thus unfinished and should be considered as work in progress. Nevertherless, given that the *EGA's (Elements de Géométrie Algébrique)* of Grothendieck and Dieudonné are a formidable and rather inpenetrable source, we feel that the material presented in this manuscript will be of some value. Indeed, some material from the *EGA's* is presented here in a more accessible form. We also hope that the exposition of spectral sequences given in this manuscript will be somewhat illuminating. In particular, Steve worked out a presentation of Serre duality which sheds some new light on its connection to spectral sequences.

We apologize for the typos and mistakes that surely occur in the manuscript (as well as unfinished sections and even unfinished proofs!). Still, our hope is that by its "freshness," this work will be of value to algebraic geometry lovers.

Please, report typos, mistakes, *etc.* (to Jean). We intend to improve and perhaps even complete this manuscript.

Philadelphia, February 2011 *Jean Gallier*

Chapter 1

Elementary Algebraic Geometry

1.1 History and Problems

Diophantus (second century A.D.) looked at simultaneous polynomial equations with $\mathbb{Z}$-coefficients, asking for $\mathbb{Z}$-solutions. For example, he looked at equations (1.1)–(1.4) among the following equations:

$$
\begin{aligned}
x^2 + y^2 &= z^2, && (1.1)\\
x^3 + y^3 + z^3 &= 0, && (1.2)\\
ax^3 + by^3 + cz^3 &= 0, && (1.3)\\
ax^3 + bx + c &= y^2, && (1.4)\\
3x^3 + 4y^3 + 5z^3 &= 0, && (1.5)\\
x^n + y^n &= z^n \ (n \geq 4), && (1.6)\\
ax^n + by^n &= cz^n (n \geq 4). && (1.7)
\end{aligned}
$$

Diophantus found all solutions for the first equation, and some answers for some special a, b, c for the third equation. Faltings proved that the last equation has only finitely many solutions in algebraic numbers (in 1983).

Cardano and Tartaglia (fifteenth century) found methods (and formulae) to solve cubic and quartic equations in one variable.

The marriage of algebra and geometry comes with Descartes (sixteenth century).

Gauss solved the linear case completely (linear equations).

After that, there were Riemann, Halphen, Max Noether, Picard and Simart (function theory), Castelnuovo, Enriques, and Severi (beginning the twentieth century), among others.

Given a field k, the standard elementary problem is the following:

Given p polynomials $f_1, \ldots, f_p \in k[x_1, \ldots, x_q]$, consider the system of simultaneous equations

$$\begin{aligned} f_1(x_1, \ldots, x_q) &= 0, \\ \ldots \quad \ldots &\quad \ldots \\ f_p(x_1, \ldots, x_q) &= 0, \end{aligned} \tag{1.8}$$

and say "something" about the solutions.

If the f_j's are linear, by Gauss-Jordan elimination, we get existence, or nonexistence, and an algorithm to solve the system. We also get a geometric description of the set of the solutions: it is the translate of some linear space.

What about existence?

Stated this way, the question is vague. For example, over $\mathbb{Q}$, the equation

$$x^2 + y^2 + 1 = 0$$

has no solutions. It also has no solutions over $\mathbb{R}$, but it has plenty of solutions over $\mathbb{C}$.

The equation

$$3x + 6y = 1$$

has no integer solutions, but it has plenty of rational solutions.

Thus, for this problem, we should ask for solutions in some algebraically closed field (at least the algebraic closure, $\overline{k}$, of k).

We observed that the problem is fully solved when the f_j's are linear. Is it easier to solve the problem when the f_j's are at most quadratic, rather than solving the general problem? The answer is **No**. If we had a method for solving Problem (1.8) in the quadratic case, then we could solve the general problem.

The proof consists in introducing new variables and new equations to lower the degree of terms to at most 2. For instance, consider the cubic terms

$$x^3, \quad x^2y, \quad xyz.$$

In the first case, let u be a new variable, and add the new equation

$$u = x^2.$$

Occurrences of x^3 are then replaced by xu.

In the second case, let u be a new variable, and add the new equation

$$u = xy.$$

Occurrences of x^2y are then replaced by xu.

In the third case, let u be a new variable, and add the new equation

$$u = yz.$$

Occurrences of xyz are also replaced by xu.

Observe that the new equations have degree 2, which is the desired goal. The generalization to terms of higher degree is straightforward.

Going back to existence, note that (1.8) clearly has no solutions if we can find some $g_j \in k[x_1, \ldots, x_q]$ such that

$$g_1 f_1 + \cdots + g_p f_p = 1. \tag{1.9}$$

Indeed, a simultaneous solution of (1.8) would yield $0 = 1$. A famous theorem of Hilbert, the *Nullstellensatz* (1893), tells us that if (1.8) is not "obviously inconsistent" (in the sense that equation 1.9 holds), then it has a solution in the algebraic closure $\overline{k}$ of k.

Given the system (1.8), assume that we have m polynomials $F_1, \ldots, F_m$ and some polynomials g_{ij} and h_{ij} such that

$$F_i = \sum_{j=1}^{p} g_{ij} f_j \quad \text{and} \quad f_i = \sum_{j=1}^{m} h_{ij} F_j,$$

then the system

$$\begin{aligned} F_1(x_1, \ldots, x_q) &= 0, \\ \cdots \quad \cdots & \quad \cdots \\ F_m(x_1, \ldots, x_q) &= 0, \end{aligned} \tag{1.10}$$

has the same set of solutions as the system (1.8). This means that if the ideals $(f_1, \ldots, f_p)$ and $(F_1, \ldots, F_m)$ (in $k[x_1, \ldots, x_q]$) are identical, then (1.8) and (1.10) have the same solutions.

Thus, the solution set of a system of polynomial equations only depends on the ideal generated by the equations. Now recall Hilbert's basis theorem (1890, see Atiyah and Macdonald [2], Theorem 7.5, Chapter 7, or Zariski and Samuel [60], Theorem 1, Chapter IV, Section 1): Every ideal

$$\mathfrak{A} \subseteq k[x_1, \ldots, x_q]$$

is finitely generated by some polynomials $f_1, \ldots, f_p$. Thus, we can talk about the "zeros" of the ideal $\mathfrak{A}$, i.e., the simultaneous solutions of (1.8) for some finite set of generators, $f_1, \ldots, f_p$, of $\mathfrak{A}$.

What do we mean by describing the solutions geometrically?

The above statement is vague. We mean, make some kind of picture of the solutions. Some relevant questions are:

1. Is the picture connected?

2. Is it compact?

3. Are there holes?

4. What are the functions on the space of solutions?

We seek to describe as well as possible some nontrivial invariants of the geometric picture.

Example 1.1 Consider the equation

$$x^2 + y^2 = 1.$$

Over $\mathbb{R}$, a good picture of the solutions is a circle.

Over $\mathbb{C}$, it is a 2-sphere without two points. This can be seen as follows. By stereographic projection from the North pole onto an equatorial plane, the complex plane $\mathbb{C}$ is in bijection with the sphere S^2 with the North pole N removed. The equation

$$x^2 + y^2 = 1$$

can be written as

$$(x + iy)(x - iy) = 1,$$

and by letting $w = x + iy$ and $z = x - iy$, we see that it is equivalent to

$$wz = 1.$$

Clearly, every $w \neq 0$ determines a unique z, and thus, the solution set is indeed $S^2 - \{N, S\}$.

Later on, we will show the following important *fact*: Systems of the form (1.8) **never** have a compact set of solutions in $\mathbb{C}$, unless the solution set is finite.

Example 1.2 Note that $\overline{k}$ is the solution set corresponding to the empty ideal in $k[x]$. Similarly, $\overline{k}^n$ is the solution set corresponding to the empty ideal in $k[x_1, \ldots, x_n]$. We also denote $\overline{k}^n$ by $\mathbb{A}^n(\overline{k})$, and call it *the points of affine n-space over $\overline{k}$.*

Can we view $\mathbb{A}^1(\overline{k}) - \{0\}$ as the solution set of some set of equations? Yes indeed. Let $\mathfrak{A}$ be the ideal, $(zw - 1)$, generated by the polynomial $zw - 1 \in k[z, w]$. The solutions of

$$zw - 1 = 0$$

are in bijection with the set of all $z \in \mathbb{A}^1(\overline{k}) - \{0\}$.

Example 1.3 We will prove later on that $\mathbb{A}^2(\overline{k}) - \{0\}$ is not the solution set of any set of equations. On the other hand, $\mathbb{A}^2(\overline{k}) - \{0\}$ is "locally" an algebraic solution set. It is possible to cover $\mathbb{A}^2(\overline{k}) - \{0\}$ with two "affine patches." Indeed,

(a) Consider $k[x, y, z]$ and the equation

$$xz = 1.$$

(b) Consider $k[x, y, t]$ and the equation

$$yt = 1.$$

The solution set, $\dagger_a$, of $xz = 1$ is in bijection with

$$\{(x, y) \in \mathbb{A}^2(\overline{k}) \mid x \neq 0\}$$

and the solution set, $\dagger_b$, of $yt = 1$ is in bijection with

$$\{(x, y) \in \mathbb{A}^2(\overline{k}) \mid y \neq 0\},$$

and thus,

$$\dagger_a \cup \dagger_b = \mathbb{A}^2(\overline{k}) - \{0\}.$$

This suggests that we define algebraic "things" (i.e., our varieties) as topological spaces that are locally solution sets of equations of the form (1.8).

Generally speaking, to "do geometry," we need

(1) A topological space.

(2) A notion of locally standard objects. For example, in the case of real manifolds, a ball in $\mathbb{R}^n$. In the case of a complex manifold, a ball in $\mathbb{C}^n$. In the case of algebraic varieties, something defined by a system of the form (1.8).

(3) Some set of functions on the space (perhaps locally defined). For example, in the real case, C^k-functions, or smooth functions, or analytic functions. In the complex case, holomorphic functions.

(4) Maps between the objects defined by (1), (2), (3).

Another theme in algebraic geometry is that of a classifying space (or moduli space). Assume that we have some geometric algebraic object X. This object X is at least a topological space.

Question: Given X, with some topological structure, "classify" all the algebraic structures it carries, compatible with the underlying topological structure.

Example 1.4 Consider the ellipic curve of equation

$$y^2 = ax^3 + bx + c, \qquad (a, b, c \in \mathbb{C}).$$

where the righthand side has distinct roots. Geometrically, this is a one-hole torus with one point missing. If we compactify, we obtain a usual torus.

Problem 1. What are the algebraic structures (up to some suitable notion of isomorphism) carried by a torus?

The collection of algebraic structures turns out to be in one-to-one correspondence with the affine line $\mathbb{A}^1_{\mathbb{C}}$, which is *again* an algebraic variety.

Problem 2. Given X (an algebraic variety), classify *all* the subobjects of X.

This problem can only be handled if we fix some discrete invariants. Then, it might be possible to classify the subobjects, and the classifying space might also be an algebraic variety.

Consider the special case where $k = \mathbb{C}$ and $X = \mathbb{A}^n$. We would like to classify all the subvarieties of $\mathbb{A}^n$ of the form (1.8). This is a very difficult problem. Let us consider the easier problem which is to classify the linear (affine) subvarieties of $\mathbb{A}^n$. Using translation, we may assume wlog[1] that they pass through the origin (that is, we have a point in $\mathbb{A}^q$ as one of the pieces of classifying data). The discrete invariant is the dimension d, where $0 \leq d \leq n$. The cases $d = 0, n$ are trivial. Let $G(n, d)$ denote the space of all linear subspaces of dimension d in $\mathbb{A}^n$ through 0.

Observe that there is an isomorphism

$$G(n, d) \cong G(n, n - d)$$

given by duality. We will treat the case $d = 1$, since it is simpler. We need to classify all the lines through the origin 0 in $\mathbb{A}^n$. Let Σ be the unit sphere in $\mathbb{A}^n$, that is,

$$\Sigma = \left\{ z \;\middle|\; \sum |z_i|^2 = 1,\ z = (z_1, \ldots, z_n) \right\} = \left\{ (x, y) \;\middle|\; \sum x_i^2 + y_i^2 = 1 \right\} = S^{2n-1}.$$

The sphere S^{2n-1} is compact in the complex topology. Given any line $L \in G(n, 1)$, consider $L \cap \Sigma$. We can define L parametrically by

$$L = \{(z_j) \mid z_j = \alpha_j t,\ t \in \mathbb{C}, \text{ some } \alpha_j \neq 0\}.$$

Then,

$$L \cap \Sigma = \left\{ t \;\middle|\; \sum |\alpha_j|^2 |t|^2 = 1 \right\} = \left\{ t \;\middle|\; |t|^2 = \frac{1}{\sum |\alpha_j|^2} \right\}.$$

[1] We use the abbreviation "wlog" for "without loss of generality."

Thus, $L \cap \Sigma$ is a circle S^1 of radius $\frac{1}{\sqrt{\sum |\alpha_j|^2}}$. Since a line through 0 is determined by just another point, observe that for any two lines $L, L' \in G(n, 1)$,

$$L = L' \quad \text{iff} \quad (L \cap \Sigma) \cap (L' \cap \Sigma) \neq \emptyset.$$

Thus, the lines through 0 are in bijection with the quotient space S^{2n-1}/S^1. The quotient space is thereby compact. Therefore, $G(n, 1)$ is **not** globally of the form (1.8). However, $G(n, 1)$ is locally of the form (1.8). To see this, it is easier to consider $G(n, n-1)$, which is isomorphic to $G(n, 1)$ (by duality).

Let $H \in G(n, n-1)$ be a hyperplane with equation

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0,$$

where $\alpha_j \neq 0$ for some j. If we define an equivalence relation $\sim$ on $\mathbb{A}^n$ so that

$$(\alpha_1, \ldots, \alpha_n) \sim (\beta_1, \ldots, \beta_n) \quad \text{iff} \quad \beta_j = \lambda\alpha_j \text{ for some } \lambda \neq 0,\ 1 \leq i \leq n,$$

then the map

$$H \mapsto [(\alpha_1, \ldots, \alpha_n)]_\sim,$$

where $[(\alpha_1, \ldots, \alpha_n)]_\sim$ denotes the equivalence class of $(\alpha_1, \ldots, \alpha_n)$, is a bijection between $G(n, n-1)$ and $\mathbb{A}^n/\sim$. Now consider

$$U_j = \{[(\alpha_1, \ldots, \alpha_n)] \mid \alpha_j \neq 0\}.$$

In each equivalence classes, there is a unique representative with $\alpha_j = 1$, and so,

$$U_j \cong \mathbb{A}^{n-1},$$

via

$$(\alpha_1, \ldots, 1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \alpha_n).$$

As a consequence,

$$G(n, n-1) = \bigcup_{j=1}^{n} U_j,$$

where each U_j is isomorphic to A^{n-1}, and thus, $G(n, n-1)$ is locally of the form (1.8).

1.2 Affine Geometry (first level of abstraction), Zariski Topology

In this section, we set up the basic correspondence between ideals in $k[X_1, \ldots, X_q]$ and subsets of $\mathbb{A}^q$. Let k be a fixed field, and let Ω be a field such that $k \subseteq \Omega$ and the following properties hold:

(1) Ω is algebraically closed.

(2) The transcendence degree of Ω over k is $\aleph_0$:

$$\mathrm{tr.d}_k\,\Omega = \aleph_0.$$

For any $q \geq 0$, we consider ideals $\mathfrak{A}$ such that $\mathfrak{A} \subseteq k[X_1, \ldots, X_q]$ or $\mathfrak{A} \subseteq \overline{k}[X_1, \ldots, X_q]$, and write

$$\mathbb{A}^q = \mathbb{A}^q_k = \Omega^q.$$

Definition 1.1 Given any ideal $\mathfrak{A} \subseteq k[X_1, \ldots, X_q]$, define $V_k(\mathfrak{A})$ by

$$V_k(\mathfrak{A}) = \{(\xi) \in \mathbb{A}^q \mid (\forall f \in \mathfrak{A})(f(\xi) = 0)\}.$$

We call $V_k(\mathfrak{A})$ the *set of Ω-points of the affine k-variety determined by* $\mathfrak{A}$. With a slight abuse of language, we call $V_k(\mathfrak{A})$ the *affine k-variety determined by* $\mathfrak{A}$. Similarly, given any ideal $\mathfrak{A} \subseteq \overline{k}[X_1, \ldots, X_q]$, define $V_{\overline{k}}(\mathfrak{A})$ by

$$V_{\overline{k}}(\mathfrak{A}) = \{(\xi) \in \mathbb{A}^q \mid (\forall f \in \mathfrak{A})(f(\xi) = 0)\}.$$

We call $V_{\overline{k}}(\mathfrak{A})$ the *set of Ω-points of the (geometric) affine $\overline{k}$-variety determined by* $\mathfrak{A}$, or for short, the *(geometric) affine variety determined by* $\mathfrak{A}$.

To ease the notation, we usually drop the subscript k or $\overline{k}$ and simply write V for V_k or $V_{\overline{k}}$. Generally, V means V_k, unless specified otherwise.

If A is a (commutative) ring (with unit 1), recall that the *radical,* $\sqrt{\mathfrak{A}}$, of an ideal, $\mathfrak{A} \subseteq A$, is defined by

$$\sqrt{\mathfrak{A}} = \{a \in A \mid \exists n \geq 1,\ a^n \in \mathfrak{A}\}.$$

A *radical ideal* is an ideal, $\mathfrak{A}$, such that $\mathfrak{A} = \sqrt{\mathfrak{A}}$.

The following properties are easily verified. Following our conventions, they are stated for $V = V_k$, but they hold as well for $V_{\overline{k}}$.

$$V((0)) = \mathbb{A}^n, \quad V((1)) = \emptyset \tag{1.11}$$

$$V(\mathfrak{A} \cap \mathfrak{B}) = V(\mathfrak{A}\mathfrak{B}) = V(\mathfrak{A}) \cup V(\mathfrak{B}) \tag{1.12}$$

$$\mathfrak{A} \subseteq \mathfrak{B} \text{ implies that } V(\mathfrak{B}) \subseteq V(\mathfrak{A}) \tag{1.13}$$

$$V(\Sigma_\alpha \mathfrak{A}_\alpha) = \bigcap_\alpha V(\mathfrak{A}_\alpha) \tag{1.14}$$

$$V(\sqrt{\mathfrak{A}}) = V(\mathfrak{A}). \tag{1.15}$$

From (1.11), (1.12), (1.14), it follows that the sets $V(\mathfrak{A}) = V_k(\mathfrak{A})$ can be taken as closed subsets of $\mathbb{A}^q$, and we obtain a topology on $\mathbb{A}^q$. This is the *k-topology on* $\mathbb{A}^q$. If we consider ideals in $\overline{k}[X_1, \ldots, X_q]$ (i.e., sets of the form $V_{\overline{k}}(\mathfrak{A})$), we obtain the *Zariski topology on* $\mathbb{A}^q$.

Remark: Each set of the form $V(\mathfrak{A})$ inherits a topology, and so, each set of the form (1.8) is topologized.

The Zariski topology is not Hausdorff (except when $V(\mathfrak{A})$ consists of a finite set of points).

Let us see that $\mathbb{A}^q$ is not Hausdorff in the Zariski topology. Let $P, Q \in \mathbb{A}^q$, with $P \neq Q$. The line PQ is isomorphic to $\mathbb{A}^1$. Thus, it is enough to show that $\mathbb{A}^1$ is not Hausdorff. Consider any ideal $\mathfrak{A} \subseteq \overline{k}[X]$. Then, $\mathfrak{A}$ is a principal ideal, and thus

$$\mathfrak{A} = (f)$$

for some polynomial f, which shows that $V(\mathfrak{A}) = V((f))$ is a finite set. As a consequence, the closed sets of $\mathbb{A}^1$ (other than $\mathbb{A}^1$) are finite. Then, the union of two closed sets (distinct from $\mathbb{A}^1$) is also finite, and thus distinct from $\mathbb{A}^1$.

The topology on $\mathbb{A}^q$ is not the product topology on $\mathbb{A}^1 \prod \cdots \prod \mathbb{A}^1$.

For example, when $n = 2$, the closed sets in $\mathbb{A}^1 \prod \mathbb{A}^1$ are those sets consisting of finitely many horizontal and vertical lines, and intersections of such sets. However

$$X^2 + Y^2 - 1 = 0$$

defines a closed set in $\mathbb{A}^2$ not of the previous form.

To go backwards from subsets of $\mathbb{A}^q$ to ideals, we make the following definition.

Definition 1.2 Given any subset $S \subseteq \mathbb{A}^q$, define $\mathfrak{I}_k(S)$ and $\mathfrak{I}_{\overline{k}}(S)$ by

$$\mathfrak{I}_k(S) = \{f \in k[X_1, \ldots, X_q] \mid (\forall s \in S)(f(s) = 0)\}$$

and

$$\mathfrak{I}_{\overline{k}}(S) = \{f \in \overline{k}[X_1, \ldots, X_q] \mid (\forall s \in S)(f(s) = 0)\}.$$

The following properties are easily shown (following our conventions, they are stated for $\mathfrak{I}$, i.e., $\mathfrak{I}_k$, but they also hold for $\mathfrak{I}_{\overline{k}}$).

$$S \subseteq T \quad \text{implies that} \quad \mathfrak{I}(T) \subseteq \mathfrak{I}(S) \tag{1.16}$$

$$\mathfrak{A} \subseteq \mathfrak{I}(V(\mathfrak{A})) \tag{1.17}$$

What about $V(\mathfrak{I}(V(\mathfrak{A})))$? By (1.17), we have

$$V(\mathfrak{I}(V(\mathfrak{A}))) \subseteq V(\mathfrak{A}).$$

If $\xi \in V(\mathfrak{A})$ and $f \in \mathfrak{I}(V(\mathfrak{A}))$, so that $f(\xi) = 0$, then, $\xi \in V(\mathfrak{I}(V(\mathfrak{A})))$. Thus,

$$V(\mathfrak{I}(V(\mathfrak{A}))) = V(\mathfrak{A}). \tag{1.18}$$

Given a set $S \subseteq \mathbb{A}^q$, we get the closed set $V(\mathfrak{I}(S))$.

Claim. The Zariski closure (resp. k-closure) of S in $\mathbb{A}^q$ is simply $V(\mathfrak{I}(S))$ (resp. $V(\mathfrak{I}_{\overline{k}}(S))$).
Proof. It is clear that
$$S \subseteq V(\mathfrak{I}(S)).$$
Pick any Zariski-closed set $V(\mathfrak{A})$ such that $S \subseteq V(\mathfrak{A})$. Then, we have
$$\mathfrak{I}(V(\mathfrak{A})) \subseteq \mathfrak{I}(S),$$
and so,
$$V(\mathfrak{I}(S)) \subseteq V(\mathfrak{I}(V(\mathfrak{A}))) = V(\mathfrak{A}),$$
proving that $V(\mathfrak{I}(S))$ is indeed the smallest closed Zariski set containing S. $\square$

In summary
$$S \subseteq V(\mathfrak{I}(S)) \quad \text{and} \quad V(\mathfrak{I}(S)) \text{ is the Zariski closure (resp. } k\text{-closure) of } S. \tag{1.19}$$
In analogy with (1.18), we also have
$$\mathfrak{I}(V(\mathfrak{I}(S))) = \mathfrak{I}(S). \tag{1.20}$$

Now, $V(\mathfrak{A}) = V(\mathfrak{B})$ implies that
$$\mathfrak{I}(V(\mathfrak{A})) = \mathfrak{I}(V(\mathfrak{B})).$$
Conversely, given $V(\mathfrak{A})$ and $V(\mathfrak{B})$, if $\mathfrak{I}(V(\mathfrak{A})) = \mathfrak{I}(V(\mathfrak{B}))$, then
$$V(\mathfrak{A}) = V(\mathfrak{I}(V(\mathfrak{A}))) = V(\mathfrak{I}(V(\mathfrak{B}))) = V(\mathfrak{B}).$$
Thus
$$\mathfrak{I}(V(\mathfrak{A})) = \mathfrak{I}(V(\mathfrak{B})) \quad \text{iff} \quad V(\mathfrak{A}) = V(\mathfrak{B}). \tag{1.21}$$

In other words, closed sets are determined by the ideals associated with them. Since ideals in $k[X_1, \ldots, X_q]$ have the ascending chain condition (ACC), we find that affine varieties in $\mathbb{A}^q$ have the descending chain condition (DCC). From this, it follows that affine varieties satisfy the Heine-Borel property (every open cover has a finite subcover). Hence, they are *quasi-compact* (reserving the term *compact* for quasi-compact *and* Hausdorff). Unfortunately, quasi-compactness is a very weak property.

A notion aligned to connectedness for non-Hausdorff topologies is the notion of irreducible set. Here is the definition for our topologies.

Definition 1.3 An affine variety $V \subseteq \mathbb{A}^q$ is *k-irreducible (resp. geometrically irreducible)* if V is not the union of two properly contained k-closed (resp. Zariski-closed) subsets of V.

A variety V may be k-irreducible but not geometrically irreducible. Consider the case where $k = \mathbb{Q}$ and $\Omega \subseteq \mathbb{C}$, let $f(X, Y) = X^2 + Y^2$, and let

$$V = \{\xi \in \Omega^2 \mid \xi_1^2 + \xi_2^2 = 0\}.$$

Clearly, V is $\mathbb{Q}$-irreducible. However, if we adjoin i to $\mathbb{Q}$, then

$$X^2 + Y^2 = (X + iY)(X - iY).$$

Let

$$W_1 = \{\xi \mid \xi_1 + i\xi_2 = 0\}, \quad W_2 = \{\xi \mid \xi_1 - i\xi_2 = 0\}.$$

Then, $V = W_1 \cup W_2$, but the W_j's are $\overline{\mathbb{Q}}$-closed, but not $\mathbb{Q}$-closed.

Proposition 1.1 *An affine variety $V \subseteq \mathbb{A}^q$ is k-irreducible iff $\mathfrak{I}_k(V)$ is a prime ideal in $k[X_1, \ldots, X_q]$.*

Proof. First, assume that $\mathfrak{I}_k(V) = \mathfrak{P}$ is not prime and that V is irreducible. If so, there are some polynomials $f, g \in k[X_1, \ldots, X_q]$ such that $f, g \notin \mathfrak{P}$ and $fg \in \mathfrak{P}$. Consider the ideals $\mathfrak{A} = (\mathfrak{P}, f)$ and $\mathfrak{B} = (\mathfrak{P}, g)$, and let $V_1 = V(\mathfrak{A})$, $V_2 = V(\mathfrak{B})$. Since

$$\mathfrak{P} \subseteq \mathfrak{A} = (\mathfrak{P}, f) \quad \text{and} \quad \mathfrak{P} \subseteq \mathfrak{B} = (\mathfrak{P}, g),$$

we have

$$V_i \subseteq V$$

for $i = 1, 2$. Furthermore, $V_1 \neq V$, because f vanishes on V_1, since $V_1 = V(\mathfrak{A})$ and $\mathfrak{A} = (\mathfrak{P}, f)$, but f does not vanish on V, since $f \notin \mathfrak{P}$ and $\mathfrak{P} = \mathfrak{I}_k(V)$. Similarly, $V_2 \neq V$. However, we claim that

$$V = V_1 \cup V_2.$$

Indeed, observe that

$$\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{P},$$

since every element of $\mathfrak{A}\mathfrak{B}$ is of the form $\sum f_i g_i$ where $f_i \in \mathfrak{A}$ and $g_i \in \mathfrak{B}$, and thus,

$$V = V(\mathfrak{P}) \subseteq V(\mathfrak{A}\mathfrak{B}) = V(\mathfrak{A}) \cup V(\mathfrak{B}) \subseteq V.$$

But then, V is reducible, a contradiction.

Conversely, assume that V is reducible and that $\mathfrak{I}(V)$ is prime. Then,

$$V = V_1 \cup V_2$$

where $V_i \neq V$ for $i = 1, 2$. So,

$$\mathfrak{I}(V_1) \supset \mathfrak{I}(V) \quad \text{and} \quad \mathfrak{I}(V_2) \supset \mathfrak{I}(V).$$

Let $f \in \mathfrak{I}(V_1) - \mathfrak{I}(V)$ and $g \in \mathfrak{I}(V_2) - \mathfrak{I}(V)$. Then fg vanishes on $V = V(\mathfrak{I}(V))$, since $f \in \mathfrak{I}(V_1)$ implies that f vanishes on V_1, $g \in \mathfrak{I}(V_2)$ implies that g vanishes on V_2, and $V = V_1 \cup V_2$. Therefore, $fg \in \mathfrak{I}(V)$, which contradicts the fact that $\mathfrak{I}(V)$ is prime. $\square$

Corresponding to the decomposition into connected components, we obtain the decomposition of an affine variety into k-irreducible varieties. First, define the notion of an *irredundant decomposition*.

Definition 1.4 Given an affine variety $V \subseteq \mathbb{A}^q$, a *decomposition of* V is a finite family of k-irreducible varieties $V_1, \ldots, V_t$ such that

$$V = V_1 \cup \cdots \cup V_t.$$

Such a decomposition is *irredundant* if for no distinct i, j do we have $V_i \subseteq V_j$.

Theorem 1.2 *Every affine variety $V \subseteq \mathbb{A}^q$ has a decomposition*

$$V = V_1 \cup \cdots \cup V_t$$

into k-irreducible varieties. The decomposition of V is unique provided it is irredundant.

Proof. Let $\mathcal{S}$ be the set of all varieties $V \subseteq \mathbb{A}^q$ such that V is not the finite union of some k-irreducible varieties. We want to show that $\mathcal{S} = \emptyset$. If not, by the (DCC), $\mathcal{S}$ has a minimal element V_0. By definition of $\mathcal{S}$, V_0 is not irreducible. Thus,

$$V_0 = V_1 \cup V_2,$$

where $V_j \neq V_0$, for $j = 1, 2$. Since V_0 is minimal, $V_1, V_2 \notin \mathcal{S}$, which implies that both V_1 and V_2 can be expressed as finite unions of k-irreducible varieties, and thus, V_0 can also be expressed as a finite union of k-irreducible varieties, a contradiction.

Let us now assume that

$$V = V_1 \cup \cdots \cup V_t = W_1 \cup \cdots \cup W_p$$

are two irredundant decompositions of V. Then,

$$W_i = V \cap W_i = \bigcup_{j=1}^{t} V_j \cap W_i.$$

Since W_i is irreducible, there is some $j = j(i)$ such that

$$W_i = V_j \cap W_i$$

which implies that

$$W_i \subseteq V_{j(i)}.$$

If we repeat the argument starting with $V_j \neq W_i$, we have some $k = k(j)$ such that

$$V_j \subseteq W_{k(j)}.$$

But then,

$$W_i \subseteq V_j \subseteq W_k,$$

contradicting the fact that the decompositions are irredundant. Thus, we must have $k = i$, and the V_j's are in bijection with the W_j's. $\square$

The structure of irreducible k-varieties can be better understood using the concept of a k-generic point. First, recall that a *k-specialization of* Ω is a ring homomorphism $\varphi\colon R \to \Omega$ (the identity on k) defined on a subring R of Ω which contains k ($k \subseteq R \subseteq \Omega$).

Definition 1.5 For an affine k-variety $V \subseteq A^q$, a *k-generic point for V* is a point $\xi \in \mathbb{A}^q$ such that:

(1) $\xi \in V$.

(2) If $f \in k[X_1, \ldots, X_q]$ and $f(\xi) = 0$, then $f \in \mathfrak{I}_k(V)$, i.e., the restriction of f to V is identically zero.

Proposition 1.3 *If $V \subseteq \mathbb{A}^q$ is an affine variety, then the variety, V, is k-irreducible iff V has a k-generic point.*

Proof. Assume that V is k-irreducible. Then, $\mathfrak{I}(V)$ is a prime ideal and $k[X_1, \ldots, X_q]/\mathfrak{I}(V)$ is a finitely generated k-algebra which is an integral domain. Let $x_1, \ldots, x_q$ be the images of $X_1, \ldots, X_q$ under the residue map. Then $k[X_1, \ldots, X_q]/\mathfrak{I}(V)$ is isomorphic to $k[x_1, \ldots, x_q]$. However, the transcendence degree, r, of the fraction field, $k(x_1, \ldots, x_q)$, is finite and $r \leq q$. We may reorder the x_i's so that $x_1, \ldots, x_r$ are algebraically independent and $k[x_1, \ldots, x_q]$ is algebraic over $k[x_1, \ldots, x_r]$. Since Ω has transcendence degree $\aleph_0$, there exist $\xi_1, \ldots, \xi_r$ in Ω which are algebraically independent. Thus, there is a k-isomorphism

$$k[x_1, \ldots, x_r] \cong k[\xi_1, \ldots, \xi_r].$$

Since $k[x_1, \ldots, x_q]$ is algebraic over $k[x_1, \ldots, x_r]$ and $k[x_1, \ldots, x_r]$ is embedded in Ω, the fact that Ω is algebraically closed implies that there is an extension

$$\theta\colon k[x_1, \ldots, x_q] \to \Omega$$

which is a k-monomorphism. Let

$$\xi_{r+j} = \theta(x_{r+j}),$$

where $1 \leq j \leq q - r$, and let

$$\xi = (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_q) \in \Omega^q.$$

Clearly, the map
$$\theta\colon k[X_1,\ldots,X_q] \longrightarrow k[x_1,\ldots,x_q] \hookrightarrow \Omega,$$
which is given by $X_j \mapsto \xi_j$, has kernel $\mathfrak{I}(V)$. We have
$$\theta(f(X_1,\ldots,X_q)) = f(\xi_1,\ldots,\xi_q)$$
for every $f \in \mathfrak{I}(V)$ and
$$\theta(f) = 0 \quad \text{iff} \quad f(\xi) = 0.$$
Therefore, ξ is k-generic for V.

Conversely, assume that ξ is k-generic for V. For any $\xi \in \mathbb{A}^q$, let
$$\mathfrak{P}(\xi) = \{f \in k[X_1,\ldots,X_q] \mid f(\xi) = 0\}.$$
Clearly, $\mathfrak{P}(\xi)$ is a prime ideal. If ξ is a k-generic point, then (1) & (2) say that
$$\mathfrak{I}_k(V) = \mathfrak{P}(\xi),$$
and thus, $\mathfrak{I}_k(V)$ is prime and V is k-irreducible. $\square$

Example 1.5 Let $f(X,Y) = X^2 + Y^2 - 1 \in \mathbb{Q}[X,Y]$. Then, $V(f)$ is geometrically irreducible. Pick $\pi \in \mathbb{C}$ (the circumference of the unit circle), and consider the map
$$X \mapsto \pi.$$
Then, we get
$$\mathbb{Q}[X,Y] = \mathbb{Q}[X][T]/(T^2 = 1 - \pi^2).$$
We can find a root, θ, of $1-\pi^2$ in $\mathbb{C}$. Then, $\xi = (\pi,\theta)$ is $\overline{\mathbb{Q}}$-generic.

Remark: Given $\xi, \eta \in \mathbb{A}^q$, we say that η *is a k-specialization of* ξ if there is a k-specialization $\varphi\colon R \to \Omega$, of Ω, denoted by
$$\xi \underset{k}{\longrightarrow} \eta,$$
so that

1. $k[\xi_1,\ldots,\xi_q] \subseteq R$.
2. $\varphi(\xi_i) = \eta_i$, $1 \leq i \leq q$.

Then, the following properties are equivalent (DX).[2]

(1) η is a k-specialization of ξ.

[2] Here and in what follows, the symbol (DX) denotes an unsupported statement whose proof should be easily suppliable by the reader. If the reader cannot supply the proof, then he or she, should turn back a few pages and reread the material.

(2) For every $f \in k[X_1, \ldots, X_q]$, if $f(\xi) = 0$ then $f(\eta) = 0$.

(3) $\mathfrak{P}(\xi) \subseteq \mathfrak{P}(\eta)$.

Remark: Given $\xi, \eta \in V$, one should think of a specialization

$$\xi \underset{k}{\longrightarrow} \eta$$

as "representing a sequence in V whose limit is η." In this representation, ξ stands for the whole sequence.

Proposition 1.4 *If $V \subseteq \mathbb{A}^q$ is a k-irreducible affine variety and ξ is a k-generic point of V, then V consists exactly of the k-specializations of ξ. Conversely, if ξ is any point of $\mathbb{A}^q$, then the collection of all its k-specializations is a k-irreducible variety in $\mathbb{A}^q$. Indeed, this variety is exactly $V(\mathfrak{P}(\xi))$ and ξ is a k-generic point of it. We have*

$$k\textit{-closure}\{\xi\} = V(\mathfrak{P}(\xi)).$$

Proof. Say ξ is a k-generic point for V, where V is a k-irreducible variety. Then,

$$\mathfrak{I}_k(V) = \mathfrak{P}(\xi).$$

Let η be a k-specialization of ξ, then, we have the isomorphisms

$$k[X_1, \ldots, X_q]/\mathfrak{I}_k(V) \cong k[X_1, \ldots, X_q]/\mathfrak{P}(\xi) \cong k[\xi_1, \ldots, \xi_q],$$

and the surjection

$$\varphi\colon k[\xi_1, \ldots, \xi_q] \longrightarrow k[\eta_1, \ldots, \eta_q]$$

given by the specialization φ. We also have the isomorphism

$$k[\eta_1, \ldots, \eta_q] \cong k[X_1, \ldots, X_q]/\mathfrak{P}(\eta),$$

and thus, if $f \in \mathfrak{I}(V)$, then $f \in \mathfrak{P}(\xi) \subseteq \mathfrak{P}(\eta)$, which implies $f(\eta) = 0$, so

$$\eta \in V(\mathfrak{I}_k(V)) = V.$$

Pick $(z_1, \ldots, z_q) \in V$. If $f \in \mathfrak{P}(\xi)$, then $f \in \mathfrak{I}(V)$, which implies $f(z) = 0$. Consequently, there is a well-defined map

$$\xi_j \mapsto z_j,$$

which means that (z) is a k-specialization of ξ. Since

$$\mathfrak{I}(V(\mathfrak{P}(\xi))) = \{g \mid g(\xi) = 0\} = \mathfrak{P}(\xi)$$

and $\mathfrak{P}(\xi)$ is prime, we see that $V(\mathfrak{P}(\xi))$ is k-irreducible and ξ is k-generic. The rest is trivial. $\square$

Remark: Observe that ξ is a k-closed (resp. Zariski-closed) point iff the coordinates of ξ are in k, (resp. the coordinates of ξ are in $\overline{k}$).

Note that the closure of a point is not necessarily a point. The Zariski topology is not T_1.

Recall that in discussing the question of existence of solutions to equation (1.8), we asserted that the criterion of "non-obvious inconsistency" was a criterion for existence (Hilbert's Nullstellensatz). The following theorem is a weak version of this theorem and is called the weak Nullstellensatz.

Remember that an ideal $\mathfrak{A}$ in a ring A is a *primary ideal* if $\mathfrak{A} \neq A$ and whenever $xy \in \mathfrak{A}$, either $x \in \mathfrak{A}$ or $y^m \in \mathfrak{A}$ for some $m \geq 1$ (see Atiyah and Macdonald [2], Chapter 4, or Zariski and Samuel [60], Chapter III, Section 9). This is equivalent to saying that $A/\mathfrak{A}$ is not the trivial ring $\{0\}$ and that every zero-divisor in $A/\mathfrak{A}$ is nilpotent. Of course, if $\mathfrak{A}$ is a primary ideal, then its radical $\sqrt{\mathfrak{A}}$ is the smallest prime ideal containing $\mathfrak{A}$.

Theorem 1.5 *(Weak Nullstellensatz) Let $\mathfrak{P}$ be a prime ideal in $k[X_1, \ldots, X_q]$. Then, $V(\mathfrak{P})$ is k-irreducible and $\mathfrak{I}(V(\mathfrak{P})) = \mathfrak{P}$. If $\mathfrak{A}$ is any ideal in $k[X_1, \ldots, X_q]$, then*

$$\mathfrak{I}(V(\mathfrak{A})) = \sqrt{\mathfrak{A}}.$$

Proof. If $\mathfrak{P}$ is prime then

$$k[X_1, \ldots, X_q]/\mathfrak{P} = k[x_1, \ldots, x_q]$$

is a finitely generated algebra which is an integral domain. By the usual argument, there are some $\xi_1, \ldots, \xi_q \in \Omega$ such that

$$k[x_1, \ldots, x_q] \cong k[\xi_1, \ldots, \xi_q],$$

i.e., $\mathfrak{P} = \mathfrak{P}(\xi)$. By Proposition 1.4,

$$V = V(\mathfrak{P}) = V(\xi)$$

is a k-irreducible variety and

$$\mathfrak{I}(V(\mathfrak{P}(\xi))) = \mathfrak{P}(\xi).$$

Let $\mathfrak{A}$ be any ideal of $k[X_1, \ldots, X_q]$. By the Lasker-Noether intersection theorem (see Zariski and Samuel [60], Theorem 4, Chapter IV, Section 4, or Atiyah and Macdonald [2], Theorem 7.13, Chapter 7),

$$\mathfrak{A} = \bigcap_{i=1}^{n} \mathfrak{Q}_i,$$

where the $\mathfrak{Q}_i$ are primary ideals, and thus, the ideals $\mathfrak{P}_i = \sqrt{\mathfrak{Q}_i}$ are prime ideals. The ideals $\mathfrak{P}_i$ are called *associated primes* of $\mathfrak{A}$ (they are uniquely determined by $\mathfrak{A}$, see Zariski and Samuel [60], Theorem 6, Chapter IV, Section 5). We have

$$V(\mathfrak{A}) = \bigcup_{i=1}^{n} V(\mathfrak{Q}_i) = \bigcup_{i=1}^{n} V(\mathfrak{P}_i).$$

We can eliminate the terms $V(\mathfrak{P}_i)$ which are already contained in some other variety $V(\mathfrak{P}_j)$ and keep only the minimal primes (called *isolated primes*, see Zariski and Samuel [60], Chapter IV, Section 5); so, we have

$$V(\mathfrak{A}) = \bigcup_i V(\mathfrak{P}_i),$$

where the $V(\mathfrak{P}_i)$'s are isolated primes. Then, we get

$$\mathfrak{I}(V(\mathfrak{A})) = \bigcap_i \{\mathfrak{I}(V(\mathfrak{P}_i)) \mid \mathfrak{P}_i \text{ is an isolated prime}\} = \bigcap_i \{\mathfrak{P}_i \mid \mathfrak{P}_i \text{ is an isolated prime}\}.$$

However, by commutative ring theory (see Zariski and Samuel [60], Theorem 10, Chapter IV, Section 6),

$$\sqrt{\mathfrak{A}} = \bigcap_i \{\mathfrak{P}_i \mid \mathfrak{P}_i \text{ is an isolated prime}\}. \qquad \square$$

From now on, we will omit $\mathfrak{A}$ in $V(\mathfrak{A})$, and just write V (if reference to $\mathfrak{A}$ is clear).

Definition 1.6 If $V \subseteq \mathbb{A}^q$ is an affine k-variety, we define the k-algebra $k[V]$ by

$$k[V] = k[X_1, \ldots, X_q]/\mathfrak{I}_k(V),$$

and call $k[V]$ the *affine coordinate ring of* V.

The k-algebras $k[V]$ satisfy the following properties.

(1) Each $k[V]$ is a finitely generated k-algebra.

(2) Each $k[V]$ possesses no nonzero nilpotent elements.

(3) Given A, a finitely generated k-algebra which is *reduced* (i.e., it has no nonzero nilpotent elements), there exists an affine k-variety $V \subseteq \mathbb{A}^q$ so that $A \cong k[V]$.

(4) Given A, a reduced, finitely generated algebra over k, A is an integral domain iff any variety V such that $A \cong k[V]$ is k-irreducible.

These properties are all easy to prove, and we only prove (4).

Proof. By definition, since A is a finitely generated k-algebra, we have

$$A \cong k[X_1, \ldots, X_q]/\mathfrak{A}$$

for some ideal $\mathfrak{A}$. Since A is reduced, we have

$$A = A_{\text{red}} = A/\mathfrak{N}(A)$$

where $\mathfrak{N}(A)$ is the nilradical ideal of A, and so,

$$\sqrt{\mathfrak{A}} = \mathfrak{A}.$$

Letting $V = V(\mathfrak{A}) \subseteq \mathbb{A}^q$, we get

$$\mathfrak{I}(V) = \sqrt{\mathfrak{A}} = \mathfrak{A},$$

and thus,

$$A \cong k[V]. \qquad \square$$

Let us agree to write, given a subfield L of the layer Ω/k (i.e., $k \subseteq L \subseteq \Omega$), $V(L)$ for the set of points of V $(= V(\Omega))$ whose coordinates all lie in L. In referring to $V(L)$, we will use the locutions

$V(L) = L$-valued points of V = points of V with values in L = L-rational points of V.

If $V \subseteq \mathbb{A}^q$ is an affine k-variety, let

$$\mathrm{Hom}_{k-\mathrm{alg}}(k[V], L)$$

denote the set of all ring homomorphisms $\theta\colon k[V] \to L$ that are the identity on k.

Proposition 1.6 *Let $V \subseteq \mathbb{A}^q$ be an affine k-variety, then, there is a bijection between $V(L)$ and the set $\mathrm{Hom}_{k-\mathrm{alg}}(k[V], L)$. Moreover, there is a bijection between k-closed subvarieties of V and radical ideals, $\mathfrak{B}$, of $k[V]$. The k-irreducible varieties of V correspond to prime ideals of $k[V]$ (under the above correspondence).*

Proof. We have $V = V(\mathfrak{A})$ and we can assume that $\mathfrak{A} = \sqrt{\mathfrak{A}}$, by choosing $\mathfrak{A} = \mathfrak{I}_k(V)$. Then,

$$k[V] = k[X_1, \ldots, X_q]/\mathfrak{A},$$

and if $\theta \in \mathrm{Hom}_{k-\mathrm{alg}}(k[V], L)$, we have the diagram

$$\Theta\colon k[X_1, \ldots, X_q] \longrightarrow k[X_1, \ldots, X_q]/\mathfrak{A} \xrightarrow{\theta} L,$$

and the composite map Θ determines (and is determined by) a point $\eta = (\eta_1, \ldots, \eta_q)$, where $\eta_i = \Theta(X_i)$. If $f \in \mathfrak{I}_k(V)$, then $f \in \mathrm{Ker}\,\Theta$. However, $\Theta(f) = f(\eta)$, and therefore, if $f \in \mathfrak{I}_k(V)$, then $f(\eta) = 0$ and $\eta \in V(L)$.

Conversely, assume that $\eta \in V(L)$. We have a k-algebra homomorphism

$$\Theta\colon k[X_1, \ldots, X_q] \longrightarrow L$$

given by $\Theta(X_j) = \eta_j$. Since $\eta \in V(L)$, whenever $f \in \mathfrak{I}_k(V)$, we have $f(\eta) = 0$. Since $\mathfrak{A} = \mathfrak{I}_k(V)$ and since $k[V] = k[X_1, \ldots, X_q]/\mathfrak{A}$, every element of $k[V]$ is a coset of the form $f + \mathfrak{A}$ for some $f \in k[X_1, \ldots, X_q]$ and we can define the function $\theta\colon k[V] \to L$ by setting

$$\theta(f + \mathfrak{A}) = \Theta(f).$$

Since $\mathrm{Ker}\,\Theta \subseteq \mathfrak{A}$, this function is well defined, and $\theta\colon k[V] \to L$ is a ring homomorphism.

Let $\rho\colon k[X_1, \ldots, X_q] \to k[V]$ be the residue map, where $k[V] = k[X_1, \ldots, X_q]/\mathfrak{A}$. Assume that W is a closed variety in V, this means that W corresponds to some ideal $\mathfrak{B}$ of $k[X_1, \ldots, X_q]$ such that $\mathfrak{B} = \sqrt{\mathfrak{B}}$, and

$$\mathfrak{A} = \mathfrak{I}(V) \subseteq \mathfrak{I}(W) = \mathfrak{B}.$$

Then, $\mathfrak{B}$ corresponds to the radical ideal $\rho(\mathfrak{B}) = \mathfrak{B}/\mathfrak{A}$ in $k[V]$. Conversely, every radical ideal, $\mathfrak{Q}$, of $k[V] = k[X_1, \ldots, X_q]/\mathfrak{A}$ corresponds to the radical ideal $\mathfrak{B} = \rho^{-1}(\mathfrak{Q})$ which contains $\mathfrak{A}$. Hence, $V(\mathfrak{B})$ is a closed variety in V. $\square$

Remarks:

(1) Given any $\eta \in V(L)$, the unique k-algebra homomorphism

$$\Theta\colon k[X_1, \ldots, X_q] \longrightarrow L$$

given by $\Theta(X_j) = \eta_j$ has the property that

$$\Theta(f) = f(\eta)$$

for every $f \in k[X_1, \ldots, X_q]$. In other words, Θ is the "evaluation homomorphism at η (on $k[X_1, \ldots, X_q]$)." The unique k-algebra homomorphism $\theta\colon k[V] \to L$ induced by Θ leads us to define $F(\eta)$ for every $F \in k[V]$ by setting

$$F(\eta) = \theta(F).$$

Again, $\eta \in V(L)$ corresponds to the evaluation homomorphism at η (on $k[V]$).

(2) The k-topology (resp. Zariski-topology) on $\mathbb{A}^q$ has as a basis the sets

$$V_f = \{\xi \in \mathbb{A}^q \mid f(\xi) \neq 0\} = V((f))^c,$$

(where X^c denotes the set-theoretic complement of X) with $f \in k[X_1, \ldots, X_q]$ (resp. $f \in \overline{k}[X_1, \ldots, X_q]$). Indeed, if U is any k-open set, then $U^c = V(\mathfrak{A})$ for some ideal $\mathfrak{A}$, and if $\mathfrak{A}$ is generated by the family of polynomials (f_α), then

$$V(\mathfrak{A}) = \bigcap_\alpha V(f_\alpha);$$

So,

$$U = \bigcup_\alpha V(f_\alpha)^c = \bigcup_\alpha V_{f_\alpha}.$$

(3) In other references, such as Hartshorne [33], Grothendieck and Dieudonné (EGA I) [30], or Dieudonné [13], the basic open sets V_f are also denoted by $D(f)$.

Proposition 1.7 *If f is any polynomial in $k[X_1, \ldots, X_q]$ (resp. $f \in \overline{k}[X_1, \ldots, X_q]$), the restriction, $f \restriction V$, of f to V is a continuous function on the affine variety $V \subseteq \mathbb{A}^q$, when we give Ω the k-topology (resp. the Zariski topology). Furthermore, the k-topology (resp. the Zariski topology) is the weakest topology for which all such functions are continuous.*

Proof. First, consider the Zariski topology. If ξ is a closed point in $\mathbb{A}^1(\Omega)$, then $\xi \in \overline{k}$. Given the polynomial $f \in \overline{k}[X_1, \ldots, X_q]$, the polynomial

$$g(X_1, \ldots, X_q) = f(X_1, \ldots, X_q) - \xi$$

has coefficients in $\overline{k}$, and its zero locus is exactly the inverse image under f of the closed point ξ. As Zariski-closed sets in $\mathbb{A}^1(\Omega)$ are merely finite sets of such ξ, we see that f^{-1} of any Zariski-closed set in Ω is Zariski-closed in V.

When k is arbitrary, Galois theory shows that an irreducible k-closed set in Ω is a finite set of k-conjugate Zariski-closed points in Ω (with p-powers multiplicities if necessary in characteristic $p > 0$). Hence, we obtain the continuity of f. $\square$

Remark: Consider the image of X_j in $k[V]$ viewed as a continuous function. This is the jth coordinate function on V, and these functions generate $k[V]$. Therefore, $k[V]$ merits its name of "coordinate ring." Suppose $V \subseteq \mathbb{A}^q$ is a k-irreducible variety, let ξ be a k-generic point; we may assume that

$$\xi = (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_q)$$

where $\xi_1, \ldots, \xi_r$ are algebraically independent over k and $\xi_{r+1}, \ldots, \xi_q$ are algebraic over $k[\xi_1, \ldots, \xi_r]$, then, to get any point, $\eta \in V$, we apply a k-specialization, φ, to ξ. We can map $\xi_1, \ldots, \xi_r$ *anywhere*. Intuitively, we have r degrees of freedom on V. This leads to the definition of dimension.

Definition 1.7 Let $V \subseteq \mathbb{A}^q$ be any k-irreducible variety. The *dimension*, $\dim_k V$, *of* V is defined by

$$\dim_k V = \mathrm{tr.d}_k(k[V]).$$

If V is reducible, then there is a unique irredundant decomposition

$$V = V_1 \cup \cdots \cup V_t$$

into k-irreducible components, and we let

$$\dim_k V = \max_i \{\mathrm{tr.d}_k(k[V_i])\}.$$

Here are some of the desirable properties of the notion of dimension; all turn out to be true, but we will not prove all of them right now.

(1) $\dim_k \mathbb{A}^q_k = q$.

(2) If $V \subseteq \mathbb{A}^q$, then $\dim_k V \le q$

(3) If $V \subseteq W$, where V and W are k-irreducible varieties, then

$$\dim_k V \le \dim_k W,$$

and, if $\dim_k V = \dim_k W$, then $V = W$.

(4) If W is a k-irreducible variety and V is a maximal k-irreducible subvariety of W, then

$$\dim_k V = \dim_k W - 1.$$

Remarks:

(1) Property (3) is false if we drop the assumption of k-irreducibility. For a counterexample, consider the case where V consists of one line and W consists of two lines.

(2) Property (4) does not make sense if we drop the assumption of k-irreducibility. Consider countably many distinct parallel lines in $\mathbb{A}^2$. Label them $l_1, l_2, \ldots$, and write

$$W_j = \bigcup_{i=1}^{j} l_i.$$

The W_j's form an infinite ascending chain of irreducible subvarieties of $\mathbb{A}^2$, and there is no maximal proper subvariety of the variety

$$W = \overline{\bigcup_{j=1}^{\infty} W_j}.$$

"There is no maximality without irreducibility."

Properties (1) and (2) are clear; we will now prove (3). (Clearly, (3) implies (2)).

Proposition 1.8 *Let $V, W \subseteq \mathbb{A}^q$ be two k-irreducible varieties. If $V \subseteq W$, then*

$$\dim_k V \le \dim_k W,$$

and if $\dim_k V = \dim_k W$, then $V = W$.

Proof. Let ξ be a k-generic point of W and η be a k-generic point of V. Then,

$$\overline{\{\xi\}} = W \quad \text{and} \quad \overline{\{\eta\}} = V.$$

There is a surjective k-specialization

$$\varphi\colon k[\xi_1, \ldots, \xi_q] \to k[\eta_1, \ldots, \eta_q],$$

and we can arrange $\eta_1, \ldots, \eta_q$ so that $\eta_1, \ldots, \eta_r$ form a transcendence base, where $r = \dim_k V$. We claim that $\xi_1, \ldots, \xi_r$ are algebraically independent. If not, there is some nonzero polynomial $F \in k[X_1, \ldots, X_r]$ such that

$$F(\xi_1, \ldots, \xi_r) = 0,$$

and by applying φ, we get

$$F(\eta_1, \ldots, \eta_r) = 0.$$

Since $\eta_1, \ldots, \eta_r$ form a transcendence base, we have $F \equiv 0$, and so $\xi_1, \ldots, \xi_r$ are algebraically independent. As a consequence

$$\dim_k V = r \leq \dim_k W.$$

Let us now assume that $\xi_1, \ldots, \xi_r$ also form a transcendence base for W. We have a surjection

$$k[W] \cong k[\xi_1, \ldots, \xi_q] \xrightarrow{\varphi} k[\eta_1, \ldots, \eta_q] \cong k[V].$$

We claim that the map φ is also injective. If $y \in k[W]$ is in the kernel of φ, then y is algebraic over $\xi_1, \ldots, \xi_r$, and y satisfies a polynomial equation

$$b_l(\xi_1, \ldots, \xi_r)y^l + b_{l-1}(\xi_1, \ldots, \xi_r)y^{l-1} + \cdots + b_0(\xi_1, \ldots, \xi_r) = 0. \tag{1.22}$$

Applying φ, since $y \in \operatorname{Ker} \varphi$, we have

$$\varphi(b_0(\xi_1, \ldots, \xi_r)) = 0.$$

However, the restriction of φ to $k[\xi_1, \ldots, \xi_r]$ is an isomorphism onto $k[\eta_1, \ldots, \eta_r]$, and therefore,

$$b_0(\xi_1, \ldots, \xi_r) = 0.$$

If we choose the polynomial in (1.22) to have minimal degree, we get a contradiction. Therefore, $k[V] \cong k[W]$ and $V = W$. $\square$

We know that there is a one-to-one correspondence among the following three entities:

1. k-irreducible k-varieties $V \subseteq \mathbb{A}^q$.

2. Prime ideals in $k[X_1, \ldots, X_q]$.

3. k-equivalence classes of points $\xi \in \mathbb{A}^q$: k-specialization.

To each such object, we can assign an integer r, $(\dim_k V)$, $0 \leq r \leq q$, and this integer behaves much like vector space dimension, in that:

(a) It is monotone:

$$V \subseteq W \quad \text{implies} \quad \dim V \leq \dim W.$$

(b) If $V \subseteq W$ and $\dim V = \dim W$, then $V = W$.

Remark: Let $f \in k[X_1, \ldots, X_q]$ and assume that U is k-dense in V. If $f \restriction U = 0$, then $f \restriction V = 0$.
Proof. The point 0 is k-closed in $\mathbb{A}^1$, and $f(U) \subseteq \{0\}$. By continuity,

$$f(\overline{U}) \subseteq \overline{f(U)},$$

and therefore

$$f(V) = f(\overline{U}) \subseteq \overline{\{0\}} = \{0\}. \qquad \square$$

Finally, we are in a position to prove Hilbert's Nullstellensatz. First, we give various equivalent statements of the Nullstellensatz.

Proposition 1.9 *Let $V = V(\mathfrak{A}) \subseteq \mathbb{A}^q$ be an affine k-variety, then the following statements are equivalent.*

(α) If $\mathfrak{A} \neq (1)$ then $V(\overline{k}) \neq \emptyset$.

(β) The point set $V(\overline{k})$ is k-dense in $V(= V(\Omega))$. (Note: $V(\overline{k})$ consists of the k-closed points of V).

(γ_1) If $f \in k[X_1, \ldots, X_q]$ and $f \restriction V(\overline{k}) = 0$, then $f \restriction V = 0$.

(γ_2) If $f \in k[X_1, \ldots, X_q]$ and $f \restriction V(\overline{k}) = 0$, then $f^\rho \in \mathfrak{A}$ for some power $\rho \geq 1$.

(δ) Every maximal ideal of the ring $\overline{k}[X_1, \ldots, X_q]$ has the form

$$\mathfrak{M} = (X_1 - \alpha_1, \ldots, X_q - \alpha_q),$$

for some $\alpha_1, \ldots, \alpha_q \in \overline{k}$.

Proof. First, we prove that (β) implies (α). If assuming (β), we prove that $V \neq \emptyset$, we are done. But if $V = \emptyset$, then by the weak Nullstellensatz (Theorem 1.5), we have $\mathfrak{I}(V) = (1) = \sqrt{\mathfrak{A}}$. As a consequence, $\mathfrak{A} = (1)$, a contradiction.

(γ_1) $\Rightarrow$ (β). Assume that if $f \restriction V(\overline{k}) = 0$, then $f \restriction V = 0$. If so, $f \neq 0$ on V implies that $f \neq 0$ on $V(\overline{k})$. Thus, $\mathbb{A}^q_f \cap V \neq \emptyset$ implies that $\mathbb{A}^q_f \cap V(\overline{k}) \neq \emptyset$. Since the sets $\mathbb{A}^q_f$ form a base of the k-topology in $\mathbb{A}^q$, $V(\overline{k})$ is k-dense in V.

(γ_2) $\Rightarrow$ (γ_1). This is trivial.

(α) $\Rightarrow$ (γ_2) (Rabinowitch trick). Given $f \in k[X_1, \ldots, X_q]$, assume that $f = 0$ on $V(\overline{k})$. We have $V = V(\mathfrak{A})$, and since $\mathfrak{A}$ is finitely generated, $\mathfrak{A} = (f_1, \ldots, f_p)$ for some $f_i \in$

$k[X_1, \ldots, X_q]$. Pick a new transcendental T independent of $X_1, \ldots, X_q$, and consider the system of equations

$$f_1 = 0, \ldots, f_p = 0, 1 - Tf = 0$$

in the variables $X_1, \ldots, X_q, T$. These equations define a variety W in $\mathbb{A}^{q+1}$. Observe that $W(\overline{k}) = \emptyset$, since if $f_i(\xi) = 0$ for $i = 1, \ldots, q$, and $\xi \in V(\overline{k})$, then the equation $1 = 0$ would hold. By (α) applied to W, the ideal

$$(f_1, \ldots, f_p, 1 - Tf)$$

is equal to (1). Thus, there is some equation

$$1 = \sum_{i=1}^{p} g_i(X_1, \ldots, X_q, T) f_i(X_1, \ldots, X_q) + h(X_1, \ldots, X_q, T)(1 - Tf).$$

If we specialize T to be $1/f$, we get

$$1 = \sum_{i=1}^{p} g_i(X_1, \ldots, X_q, 1/f) f_i(X_1, \ldots, X_q).$$

By clearing denominators, we find some $\rho > 0$ so that

$$f^\rho = \sum_{i=1}^{p} g_i(X_1, \ldots, X_q, f) f_i(X_1, \ldots, X_q),$$

but each $g_i(X_1, \ldots, X_q, f)$ is equal to some polynomial $G_i(X_1, \ldots, X_q)$, and so

$$f^\rho = \sum_{i=1}^{p} G_i(X_1, \ldots, X_q) f_i(X_1, \ldots, X_q),$$

which means that $f^\rho \in \mathfrak{A}$.

$(\alpha) \Rightarrow (\delta)$. Given a maximal ideal $\mathfrak{M}$ of $\overline{k}[X_1, \ldots, X_q]$, since $\mathfrak{M} \neq (1)$, using (α), we have $V(\mathfrak{M})(\overline{k}) \neq \emptyset$. Assume that $\xi = (\xi_1, \ldots, \xi_q) \in \overline{k}^q$ is in $V(\mathfrak{M})(\overline{k})$. Now, $\xi \in V(\mathfrak{M})$, so

$$\mathfrak{M} \subseteq \mathfrak{I}(\{\xi\}).$$

But $\{\xi\}$ is k-closed, thus

$$\mathfrak{I}(\{\xi\}) = (X_1 - \xi_1, \ldots, X_q - \xi_q).$$

However, $\mathfrak{M}$ is maximal, so we must have

$$\mathfrak{M} = (X_1 - \xi_1, \ldots, X_q - \xi_q).$$

$(\delta) \Rightarrow (\alpha)$. Let $V = V(\mathfrak{A})$ for some ideal $\mathfrak{A}$ in $k[X_1, \ldots, X_q]$, and assume $\mathfrak{A} \neq (1)$. Then, it is easily shown that $\overline{\mathfrak{A}}$ in $\overline{k}[X_1, \ldots, X_q]$ is also different from (1) (DX). There is some maximal ideal $\mathfrak{M}$ of $\overline{k}[X_1, \ldots, X_q]$ such that $\overline{\mathfrak{A}} \subseteq \mathfrak{M}$. By (δ), we have

$$\mathfrak{M} = (X_1 - \xi_1, \ldots, X_q - \xi_q)$$

for some $\xi_1, \ldots, \xi_1 \in \overline{k}$, and thus,

$$\overline{\mathfrak{A}} \subseteq (X_1 - \xi_1, \ldots, X_q - \xi_q),$$

which implies that

$$\{\xi\} = V((X_1 - \xi_1, \ldots, X_q - \xi_q)) \subseteq V(\overline{\mathfrak{A}}) = V(\mathfrak{A}),$$

and $\xi \in V(\overline{k})$. $\square$

Here is Hilbert's Nullstellensatz.

Theorem 1.10 *(Hilbert's Nullstellensatz (1893)) Given any affine k-variety $V \subseteq \mathbb{A}^q$, all the statements of Proposition 1.9 hold.*

Proof. Assume that $V = V(\mathfrak{A})$ and that $\mathfrak{A} \neq (1)$. There is a maximal ideal $\mathfrak{M}$ such that $\mathfrak{A} \subseteq \mathfrak{M}$. Since $V(\mathfrak{M}) \subseteq V(\mathfrak{A})$, we may assume that $\mathfrak{A}$ is maximal. But then, $\mathfrak{A}$ is prime, so V is k-irreducible, and there is some k-generic point $\xi \in V(= V(\Omega))$. Let $r = \dim_k V$, $0 \leq r \leq q$, and, as usual, arrange the notation so that $\xi_1, \ldots, \xi_r$ form a transcendence base for $k[\xi_1, \ldots, \xi_q]$ over k. Then, $\xi_{r+1}, \ldots, \xi_q$ are algebraic over $k[\xi_1, \ldots, \xi_r]$, and we have some minimal equations for the ξ_j's $(r + 1 \leq j \leq q)$ of the form

$$g^{(j)}_{m_j}(\xi_1, \ldots, \xi_r)\xi_j^{m_j} + \cdots + g^{(j)}_0(\xi_1, \ldots, \xi_r) = 0. \tag{$*$}$$

Since $\overline{k}$ is algebraically closed, it is infinite; so, by elementary algebra, there are some $\eta_1, \ldots, \eta_r \in \overline{k}$ so that

$$g^{(j)}_{m_j}(\eta_1, \ldots, \eta_r) \neq 0 \tag{1.23}$$

for all j, where $r+1 \leq j \leq q$. By mapping $\xi_1, \ldots, \xi_r$ to $\eta_1, \ldots, \eta_r$, we have a homomorphism

$$\varphi\colon \overline{k}[\xi_1, \ldots, \xi_r] \longrightarrow \overline{k},$$

so by the place-extension theorem, φ extends to a $\overline{k}$-valued place of the field $\overline{k}(\xi_1, \ldots, \xi_q)$, which we also denote by φ. The domain of φ is a subring of $\overline{k}(\xi_1, \ldots, \xi_q)$ and φ is not extendable further with values in $\overline{k}$. It is known that if $x \in \overline{k}(\xi_1, \ldots, \xi_q)$ and $x \notin \mathrm{dom}(\varphi)$, then $1/x \in \mathrm{dom}(\varphi)$ and $\varphi(1/x) = 0$ (see Zariski and Samuel [61], Theorem 5′, Chapter VI, Section 4). I claim that φ is finite on $\xi_{r+1}, \ldots, \xi_q$, i.e., that $\xi_{r+1}, \ldots, \xi_q$ belong to $\mathrm{dom}(\varphi)$.

Otherwise, there is some ξ_j such that $\xi_j \notin \mathrm{dom}(\varphi)$, where $r + 1 \leq j \leq q$. But then, $1/\xi_j \in \mathrm{dom}(\varphi)$ and $\varphi(1/\xi_j) = 0$. Consider the j-th equation of $(*)$ and divide each term by $\xi_j^{m_j}$. We get

$$g^{(j)}_{m_j}(\xi_1, \ldots, \xi_r) + g^{(j)}_{m_{j-1}}(\xi_1, \ldots, \xi_r)\left(\frac{1}{\xi_j}\right) + \cdots + g^{(j)}_0(\xi_1, \ldots, \xi_r)\left(\frac{1}{\xi_j}\right)^{m_j} = 0.$$

Applying φ, we get

$$g^{(j)}_{m_j}(\eta_1, \ldots, \eta_r) = 0,$$

which contradicts (1.23).

Therefore, $\varphi(\xi_j) \in \overline{k}$, where $r+1 \leq j \leq q$. Letting $\eta_{r+1} = \varphi(\xi_{r+1}), \ldots, \eta_q = \varphi(\xi_q)$, we obtain a $\overline{k}$-point

$$\varphi\colon \overline{k}[\xi_1, \ldots, \xi_q] \longrightarrow \overline{k}. \qquad \square$$

Remark: The key part of the argument involving the existence of $\eta_1, \ldots, \eta_r \in \overline{k}$ so that equation (1.23) holds is Hilbert's original argument. The rest of the argument follows Chevalley. Other proofs of Hilbert's Nullstellensatz can be found in Zariski and Samuel [61] (Theorem 14, Chapter VII, Section 3), or Mumford [43], Fulton [17], Eisenbud [14], Bourbaki [7], Atiyah and Macdonald [2], Matsumura [40].

1.3 Functions and Morphisms

We begin with some remarks about k-density. Let $V \subseteq \mathbb{A}^q$ be an affine k-variety.

(1) If V is k-irreducible, then any nonempty k-open subset U of V is k-dense.

(2) If V is k-irreducible, then any two nonempty k-open subsets of V have a nonempty (and thus, k-dense) intersection.

(3) If U is k-dense in V and U is k-irreducible in the relative topology, then V is k-irreducible (DX).

(4) If U is k-dense in V, f, g are any two polynomials in $k[X_1, \ldots, X_q]$, and $f \restriction U = g \restriction U$, then $f = g$ on V.

Statement (3) being a diagnostic exercise, we prove (1), (2), and (4).

Proofs. (1) Let U be nonempty and open in V, where V is k-irreducible. Then, $Z = U^c$ is closed and $Z \neq V$. Let $W = \overline{U}$ be the k-closure of U, then

$$V = U^c \cup U = Z \cup U \subseteq Z \cup W \subseteq V.$$

So, $V = Z \cup W$. Since V is k-irreducible, we must have $V = W$ and U is k-dense in V.

(2) Let U_1, U_2 be two nonempty open subsets of V. Then, U_1^c and U_2^c are closed subsets distinct from V. Since V is k-irreducible, we must have

$$U_1^c \cup U_2^c \neq V,$$

and thus, $U_1 \cap U_2 \neq \emptyset$.

(4) Consider the map $(f, g)\colon U \to \mathbb{A}^2$ defined by

$$(f, g)(\xi) = (f(\xi), g(\xi)),$$

and give $\mathbb{A}^2 = \Omega \times \Omega$ the Zariski topology (resp. k-topology). It is easily checked that (f, g) is continuous (DX). If

$$\Delta = \{(\eta, \eta) \mid \eta \in \Omega\},$$

the hypothesis $f \upharpoonright U = g \upharpoonright U$ is expressed by

$$(f, g)(U) \subseteq \Delta.$$

However, Δ is k-closed (resp. Zariski-closed) in $\mathbb{A}^2$, since it is given by the equation

$$x - y = 0,$$

and thus, by continuity,

$$(f, g)(V) = (f, g)(\overline{U}) \subseteq \overline{(f, g)(U)} \subseteq \overline{\Delta} = \Delta,$$

and $f = g$ on V. $\square$

Remark: Property (4) can also be proven by considering the function $f - g$ which vanishes on U. By the remark just before Proposition 1.9, $f - g$ vanishes on V, i.e., $f = g$. The reason we gave the longer proof above is because it is the archtype of the proof we must use in the more general situation where subtraction is not available.

Recall that to do geometry, we need a topological space and, locally defined, functions. Here is our definition of the functions on an affine variety V. Given an affine variety $V \subseteq \mathbb{A}^q$, a *function* $\varphi\colon V \to \Omega$ on V is a set-theoretic function that is locally defined by rational functions. This is analogous to the definition of a complex holomorphic function which is locally defined by convergent power series.

Definition 1.8 Let $V \subseteq \mathbb{A}^q$ be an affine k-variety. A *locally defined holomorphic function* on V is a triple (f, g, U) where $f, g \in k[V]$ and U is a k-open subset of V such that g does not vanish on U. (That is, $U \subseteq V_g = \{\xi \in V \mid g(\xi) \neq 0\}$).

Given such a triple, we get a set-theoretic function on U with values in Ω also denoted by (f, g, U), namely, the function such that

$$\xi \mapsto \frac{f(\xi)}{g(\xi)}$$

for every $\xi \in U$. By Proposition 1.7 the function (f, g, U) is Zariski-continuous (resp. k-continuous).

To actually define functions on V, we need to introduce an equivalence relation on triples (f, g, U):

Two locally defined holomorphic functions (f, g, U) and $(\widetilde{f}, \widetilde{g}, \widetilde{U})$ are *equivalent*, denoted by

$$(f, g, U) \sim (\widetilde{f}, \widetilde{g}, \widetilde{U})$$

if

$$U = \widetilde{U} \quad \text{and} \quad f(\xi)\widetilde{g}(\xi) = g(\xi)\widetilde{f}(\xi),$$

for all $\xi \in U$.

Finally, the class of "good" functions on an affine variety can be defined.

Definition 1.9 Let $V \subseteq \mathbb{A}^q$ be an affine k-variety. A *(global) holomorphic function* on V is a set-theoretic function $\varphi\colon V \to \Omega$ satisfying the following condition: For every $\xi \in V$, there is some k-open subset U_ξ of V with $\xi \in U_\xi$ and some locally defined holomorphic function (f, g, U_ξ) so that

$$\varphi(\eta) = (f, g, U_\xi)(\eta)$$

for all $\eta \in U_\xi$.

Informally, φ looks like the rational function $f(\xi)/g(\xi)$ on U_ξ. Holomorphic functions on V are continuous, since they are defined locally by continuous functions.

Remarks:

(1) Every $f \in k[V]$ gives rise to a (global) holomorphic function on V. Indeed, we can cover V by itself and take $g \equiv 1$. Then, f corresponds to the function $(f, 1, V)$.

(2) The map from $k[V]$ to the set of (global) holomorphic functions on V defined by the assignment

$$f \mapsto (f, 1, V)$$

is an injection. This follows because from $(f, 1, V) \sim (g, 1, V)$, we find

$$(f - g)(\xi) = 0$$

for all $\xi \in V$. Thus, $(f - g) \in \mathfrak{I}_k(V)$, and $f - g \equiv 0$ on $k[V] = k[X_1, \ldots, X_q]/\mathfrak{I}_k(V)$.

Now that we have a class of good functions on the set of points of an affine variety, we can give the definition of an affine variety that implicitly incorporates the important concept of a sheaf.

Definition 1.10 An *affine k-variety* is a pair consisting of

(1) The set $V = V(\Omega)$ viewed as a topological space (where V is of the form $V_k(\mathfrak{A})$, as in Definition 1.1), and

(2) The collection of all (equivalence classes of) locally defined holomorphic functions on V.

Now that we have the basic objects of affine algebraic geometry (at the first level of abstraction), we can define morphisms between them.

Definition 1.11 If V, W are two affine k-varieties, a *k-morphism* $\varphi\colon V \to W$ is a set-theoretic map $\varphi\colon V(\Omega) \to W(\Omega)$ so that for every locally defined holomorphic function (f, g, U) on W, the pull-back $\varphi^*(f, g, U)$ (that is, the composition $(f, g, U) \circ \varphi$) is again a locally defined holomorphic function on V.

Even though Definition 1.11 is nice and clean, it does not say what a k-morphism really is. This is the object of the next proposition.

Proposition 1.11 *(Explicit form of k-morphisms) Let V and W be affine k-varieties with $W \subseteq \mathbb{A}^q$. A set-theoretic map $\varphi\colon V(\Omega) \to W(\Omega)$ is a k-morphism iff for every $\xi \in V$, there is some k-open subset U_ξ of V with $\xi \in U_\xi$ and there are some $f_1, \ldots, f_q$, $g_1, \ldots, g_q \in k[V]$ so that the following properties hold:*

$$(a) \quad \prod_{i=1}^{q} g_i \not\equiv 0 \text{ on } U_\xi, \quad \text{i.e.,} \quad U_\xi \subseteq \bigcap_{j=1}^{q} V_{g_j}, \text{ and}$$

$$(b) \quad \varphi(\eta) = \left\langle \frac{f_1(\eta)}{g_1(\eta)}, \ldots, \frac{f_q(\eta)}{g_q(\eta)} \right\rangle, \text{ for all } \eta \in U_\xi.$$

Proof. Assume that (a) and (b) hold. Let (F, G, O) be a locally defined holomorphic function on W. By (b), the function φ is k-continuous (or Zariski-continuous). Therefore, $U = \varphi^{-1}(O)$ is k-open (Zariski-open) in V. For any $\eta \in U$, we have

$$\varphi^*(F, G, O)(\eta) = \frac{F\left(\dfrac{f_1(\eta)}{g_1(\eta)}, \ldots, \dfrac{f_q(\eta)}{g_q(\eta)}\right)}{G\left(\dfrac{f_1(\eta)}{g_1(\eta)}, \ldots, \dfrac{f_q(\eta)}{g_q(\eta)}\right)} = \frac{h(\eta)}{l(\eta)}$$

for some $h, l \in k[V]$.

Conversely, assume that $\varphi\colon V(\Omega) \to W(\Omega)$ is a k-morphism. On W, we have the coordinate functions $Y_1, \ldots, Y_q$. Thus, $\varphi^*(Y_j)$ is a locally defined holomorphic function on all of V for every j, $1 \leq j \leq q$. This means that $\varphi^*(Y_j)$ is of the form $f_j^{(\xi)}/g_j^{(\xi)}$ on some suitable k-open U_ξ. But then,

$$\varphi(\eta) = \left\langle \frac{f_1^{(\xi)}(\eta)}{g_1^{(\xi)}(\eta)}, \ldots, \frac{f_q^{(\xi)}(\eta)}{g_q^{(\xi)}(\eta)} \right\rangle$$

on U_ξ. $\square$

Remarks:

(1) From Definition 1.11, the composition of k-morphisms is a k-morphism. Thus, affine k-varieties form a category.

(2) The explicit form (Proposition 1.11) implies that a k-morphism is a continuous map of topological spaces.

(3) The explicit form implies that k-morphisms preserve "rationality," i.e., for every field L such that $k \subseteq L \subseteq \Omega$, if $\varphi\colon V \to W$ is a k-morphism, then $\varphi\colon V(L) \to W(L)$.

(4) Let $\mathrm{Hom}_k(V, W)$ denote the set of all k-morphisms $\varphi\colon V \to W$. Then, $\mathrm{Hom}_k(V, \mathbb{A}^1)$ is the collection of global holomorphic functions on V.

Let A be an affine k-algebra, i.e., a k-algebra which is finitely generated and reduced. Thus, $A = k[V]$, where $V = V(\mathfrak{A})$ for some radical ideal, $\mathfrak{A}$, of $k[X_1, \ldots, X_q]$, and $A = k[X_1, \ldots, X_q]/\mathfrak{A}$.

Definition 1.12 Any affine k-algebra A as above completely determines an affine k-variety $\mathrm{Spec}\, A$ as follows:

(i) The set of points $V(\Omega)$ of $\mathrm{Spec}\, A$ is

$$V(\Omega) = \mathrm{Hom}_{k-\mathrm{alg}}(A, \Omega).$$

Define $h(\theta) = \theta(h)$, for all $\theta \in V(\Omega)$ and all $h \in A$.

(ii) The k-closed sets of $V(\Omega)$ correspond to radical ideals of A:

$$\mathfrak{A} \longleftrightarrow V(\mathfrak{A}) = \{\theta \in \mathrm{Hom}_{k-\mathrm{alg}}(A, \Omega) \mid \mathrm{Ker}\ \theta = \mathfrak{A}\}.$$

(iii) The locally defined holomorphic functions are the triples (f, g, U) where U is open (in the topology defined in (ii)) and f and g are in A, with $g \restriction U$ never zero.

The following proposition is the key step in showing that the mapping

$$A \mapsto \mathrm{Spec}\, A$$

is a functor from the category of finitely generated reduced (no nonzero nilpotent elements) k-algebras to the category of affine k-varieties.

Proposition 1.12 *Let A and B be affine k-algebras, and write V and W for* $\mathrm{Spec}\, A$ *and* $\mathrm{Spec}\, B$ *respectively. Every k-algebra homomorphism $\theta\colon A \to B$ determines a k-morphism $\widetilde{\theta}\colon W \to V$.*

Proof. Let $\theta\colon A \to B$ be a k-algebra homomorphism. Say $A \cong k[X_1, \ldots, X_q]/\mathfrak{A}$ where $\mathfrak{A} = \sqrt{\mathfrak{A}}$. Then, we have the composite map

$$\Theta\colon k[X_1, \ldots, X_q] \xrightarrow{\rho} A \xrightarrow{\theta} B.$$

Let $F_j = \Theta(X_j)$, so that $F_j \in k[W] = B$. We define the map $\widetilde{\theta}\colon W \to V$ as follows: For every $w \in W$,

$$\widetilde{\theta}(w) = \langle F_1(w), \ldots, F_q(w)\rangle.$$

Clearly, $\widetilde{\theta}(w) \in \mathbb{A}^q$, and $\widetilde{\theta}\colon W \to \mathbb{A}^q$ is a k-morphism. We need to prove that $\widetilde{\theta}(w) \in V$.

We know from Proposition 1.6 that each $w \in W$ corresponds to the unique k-algebra homomorphism $\lambda\colon B \to \Omega$ such that

$$\lambda(b) = b(w)$$

for all $b \in B$. Choose $f \in \mathfrak{A}$, we have

$$\begin{aligned}
f(\widetilde{\theta}(w)) &= f(F_1(w), \ldots, F_q(w)) \\
&= f(\lambda(F_1), \ldots, \lambda(F_q)) \\
&= f(\lambda(\Theta(X_1)), \ldots, \lambda(\Theta(X_q))) \\
&= f(\lambda(\theta(\rho(X_1))), \ldots, \lambda(\theta(\rho(X_q)))) \\
&= \lambda \circ \theta \circ \rho(f(X_1, \ldots, X_q)) \\
&= \lambda \circ \theta \circ \rho(f) \\
&= \lambda(\Theta(f)).
\end{aligned}$$

Since $f \in \mathfrak{A}$, we have $\Theta(f) = 0$; thus, $\lambda(\Theta(f)) = 0$ and $f(\widetilde{\theta}(w)) = 0$. Hence, $\widetilde{\theta}(w) \in V(\mathfrak{A}) = V$. $\square$

Remark: Given an affine k-algebra B, there is an isomorphism

$$\mathrm{Hom}_{k-\mathrm{alg}}(k[X], B) \cong B.$$

If $W = \mathrm{Spec}\, B$, we have the map $\theta \mapsto \widetilde{\theta}$ from $\mathrm{Hom}_{k-\mathrm{alg}}(k[X], B)$ to $\mathrm{Hom}_k(W, \mathbb{A}^1)$. We also have a map from B to $\mathrm{Hom}_k(W, \mathbb{A}^1)$; we claim that the following diagram is commutative (DX):

$$\begin{tikzcd}
B \arrow[r] & \mathrm{Hom}_k(W, \mathbb{A}^1) \\
\mathrm{Hom}_{k-\mathrm{alg}}(k[X], B) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Given a k-algebra homomorphism $\varphi\colon A \to B$ between two k-algebras A and B, we get a k-morphism $\widetilde{\varphi}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$. Furthermore, if φ is a k-isomorphism, then $\widetilde{\varphi}$ is also an isomorphism of k-varieties. This shows that affine k-varieties can be k-isomorphic even if they live in different ambient spaces.

The following situation also arises, and yields a broader concept of an affine variety. Given an affine variety $V \subseteq \mathbb{A}^q$, we may have a subset X such that

$$X(L) \subseteq V(L)$$

for every field L with $k \subseteq L \subseteq \Omega$ compatibly, in the sense that the following diagram commutes

$$\begin{tikzcd} X(L') \arrow[r, hook] & V(L') \\ X(L) \arrow[r, hook] \arrow[u] & V(L) \arrow[u] \end{tikzcd}$$

for all L, L' with $k \subseteq L \subseteq L' \subseteq \Omega$. It can happen that there is some affine k-variety W and a compatible bijection

$$X(L) \longleftrightarrow W(L).$$

In this case, the functions on $W(L)$ can be pulled back from W to X, and we get a collection of functions on X. This gives X the structure of an affine variety. If X is open, we want these new functions on X to agree with the ones we get by restricting the locally defined holomorphic functions on V to X. If the latter condition holds, we say that X is an *open affine k-subvariety of V*.

Example 1.6 Choose V$= \mathbb{A}^1$ and let the open X be

$$X = \{\xi \in \mathbb{A}^1 \mid \xi \neq 0\}.$$

For W, we choose the following affine variety in $\mathbb{A}^2$:

$$W = \{(\xi, \eta) \in \mathbb{A}^2 \mid \xi\eta - 1 = 0.\}$$

The bijection is given by

$$\xi \mapsto \left(\xi, \frac{1}{\xi}\right).$$

The functions on $X(L)$ are the functions from $k[X]$ where the denominators are of the form $X^k g(X)$, with $g(0) \neq 0$. On $W(L)$, the functions are locally of the form

$$\frac{g\left(X, \dfrac{1}{X}\right)}{h\left(X, \dfrac{1}{X}\right)},$$

and these indeed have denominators of the form $X^k g(X)$.

We can generalize the previous example. Given an affine k-variety $V \subseteq \mathbb{A}^q$, recall that

$$V_f = \{\xi \in V \mid f(\xi) \neq 0\}.$$

Proposition 1.13 *Let $V \subseteq \mathbb{A}^q$ be an affine k-variety and let f be in $k[V]$. The open set V_f is an open k-subvariety of V. In fact, V_f is k-isomorphic to* Spec $(k[V]_f)$, *where*

$$k[V]_f = k[V][T]/(fT - 1)$$

is the localization of $k[V]$ at f.

Proof. First, we prove that $k[V]_f$ is reduced. Pick $\frac{g}{f^r} \in k[V]_f$. If

$$\left(\frac{g}{f^r}\right)^N = 0,$$

then $g^N = 0$, since f is a unit (the reasoning takes place in $k[V]_f$). This means that there is some $s > 0$ such that

$$f^s g^N = 0$$

in $k[V]$. Letting $M = \max(s, N)$, we get

$$(fg)^M = 0$$

in $k[V]$. Since $k[V]$ is reduced, we get $fg = 0$, and thus $g/1 = 0$ in $k[V]_f$. As a consequence,

$$\frac{g}{f^r} = 0,$$

and $k[V]_f$ is reduced. Since $k[V]_f$ is finitely generated and reduced, if $W = \operatorname{Spec} k[V]_f$, then W is an affine k-variety. So,

$$\begin{aligned} W(L) &= \{\lambda\colon k[V] \to L \mid \lambda(f) \neq 0\} \\ &= \{\xi \in V(L) \mid f(\xi) \neq 0\} = V_f(L). \end{aligned}$$

We leave it as an exercise to check that the functions are the same on V_f and W. $\square$

We need two technical lemmas for the proof of the next theorem. In preparation for this theorem (Theorem 1.16), you may want to review the definition of an equivalence of categories, say in Grothendieck [21] or Mac Lane [39], Chapter IV, Section 4, especially Theorem 1.

Lemma 1.14 *Let V be an affine k-variety and let $g, h \in k[V]$. If $h \upharpoonright V_g$ is never zero, then h is a unit of the ring $k[V]_g$, so that h and $\frac{1}{h}$ have the form*

$$\frac{\alpha}{g^r},$$

where $\alpha \in k[V]$ is a unit of $k[V]_g$ and $r > 0$.

Proof. We know that V_g is an open affine k-variety, and

$$\frac{h}{1} \in k[V]_g.$$

By hypothesis, $h/1$ is never zero on V_g, so, by the Nullstellensatz, we must have

$$\mathfrak{I}\left(V\left(\frac{h}{1}\right)\right) = V_g.$$

This implies that the ideal $(h/1)$, generated by $h/1$, is the unit ideal in $k[V]_g$. Thus, h is a unit of $k[V]_g$. $\square$

The second lemma gives a very useful normal form for k-morphisms; its proof will be given after the proof of Theorem 1.16.

Lemma 1.15 *Choose two affine k-varieties V, W, with $V \subseteq \mathbb{A}^q$. Given any k-morphism $\theta\colon W \to V$, there are some $F_1, \ldots, F_q \in k[W]$ such that*

$$\theta(w) = \langle F_1(w), \ldots, F_q(w)\rangle$$

for all $w \in W$.

Theorem 1.16 *(Fundamental theorem of affine geometry) The contravariant functor*

$$A \mapsto \mathrm{Spec}\, A$$

from the category of affine k-algebras to the opposite of the category of affine k-varieties (i.e., the category with the arrows reversed) is an equivalence of categories.

Proof. We already know that a k-algebra homomorphism $\varphi\colon A \to B$ gives rise to a k-morphism $\widetilde{\varphi}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, and this is functorial. Thus, we get a map

$$\mathrm{Hom}_{k-\mathrm{alg}}(A, B) \mapsto \mathrm{Hom}_k(\mathrm{Spec}\, B, \mathrm{Spec}\, A);$$

It is given by

$$\varphi \mapsto \widetilde{\varphi}.$$

We need to prove that this is a bijection. Let $W = \mathrm{Spec}\, B$ and $V = \mathrm{Spec}\, A$. If $\theta\colon W \to V$ is a k-morphism, by Lemma 1.15, there are some $F_1, \ldots, F_q \in k[W] = B$ such that

$$\theta(w) = \langle F_1(w), \ldots, F_q(w)\rangle$$

for all $w \in W$. We may assume that $V \subseteq \mathbb{A}^q$ and that $A = k[X_1, \ldots, X_q]/\mathfrak{A}$, where $\mathfrak{A} = \sqrt{\mathfrak{A}}$. Now, define the k-algebra map

$$\varphi\colon k[X_1, \ldots, X_q] \longrightarrow B$$

by setting $\varphi(X_j) = F_j \in B$. Each $w \in W$ corresponds to a unique k-algebra homomorphism $\lambda\colon B \to \Omega$ such that

$$\lambda(F) = F(w)$$

for all $F \in B$. Let $f \in \mathfrak{A}$. We have

$$\begin{aligned}
\lambda(\varphi(f)) &= \lambda(\varphi(f(X_1, \ldots, X_q)) \\
&= \lambda(f(\varphi(X_1), \ldots, \varphi(X_q))) \\
&= \lambda(f(F_1, \ldots, F_q)) \\
&= f(\lambda(F_1), \ldots, \lambda(F_q)) \\
&= f(F_1(w), \ldots, F_q(w)).
\end{aligned}$$

Since $f \in \mathfrak{A}$, we have

$$f(F_1(w), \ldots, F_q(w)) = 0,$$

and so

$$\lambda(\varphi(f)) = 0$$

for all λ. This implies that

$$\varphi(f)(w) = 0$$

for all $w \in W$. Therefore, $\varphi(f) \equiv 0$ in B, since $B \subseteq \mathrm{Hom}_k(W, \mathbb{A}^1)$, and we have proved that φ induces a k-algebra homomorphism $\widehat{\theta}\colon A \to B$. We leave it as an exercise to check that the maps

$$\varphi \mapsto \widetilde{\varphi} \quad \text{and} \quad \theta \mapsto \widehat{\theta}$$

are mutual inverses. $\square$

Here is the proof of Lemma 1.15.

Proof. By the definition of a k-morphism, for every $\xi \in W$, there is some k-open subset $U(\xi)$ of W with $\xi \in U(\xi)$ and there are some $\beta_1^{(\xi)}, \ldots, \beta_q^{(\xi)}$, $\gamma_1^{(\xi)}, \ldots, \gamma_q^{(\xi)} \in k[W]$ so that

$$\theta(\eta) = \left\langle \frac{\beta_1^{(\xi)}(\eta)}{\gamma_1^{(\xi)}(\eta)}, \ldots, \frac{\beta_q^{(\xi)}(\eta)}{\gamma_q^{(\xi)}(\eta)} \right\rangle$$

for all $\eta \in U(\xi)$, where $\gamma_i^{(\xi)}$ is never zero on $U(\xi)$ for all i, $1 \leq i \leq q$. Since the W_g's form a basis of the k-topology, we can pick some g so that $\xi \in W_g \subseteq U(\xi)$, and then, Lemma 1.14 implies that there are some $f_1^{(\xi)}, \ldots, f_q^{(\xi)} \in k[W]$, so that

$$\theta(\eta) = \left\langle \frac{f_1^{(\xi)}(\eta)}{g^{\nu_1}(\eta)}, \ldots, \frac{f_q^{(\xi)}(\eta)}{g^{\nu_1}(\eta)} \right\rangle.$$

If ν is the maximum of the ν_i's, since $W_{g^\nu} = W_g$, we may assume that $\nu = 1$, and we have

$$\theta(\eta) = \left\langle \frac{f_1^{(\xi)}(\eta)}{g(\eta)}, \ldots, \frac{f_q^{(\xi)}(\eta)}{g(\eta)} \right\rangle.$$

Now, the W_g's cover W, and by quasi-compactness, there is a finite subfamily $\{W_{g_1}, \ldots, W_{g_t}\}$ that covers W. Thus, on each W_{g_j}, we have

$$\theta(\eta) = \left\langle \frac{f_1^{(j)}(\eta)}{g_j(\eta)}, \ldots, \frac{f_q^{(j)}(\eta)}{g_j(\eta)} \right\rangle. \tag{$*$}$$

Since θ is well-defined, the local definitions of θ must agree on $W_{g_i} \cap W_{g_j} = W_{g_i g_j}$, and we have

$$\frac{f_l^{(j)}(\eta)}{g_j(\eta)} = \frac{f_l^{(i)}(\eta)}{g_i(\eta)}$$

for all $\eta \in W_{g_i g_j}$ and all l, $1 \leq l \leq q$. As a consequence,

$$f_l^{(j)} g_i - f_l^{(i)} g_j = 0 \quad \text{on } W_{g_i g_j},$$

which implies that

$$f_l^{(j)} g_i - f_l^{(i)} g_j = 0 \quad \text{in } k[W]_{g_i g_j}.$$

Therefore, there are some integers n_{ijl} so that

$$(g_i g_j)^{n_{ijl}} (f_l^{(j)} g_i - f_l^{(i)} g_j) = 0 \quad \text{in } k[W].$$

Let $N = \max\{n_{ijl}\}$, where $1 \leq i, j, \leq t$, $1 \leq l \leq q$. We have

$$(g_i g_j)^N f_l^{(j)} g_i = (g_1 g_j)^N f_l^{(i)} g_j, \tag{$**$}$$

for all i, j, l, with $1 \leq i, j, \leq t$, $1 \leq l \leq q$. Now, the W_{g_i} cover W. Hence, the g_i have no common zero, and neither do the g_i^{N+1} (since $W_{g^N} = W_g$). By the Nullstellensatz,

$$(g_1^{N+1}, \ldots, g_t^{N+1}) = (1),$$

the unit ideal in $k[W]$, and thus, there are some h_i so that

$$1 = \sum_{i=1}^{t} h_i g_i^{N+1}.$$

But, we have

$$\begin{aligned}
g_i^N f_l^{(i)} &= g_i^N f_l^{(i)} \left(\sum_{r=1}^{t} h_r g_r^{N+1} \right) \\
&= \sum_{r=1}^{t} h_r g_r^{N+1} g_i^N f_l^{(i)} \\
&= \sum_{r=1}^{t} h_r g_i^{N+1} g_r^N f_l^{(r)} \qquad \text{by } (**) \\
&= g_i^{N+1} \left(\sum_{r=1}^{t} h_r g_r^N f_l^{(r)} \right).
\end{aligned}$$

Letting

$$F_l = \sum_{r=1}^{t} h_r g_r^N f_l^{(r)},$$

we have $F_l \in k[W]$, and

$$g_i^N f_l^{(i)} = g_i^{N+1} F_l \quad \text{in } k[W].$$

For any $\eta \in W_{g_i}$, we get

$$\frac{f_l^{(i)}(\eta)}{g_i(\eta)} = F_l(\eta),$$

and by $(*)$,

$$\theta(\eta) = \langle F_1(\eta), \ldots, F_1(\eta) \rangle. \qquad \square$$

Theorem 1.16 has many corollaries. Here are two.

Corollary 1.17 *The global holomorphic functions on an affine k-variety V are exactly the elements of the coordinate ring $k[V]$.*

Proof. By definition, the ring of global holomorphic functions is $\mathrm{Hom}_k(V, \mathbb{A}^1)$. By Theorem 1.16,

$$\mathrm{Hom}_k(V, \mathbb{A}^1) \cong \mathrm{Hom}_{k-\mathrm{alg}}(k[T], k[V]) \cong k[V]. \qquad \square$$

Corollary 1.18 *Let A and B be affine k-algebras, $V = \mathrm{Spec}\, A$, $W = \mathrm{Spec}\, B$, and let $\varphi\colon A \to B$ be a k-algebra homomorphism. Then:*

(1) φ is surjective iff $\widetilde{\varphi}$ maps W isomorphically onto a k-closed subvariety of V.

(2) φ is injective iff $\mathrm{Im}\, \widetilde{\varphi}$ is k-dense.

Proof. (1) The morphism φ is surjective iff $B \cong A/\mathfrak{A}$ for a radical ideal $\mathfrak{A}$. By Theorem 1.16, we have

$$\mathrm{Spec}\, B \cong \mathrm{Spec}(A/\mathfrak{A}),$$

and $\mathrm{Spec}(A/\mathfrak{A})$ is a k-closed subvariety of V (namely, $V(\mathfrak{A})$).

(2) Let $\mathfrak{A} = \mathrm{Ker}\, \varphi$ and assume that $\mathrm{Im}\, \widetilde{\varphi}$ is k-dense. We have the following commutative diagram in which θ is an injection

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & B \\
& A/\mathfrak{A} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

and B being reduced, $A/\mathfrak{A}$ is also reduced. This implies that $\mathfrak{A} = \sqrt{\mathfrak{A}}$. By Theorem 1.16, we have the diagram

$$\begin{tikzcd} W \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\widetilde{\theta}"'] & & V \\ & Z \arrow[ur, "\widetilde{\pi}"'] & \end{tikzcd}$$

where $Z = \mathrm{Spec}(A/\mathfrak{A})$ is k-closed in V. Now, the image of $\widetilde{\varphi}$ is k-dense and $\mathrm{Im}\,\widetilde{\varphi} \subseteq Z$; so, $Z = V$, and then $\mathfrak{A} = (0)$, by the Nullstellensatz.

Conversely, assume that φ is injective and let Z be the closure of the image of $\widetilde{\varphi}$. Then $Z = \mathrm{Spec}(A/\mathfrak{A})$ for some radical ideal $\mathfrak{A}$. We have the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, B = W \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\widetilde{\theta}"'] & & V = \mathrm{Spec}\, A \\ & Z = \mathrm{Spec}(A/\mathfrak{A}) \arrow[ur, "\widetilde{\pi}"'] & \end{tikzcd}$$

and (by Theorem 1.16) we get the diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & B \\ & A/\mathfrak{A} \arrow[ur, "\theta"'] & \end{tikzcd}$$

But φ is injective and $\mathfrak{A} = \mathrm{Ker}\,\varphi$, by the first part of the proof of (2); so, $\mathfrak{A} = (0)$, and thus, $Z = V$. $\square$

1.4 Integral Morphisms, Products, Diagonal, Fibres

Let A, B be some k-algebras (not necessarily affine algebras) and let $\varphi\colon A \to B$ be a k-algebra homomorphism. Then, we can view B as an A-algebra.

Definition 1.13 We say that B *is integral over* A (or B *is an integral* A*-algebra*) if for every $b \in B$, there are $a_0, \ldots, a_{n-1} \in A$ such that

$$b^n + \varphi(a_{n-1})b^{n-1} + \cdots + \varphi(a_1)b + \varphi(a_0) = 0.$$

If V and W are affine k-varieties, with $V = \mathrm{Spec}\, A$ and $W = \mathrm{Spec}\, B$, and if $\widetilde{\varphi}\colon W \to V$ is a k-morphism (with corresponding k-algebra homomorphism $\varphi\colon A \to B$), then $\widetilde{\varphi}$ is an *integral* k*-morphism* (or W *is integral over* V) if B is integral over A.

Remark: Lots of morphisms $\theta\colon W \to V$ are not integral. For example, the composed map

$$\widetilde{\varphi}\colon V(XY = 1) \hookrightarrow \mathbb{A}^2 \xrightarrow{pr_1} \mathbb{A}^1$$

from a hyperbola to the affine line (via the first projection $pr_1\colon \mathbb{A}^2 \to \mathbb{A}^1$) is not integral. Indeed, the corresponding k-algebra homomorphism is

$$\varphi\colon k[X] \longrightarrow k[X,Y]/(XY-1)$$

given by

$$X \mapsto X.$$

However, Y is not integral over $k[X]$. Also note that the image $\mathbb{A}^1 - \{0\}$ of $\widetilde{\varphi}$ is not closed. This is the general situation. A morphism of affine k-varieties is almost never a closed map (i.e., maps closed sets to closed sets). However, for integral morphisms, we have the following theorem.

Theorem 1.19 *Let V, W be affine k-varieties and let $\widetilde{\varphi}\colon W \to V$ be a k-morphism. If $\widetilde{\varphi}$ is an integral k-morphism, then it is a closed map.*

Proof. Write $W = \operatorname{Spec} B$, $V = \operatorname{Spec} A$, and let W' be a k-closed subvariety in W

$$W' \hookrightarrow W \xrightarrow{\widetilde{\varphi}} V.$$

By Theorem 1.16, we have k-homomorphisms

$$A \longrightarrow B \longrightarrow B/\mathfrak{B},$$

where $\mathfrak{B}$ is a radical ideal. Since, by hypothesis, B is integral over A, and since $B/\mathfrak{B}$ is trivially integral over B, it follows that $B/\mathfrak{B}$ is integral over A, and we may assume that $W' = W$. We must show that $\widetilde{\varphi}(W)$ is k-closed in V. Let V' be the k-closure of $\operatorname{Im}\widetilde{\varphi}$. Then,

$$V' = \operatorname{Spec}(A/\mathfrak{A})$$

for some radical ideal $\mathfrak{A}$ and we get the diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr] & & B \\ & A/\mathfrak{A} \arrow[ur] & \end{tikzcd}$$

Since φ is integral, B is integral over $A/\mathfrak{A}$. Therefore, we may also assume that $V' = V$, and we are in the situation where $\widetilde{\varphi}\colon W \to V$ has dense image and is an integral morphism, and we must show that $\widetilde{\varphi}$ is surjective. Pick $\xi \in V$, so that

$$\mathfrak{P}(\xi) = \mathfrak{I}(\overline{\{\xi\}})$$

is a prime ideal. Since the map $\varphi\colon A \to B$ is injective (by Proposition 1.18 (2)) and B is integrally dependent on A, by the first Cohen-Seidenberg theorem (the "Lying over theorem", Zariski and Samuel [60], Theorem 3, Chapter V, Section 2, Bourbaki [7], Theorem 1, Chapter V, Section 2, or Atiyah and Macdonald [2], Chapter 5), there is a prime ideal $\mathfrak{P}$ of B with

$$\mathfrak{P}(\xi) = \mathfrak{P} \cap A.$$

This gives the commutative diagram

$$\begin{tikzcd}
B \arrow[r] & B/\mathfrak{P} \arrow[r, hook] & \mathrm{Frac}(B/\mathfrak{P}) \\
A \arrow[r] \arrow[u] & A/\mathfrak{P}(\xi) \arrow[r, "\xi"] \arrow[u] & \Omega
\end{tikzcd}$$

where $\mathrm{Frac}(B/\mathfrak{P})$ is the fraction field of $B/\mathfrak{P}$ and where the two vertical left arrows are injections. Since $A/\mathfrak{P}(\xi) \subseteq \mathrm{Frac}(B/\mathfrak{P})$, by the place extension theorem (Zariski and Samuel [61], Theorem 5′, Chapter VI, Section 4), we may extend ξ to a place η of $\mathrm{Frac}(B/\mathfrak{P})$ with values in Ω. However, B is integral over A, and it follows that $B/\mathfrak{P}$ is integral over $A/\mathfrak{P}(\xi)$. We will use this to show that $B/\mathfrak{P}$ lies in the domain of η.

If not, there is some $b \in B$ such that $\overline{b} \notin \mathrm{dom}\,\eta$, which implies that $1/\overline{b} \in \mathrm{dom}\,\eta$ and $\eta(1/\overline{b}) = 0$. However, $B/\mathfrak{P}$ is integral over $A/\mathfrak{P}(\xi)$, so we have some integral equation

$$\overline{b}^n + \overline{a}_{n-1}\overline{b}^{n-1} + \cdots + \overline{a}_1\overline{b} + \cdots + \overline{a}_0 = 0$$

with the $\overline{a_j} \in A/\mathfrak{P}(\xi)$. Dividing by $\overline{b}$, we get

$$1 + \overline{a}_{n-1}\left(\frac{1}{\overline{b}}\right) + \cdots + \overline{a}_1\left(\frac{1}{\overline{b}}\right)^{n-1} + \overline{a}_0\left(\frac{1}{\overline{b}}\right)^n = 0.$$

Now apply η. It follows that $1 = 0$, a contradiction. So, $B/\mathfrak{P} \subseteq \mathrm{dom}\,\eta$, as contended.

Now, the map $\widetilde{\varphi}\colon W \to V$ is of the form

$$\widetilde{\varphi}(w) = \langle F_1(w), \ldots, F_q(w)\rangle$$

where $\varphi(x_j) = F_j \in B$ and $A = k[x_1, \ldots, x_q]$. But, the map

$$B \longrightarrow B/\mathfrak{P} \xrightarrow{\eta} \Omega$$

corresponds to a point η of W; the value $\widetilde{\varphi}(\eta)$ is obtained by following the x_j's in the diagram

$$\begin{tikzcd}
B \arrow[r] & B/\mathfrak{P} \arrow[r, hook] & \mathrm{Frac}(B/\mathfrak{P}) \arrow[d, "\eta"] \\
A = k[x_1, \ldots, x_q] \arrow[r] \arrow[u] & A/\mathfrak{P}(\xi) \arrow[r, "\xi"] \arrow[u] & \Omega
\end{tikzcd}$$

which commutes. So,

$$x_j \mapsto \xi_j$$

by the lower line, and $\widetilde{\varphi}(\eta) = \xi$, which proves the surjectivity of $\widetilde{\varphi}$. $\square$

For use below, we need to review some basic categorical concepts. Detailed presentations can be found in Grothendieck and Dieudonné (EGA I) [30], Mac Lane [39], or Grothendieck [21]. Suppose that $\mathcal{C}$ is a category, we denote its set of objects by $\mathrm{Ob}(\mathcal{C})$, and for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the set of morphisms from A to B is $\mathrm{Mor}_{\mathcal{C}}(A, B)$ or $\mathrm{Hom}_{\mathcal{C}}(A, B)$. We often drop the subscript $\mathcal{C}$. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is denoted by an arrow, viz $f\colon A \to B$. Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^o$ has the same objects as $\mathcal{C}$ but has reversed morphisms (arrows), i.e.,

$$\mathrm{Hom}_{\mathcal{C}^o}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A),$$

or equivalently, $f\colon A \to B$ is a morphism of $\mathcal{C}^o$ iff $f\colon B \to A$ is a morphism of $\mathcal{C}$. For two categories $\mathcal{C}$ and $\mathcal{D}$, a *contravariant functor* or *cofunctor* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $F\colon \mathcal{C}^o \to \mathcal{D}$ (i.e., it is arrow-reversing). We let **Sets** denote the category of sets.

If $\mathcal{C}$ is a category, for every object X of $\mathcal{C}$ we can define a cofunctor $h_X\colon \mathcal{C} \to \mathbf{Sets}$ as follows: For a "test object" $T \in \mathcal{C}$,

$$h_X(T) = \mathrm{Hom}\,(T, X),$$

the set of all morphisms from T to X, and for any two objects $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and every morphism $f \in \mathrm{Hom}\,(Y, Z)$, the action of the functor h_X on arrows is the map

$$h_X(f)\colon \mathrm{Hom}\,(Z, X) \to \mathrm{Hom}\,(Y, X)$$

defined by

$$h_X(f)(g) = g \circ f$$

for all $g \in \mathrm{Hom}\,(Z, X)$. The functor h_X is also denoted by $\mathrm{Hom}\,(-, X)$.

Let $F\colon \mathcal{C} \to \mathbf{Sets}$ be a cofunctor. Is it of the form $h_X = \mathrm{Hom}\,(-, X)$ for some object $X \in \mathrm{Ob}(\mathcal{C})$? If so, F is said to be *representable by* X. More precisely, this means that F and $\mathrm{Hom}\,(-, X)$ are isomorphic functors. By definition, an isomorphism of functors means that there is a natural transformation, $\theta\colon h_X \to F$, i.e., there is a family (θ_T) of bijections

$$\theta_T\colon \mathrm{Hom}\,(T, X) \to F(T)$$

for every object T of $\mathcal{C}$ such that the following diagram commutes for every morphism $f\colon Y \to Z$ of $\mathcal{C}$:

$$\begin{tikzcd}
\mathrm{Hom}\,(Z, X) \arrow[r, "\theta_Z"] \arrow[d, "h_X(f)"'] & F(Z) \arrow[d, "F(f)"] \\
\mathrm{Hom}\,(Y, X) \arrow[r, "\theta_Y"'] & F(Y)
\end{tikzcd}$$

The isomorphism θ is uniquely determined by the element $x \in F(X)$ defined by

$$x = \theta_X(\mathrm{id}_X).$$

Indeed, setting $Z = X$, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}\,(X,X) \arrow[r, "\theta_X"] \arrow[d, "h_X(f)"'] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Hom}\,(Y,X) \arrow[r, "\theta_Y"'] & F(Y) \end{tikzcd}$$

and since $h_X(f)(\mathrm{id}_X) = \mathrm{id}_X \circ f = f$ (recall that $f\colon Y \to X$), we get

$$\theta_Y(h_X(f)(\mathrm{id}_X)) = F(f)(\theta_X(\mathrm{id}_X)),$$

that is,

$$\theta_Y(f) = F(f)(x),$$

which shows that θ_Y is completely determined for every object Y of $\mathcal{C}$.

Actually, the above diagram shows that there is a bijection between the set of all natural transformations $\theta\colon h_X \to F$ and $F(X)$ given by the map

$$\theta \mapsto \theta_X(\mathrm{id}_X).$$

The inverse map assigns to every $x \in F(X)$ the natural transformation θ defined by

$$\theta_T(f) = F(f)(x)$$

for all $T \in \mathrm{Ob}(\mathcal{C})$ and for all $f\colon T \to X$.

Consequently, to fix matters, we make the following definition for representability. The cofunctor F is *represented by the pair* (X, x) (where $x \in F(X)$) when the natural transformation $\theta\colon \mathrm{Hom}\,(-, X) \to F$ corresponding to x is an isomorphism of functors. It is easy to show that the pair, (X, x), representing a cofunctor F is unique up to (a unique) isomorphism

A similar treatment applies to the (covariant) functor $\mathrm{Hom}\,(X, -)\colon \mathcal{C} \to \mathbf{Sets}$ defined by

$$\mathrm{Hom}\,(X,-)(T) = \mathrm{Hom}\,(X,T)$$

for every "test object" $T \in \mathcal{C}$, and

$$\mathrm{Hom}\,(X,-)(f)(g) = f \circ g$$

for every $f \in \mathrm{Hom}\,(Y,Z)$ and every $g \in \mathrm{Hom}\,(X,Y)$. A (covariant) functor $F\colon \mathcal{C} \to \mathbf{Sets}$ is representable by some pair (X, x) with $X \in \mathcal{C}$ and $x \in F(X)$ when the natural transformation $\theta\colon \mathrm{Hom}\,(X,-) \to F$ corresponding to x is an isomorphism of functors.

Representable functors (and cofunctors) allow us to define products, coproducts, fibred products, fibred coproducts, in any category.

Let X, Y, Z be some objects in $\mathcal{C}$. Consider the cofunctor

$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, Y).$$

If this functor is representable, there is an object $P \in \mathcal{C}$ and an element in

$$\mathrm{Hom}_{\mathcal{C}}(P, X) \times \mathrm{Hom}_{\mathcal{C}}(P, Y),$$

i.e., a pair of maps $p_X \colon P \to X$ and $p_Y \colon P \to Y$ so that we have a bijection

$$\theta_T \colon \mathrm{Hom}_{\mathcal{C}}(T, P) \cong \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, Y),$$

via

$$\theta_T(f) = \langle p_X \circ f, p_Y \circ f \rangle.$$

In this case, the representing triple (P, p_X, p_Y) is called the *product of X and Y in $\mathcal{C}$*. It has the usual universal property of Cartesian products. We also denote P by $X \prod Y$.

If we consider the functor

$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(X, T) \times \mathrm{Hom}_{\mathcal{C}}(Y, T),$$

and if this functor is representable, the representing triple (C, i_X, i_Y) is called the *coproduct of X and Y in $\mathcal{C}$*. It has the usual universal property of disjoint sums. We also denote C by $X \coprod Y$.

We also have two categories $\mathcal{C}_Z$ and $\mathcal{C}^Z$ associated with Z, called *comma categories* or *slice categories*. The objects of $\mathcal{C}_Z$ are the pairs (T, θ_T) where $T \in \mathrm{Ob}(\mathcal{C})$ and $\theta_T \colon T \to Z$. A morphism $\Phi \colon (T, \theta_T) \to (S, \theta_S)$ is a morphism $\Phi \colon T \to S$ such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[rr, "\Phi"] \arrow[dr, "\theta_T"'] & & S \arrow[dl, "\theta_S"] \\ & Z & \end{tikzcd}$$

The category $\mathcal{C}^Z$ is defined by turning the arrows around, i.e., the objects are the pairs (T, θ_T) where $T \in \mathrm{Ob}(\mathcal{C})$ and $\theta_T \colon Z \to T$, etc. The category $\mathcal{C}_Z$ is called $\mathcal{C}$ *over* Z, and the category $\mathcal{C}^Z$ is called $\mathcal{C}$ *co-over* Z.

In $\mathcal{C}_Z$, if the product of (X, θ_X) and (Y, θ_Y) exists, it is called the *fibred product of X and Y over Z*, denoted by $X \prod_Z Y$. Similarly, in $\mathcal{C}^Z$, if the coproduct of (X, η_X) and (Y, η_Y) exists, it is called the *fibred coproduct of X and Y over Z*, denoted by $X \coprod^Z Y$.

Examples 1.7

(1) Let $\mathcal{C} = \mathbf{Sets}$. Then, $X \prod Y$ is the ordinary Cartesian product $X \times Y$ of X and Y with its projections, and $X \coprod Y$ is the disjoint union of X and Y with its injections. Also,

$$X \prod_Z Y = \{(x, y) \in X \times Y \mid \theta_X(x) = \theta_Y(y)\}$$

with the obvious maps:

$$\begin{tikzcd}
 & X \prod_Z Y \arrow[dl] \arrow[dr] & \\
X \arrow[dr, "\theta_X"'] & & Y \arrow[dl, "\theta_Y"] \\
 & Z &
\end{tikzcd}$$

(2) If $\mathcal{C} = \mathbf{Grp}$, the category of groups, or $\mathcal{C} = \mathbf{Ab}$, the category of abelian groups, then products, coproducts, fibred products, and fibred coproducts all exist (DX).

(3) If $\mathcal{C} = \mathbf{CRng}$, the category of commutative rings (with unit), then products, coproducts, fibred products, and fibred coproducts all exist (DX). For example, the product $A \prod B$ of two rings A, B is the set-theoretic product $A \times B$ with coordinatewise operations. For the coproduct, check that

$$A \coprod B = A \otimes_{\mathbb{Z}} B.$$

Given three commutative rings, the fibred coproduct $A \coprod^{C} B$ is given by

$$A \coprod^{C} B = A \otimes_C B,$$

and the fibred product $A \prod_C B$ by

$$A \prod_C B = \{(a,b) \in A \prod B \mid \theta_A(a) = \theta_B(b)\},$$

where $\theta_A \colon A \to C$ and $\theta_B \colon B \to C$.

(4) If $\mathcal{C} = \textbf{k-alg}$, the category of commutative k-algebras (where k is a field), then products, coproducts, fibred products, and fibred coproducts all exist (DX).

(5) If $\mathcal{C} = \textbf{k-affalg}$, the category of affine k-algebras (where k is a field), then products, coproducts, fibred products, and fibred coproducts all exist (DX). In each construction, we divide by the nilradical. For example

$$A \coprod B = (A \otimes_k B)/\mathfrak{N} = (A \otimes_k B)_{\mathrm{red}},$$

the product

$$A \prod B = A \times B,$$

as usual, and if C is an affine k-algebra with maps, $\eta_A \colon C \to A$ and $\eta_B \colon C \to B$, then

$$A \coprod^{C} B = (A \otimes_C B)/\mathfrak{N} = (A \otimes_C B)_{\mathrm{red}}.$$

The fundamental theorem implies the following proposition.

Proposition 1.20 *If V, W, Z are affine k-varieties and $\theta_V \colon V \to Z$, $\theta_W \colon W \to Z$, or $\eta_V \colon Z \to V$, $\eta_W \colon Z \to W$ are k-morphisms, the following objects all exist:*

$$
\begin{aligned}
V \prod W &= \operatorname{Spec}\left((k[V] \otimes_k k[W])_{\mathrm{red}}\right), \\
V \prod_Z W &= \operatorname{Spec}\left((k[V] \otimes_{k[Z]} k[W])_{\mathrm{red}}\right), \text{ and} \\
V \coprod W &= \operatorname{Spec}(k[V] \times k[W]), \\
V \overset{Z}{\coprod} W &= \operatorname{Spec}\left(k[V] \prod_{k[Z]} k[W]\right).
\end{aligned}
$$

It is interesting to consider the effect of extending the field k.

Definition 1.14 Let V be a k-variety and let K be a field such that $k \subseteq K \subseteq \Omega$. We define the K-variety $V \otimes_k K$ by

$$V \otimes_k K = \operatorname{Spec}\left((k[V] \otimes_k K)_{\mathrm{red}}\right).$$

(1) If K is not a finite extension over k, then $V \otimes_k K$ is **not** a k-variety.

(2) Even if K is finite algebraic extension of k, the structure of $V \otimes_k K$ as a K-variety may be different from the structure of V as a k-variety. For example, V may have $V(k) = \emptyset$, and yet $(V \otimes_k K)(K) \neq \emptyset$.

The following proposition shows that the points of the varieties $V \prod W$ and $V \prod_Z W$ are just what should be expected.

Proposition 1.21 *Given any k-varieties V, W, Z, we have*

$$(V \prod W)(\Omega) = V(\Omega) \times W(\Omega)$$

where $\times$ is the Cartesian product of sets, and

$$(V \prod_Z W)(\Omega) = V(\Omega) \prod_{Z(\Omega)} W(\Omega).$$

Proof. By definition

$$V(\Omega) = \operatorname{Hom}_{k-\mathrm{alg}}(k[V], \Omega) \quad \text{and} \quad W(\Omega) = \operatorname{Hom}_{k-\mathrm{alg}}(k[W], \Omega).$$

Because Ω is a field,

$$\operatorname{Hom}_{k-\mathrm{alg}}((k[V] \otimes_k k[W])_{\mathrm{red}}, \Omega) = \operatorname{Hom}_{k-\mathrm{alg}}(k[V] \otimes_k k[W], \Omega),$$

and then

$$\mathrm{Hom}_{k-\mathrm{alg}}(k[V]\otimes_k k[W],\Omega) = \mathrm{Hom}_{k-\mathrm{alg}}(k[V],\Omega)\times \mathrm{Hom}_{k-\mathrm{alg}}(k[W],\Omega) = V(\Omega)\times W(\Omega).$$

The proof of the other identity is similar. □

A more general notion of point is suggested by the fundamental theorem (Theorem 1.16).

Definition 1.15 Given any (commutative) k-algebra A, let

$$V(A) = \mathrm{Hom}_{k-\mathrm{alg}}(k[V], A),$$

and call $V(A)$ the *points of V with values in A*. If A is an affine k-algebra and $T = \mathrm{Spec}\, A$, then we write $V(T)$ instead of $V(A)$, sometimes. We call $V(T)$ the *T-valued points of V*, or *points of V with values in T*.

We would like to understand how V and $V\otimes_k K$ are similar and different. Here are some partial answers.

Proposition 1.22 *Let K and L be fields such that $k\subseteq K\subseteq L\subseteq \Omega$, and let V be an affine k-variety.*

(1) We have

$$(V\otimes_k K)(L) = V(L),$$

where $V\otimes_k K$ is viewed as a K-variety and V as a k-variety.

(2) If V is k-irreducible and the extension K/k is normal algebraic, then $V\otimes_k K$ is a finite union of k-conjugate K-irreducible varieties. Hence, in general, V is equidimensional over k iff $V\otimes_k K$ is equidimensional over K.

Proof. (1) Since L is a field, we have

$$\begin{aligned}(V\otimes_k K)(L) &= \mathrm{Hom}_{K-\mathrm{alg}}((k[V]\otimes_k K)_{\mathrm{red}}, L)\\ &= \mathrm{Hom}_{K-\mathrm{alg}}(k[V]\otimes_k K, L)\\ &= \mathrm{Hom}_{k-\mathrm{alg}}(k[V], L)\\ &= V(L).\end{aligned}$$

(2) Assume that V is k-irreducible. The K-variety $V\otimes_k K$ is a finite union of K-irreducible varieties

$$V\otimes_k K = V_1\cup\cdots\cup V_t.$$

Let ξ_j be a K-generic point of V_j. We have

$$\xi_j\in V_j(\Omega)\subseteq (V\otimes_k K)(\Omega) = V(\Omega),$$

and thus, ξ_j is a specialization of some ξ, where ξ is k-generic for V. Then, there is a surjection (identity on k)

$$\theta_j \colon k[\xi] \longrightarrow k[\xi_j]$$

for every j, $1 \leq j \leq t$. Hence,

$$\mathrm{tr.d}_k\, \xi_j \leq \mathrm{tr.d}_k\, \xi \tag{$*$}$$

for every j, $1 \leq j \leq t$. Now,

$$\xi \in V(\Omega) = (V \otimes_k K)(\Omega) = \bigcup_{j=1}^{t} V_j(\Omega).$$

For instance, assume that $\xi \in V_1(\Omega)$. Then, ξ is a K-specialization of ξ_1. Thus there is a surjection $\pi \colon K[\xi_1] \to K[\xi]$, and as a consequence,

$$\mathrm{tr.d}_K\, \xi \leq \mathrm{tr.d}_K\, \xi_1. \tag{$**$}$$

Since the extension K/k is algebraic, by ($**$), we get

$$\mathrm{tr.d}_k\, \xi \leq \mathrm{tr.d}_k\, \xi_1. \tag{$***$}$$

Then, by ($*$) and ($***$), we get

$$\mathrm{tr.d}_k\, \xi = \mathrm{tr.d}_k\, \xi_1.$$

Our previous work implies that θ_1 is a k-isomorphism. Hence, we get the k-surjections

$$\varphi_j \colon k[\xi_1] \xrightarrow{\theta_1^{-1}} k[\xi] \xrightarrow{\theta_j} k[\xi_j].$$

Consider $K(\xi_1) = \mathrm{Frac}(K[\xi_1])$ (the fraction field of $K[\xi_1]$). We have the following commutative diagram:

$$\begin{tikzcd}
K(\xi_1) \arrow[dr, "\widehat{\varphi_j}"] & \\
K[\xi_1] \arrow[u] & \Omega \\
k[\xi_1] \arrow[u] \arrow[r, "\varphi_j"] & k[\xi_j] \arrow[u]
\end{tikzcd}$$

By the place extension theorem, φ_j extends to a place $\widehat{\varphi_j}$, and we have

$$\widehat{\varphi_j}(\xi_1) = \xi_j.$$

Since K is algebraic over k and since k and K are fields, K is integral over k, and by the usual argument about integrality, $\widehat{\varphi_j}$ is defined on K. Since K/k is a normal extension, we have

$$\widehat{\varphi_j}(K) = K,$$

i.e., $\widetilde{\varphi_j}$, the restriction of $\widehat{\varphi_j}$ to K, is an automorphism of K. We have the diagram

$$\begin{tikzcd}
K(\xi_1) & \\
K[\xi_1] \arrow[u] \arrow[r, "\widehat{\varphi_j}"] & K[\eta] \subseteq \Omega \\
K \arrow[u] \arrow[r, "\widetilde{\varphi_j}"] & K \arrow[u]
\end{tikzcd}$$

where $\varphi_j(\xi_1) = \xi_j$, and where $\widehat{\varphi_j}(\xi) = \eta$ (η depending on j). Observe that the map

$$K[\eta] \xrightarrow{\widehat{\varphi_j}^{-1}} K[\xi_1] \xrightarrow{\varphi_j} K[\xi_j]$$

is a K-surjection, by construction. Therefore, $\xi_j \in \overline{\{\eta\}}$, and as η is a specialization of ξ, we get $\eta \in V \otimes_k K$. However,

$$V \otimes_k K = \bigcup_{j=1}^{t} V_j$$

where the decomposition is irredundant, and each V_j is K-irreducible, so we must have

$$V_j = \overline{\{\eta\}}.$$

Thus,

$$\mathrm{tr.d}_K\, \eta = \mathrm{tr.d}_K\, \xi_j;$$

So,

$$\mathrm{tr.d}_K\, \xi_1 = \mathrm{tr.d}_K\, \eta.$$

It follows that

$$\mathrm{tr.d}_K\, \eta = \mathrm{tr.d}_K\, \xi_i \tag{1.24}$$

for all i, $1 \leq i \leq t$. This shows that the components V_i have the same dimension. By (1.24), because the extension K/k is algebraic and φ_i is a k-surjection, φ_i is a k-isomorphism

$$\varphi_i \colon k[\xi_1] \to k[\xi_i]$$

for all i, $1 \leq i \leq t$. Hence, V_1 and V_i are k-conjugates. The last part is trivial. $\square$

The above Proposition suggests two kinds of questions.

1. Given a property (P) of k-varieties, does $V \otimes_k K$ possess (P) as a K-variety? This question is also phrased as: "(P) is stable under *base extension*." Generally, it is not hard.

2. Assume that $V \otimes_k K$ has the property (P) as a K-variety. Does V already have (P) as a k-variety? This question is also phrased as: "(P) is stable under *descent*." For example, if K/k is normal algebraic and $V \otimes_k K$ is K-irreducible, then V is k-irreducible. Also, from the proof of Proposition 1.22,
$$\dim_k V = \dim_K(V \otimes_k K).$$
Thus, the notion of dimension is stable under base extension and descent, by normal algebraic field extension.

Let (P) be a property of k-varieties. We say that V *is geometrically* (P) if $V \otimes_k \overline{k}$ has (P) as a $\overline{k}$-variety. For example, dimension is a geometric property.

However, note that V geometrically irreducible is different from V k-irreducible.

Proposition 1.23 *Let V be an affine k-variety and assume that V is geometrically irreducible. Then, for any field extension K/k, with $k \subseteq K \subseteq \Omega$, the K-variety $V \otimes_k K$ is K-irreducible.*

Proof. Let $\widetilde{K}$ be the smallest subfield of Ω containing K and $\overline{k}$ (i.e., the compositum of K and $\overline{k}$). We have
$$(V \otimes_k \overline{k}) \otimes_{\overline{k}} \widetilde{K} = V \otimes_k \widetilde{K}.$$
Next, we use the following result from commutative algebra: If A is an integral domain over k and k is algebraically closed, then for any extension K/k, $A \otimes_k K$ is again an integral domain (see Bourbaki (Algèbre) [6], Chapter 5, Section 17, No. 5, Corollary 3 to Proposition 9). Then, by considering the corresponding coordinate rings and the fact that $V \otimes_k \overline{k}$ is $\overline{k}$-irreducible, we see that
$$V \otimes_k \widetilde{K} = (V \otimes_k \overline{k}) \otimes_{\overline{k}} \widetilde{K}$$
is $\widetilde{K}$-irreducible. But $\widetilde{K}$ lies over K, and $\widetilde{K}$-irreducibility descends, so that $V \otimes_k K$ is K-irreducible. $\square$

Proposition 1.24 *Let V and W be irreducible affine k-varieties. The following properties hold.*

(1) If k is algebraically closed, then $V \prod_k W$ is k-irreducible.

(2) If k is algebraically closed, then
$$\mathfrak{I}_k(V \prod_k W) = (\mathfrak{I}_k(V), \mathfrak{I}_k(W)),$$
where $\mathfrak{I}_k(V) \subseteq k[X_1, \ldots, X_q]$; $\mathfrak{I}_k(W) \subseteq k[Y_1, \ldots, Y_r]$; $\mathfrak{I}_k(V \prod_k W) \subseteq k[X_1, \ldots, X_q, Y_1, \ldots, Y_r]$; and where $(\mathfrak{I}_k(V), \mathfrak{I}_k(W))$ is the ideal generated by $\mathfrak{I}_k(V)$ and $\mathfrak{I}_k(W)$.

(3) If k is algebraically closed and x, y are respectively k-generic points in V, W, then $\langle x, y\rangle$ is k-generic in $V \prod_k W$.

(4) If k is arbitrary,

$$\dim_k(V \prod_k W) = \dim_k(V) + \dim_k(W).$$

Proof. (1) By definition,

$$k[V \prod_k W] = (k[V] \otimes_k k[W])_{\mathrm{red}}.$$

However, $k[V]$ is an integral domain, k is algebraically closed, and $k[W] \subseteq k(W)$. By the fact about tensor products stated during the the proof of Proposition 1.23, $k[V] \otimes_k k[W]$ is an integral domain, and $V \prod_k W$ is k-irreducible.

(2) Since k is algebraically closed, $k[V] \otimes_k k[W]$ has no nilpotents, and so,

$$\mathrm{Spec}(k[V] \otimes_k k[W]) = V \prod_k W.$$

Now, we have the exact sequences

$$0 \longrightarrow \mathfrak{I}_k(V) \longrightarrow k[X_1, \ldots, X_q] \longrightarrow k[V] \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{I}_k(W) \longrightarrow k[Y_1, \ldots, Y_r] \longrightarrow k[W] \longrightarrow 0.$$

By tensoring the first exact sequence with $\mathfrak{I}_k(W)$ and the second exact sequence with $\mathfrak{I}_k(V)$, we get

$$0 \longrightarrow \mathfrak{I}_k(V) \otimes_k \mathfrak{I}_k(W) \longrightarrow k[X_1, \ldots, X_q] \otimes_k \mathfrak{I}_k(W) \longrightarrow k[V] \otimes_k \mathfrak{I}_k(W) \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{I}_k(V) \otimes_k \mathfrak{I}_k(W) \longrightarrow \mathfrak{I}_k(V) \otimes_k k[Y_1, \ldots, Y_r] \longrightarrow \mathfrak{I}_k(V) \otimes_k k[W] \longrightarrow 0.$$

Similarly, we can tensor the exact sequences with $k[Y_1, \ldots, Y_r]$, resp. $k[X_1, \ldots, X_q]$, and with $k[W]$, resp. $k[V]$. This gives the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{I}_k(V) \otimes_k \mathfrak{I}_k(W) \arrow[r] \arrow[d] & k[X_i\text{'s}] \otimes_k \mathfrak{I}_k(W) \arrow[r] \arrow[d] & k[V] \otimes_k \mathfrak{I}_k(W) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{I}_k(V) \otimes_k k[Y_j\text{'s}] \arrow[r] \arrow[d] & k[X_i\text{'s}] \otimes_k k[Y_j\text{'s}] \arrow[r] \arrow[d] & k[V] \otimes_k k[Y_j\text{'s}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{I}_k(V) \otimes_k k[W] \arrow[r] \arrow[d] & k[X_i\text{'s}] \otimes_k k[W] \arrow[r] \arrow[d] & k[V] \otimes_k k[W] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A simple diagram chase shows that (2) holds.

(3) The affine k-algebras $k[V]$ and $k[W]$ are finitely generated, and

$$k[x_1, \ldots, x_q] \cong k[X_1, \ldots, X_q]/\mathfrak{I}_k(V) \quad \text{and} \quad k[y_1, \ldots, y_r] \cong k[Y_1, \ldots, Y_r]/\mathfrak{I}_k(W).$$

The result then follows from (2).

(4) Expressing V and W in terms of irredundant decompositions, we have

$$V = \bigcup_{j=1}^{s} V_j \quad \text{and} \quad W = \bigcup_{l=1}^{t} W_l.$$

Then

$$V \prod_k W = \bigcup_{j=1,l=1}^{s,t} V_j \prod_k W_l.$$

Thus, it is enough to assume that V and W are k-irreducible. We also have irredundant decompositions

$$V \otimes_k \overline{k} = \bigcup_{j=1}^{s} \widetilde{V_j}$$

where the $\widetilde{V_j}$ are $\overline{k}$-irreducible and all have the same dimension, and similarly,

$$W \otimes_k \overline{k} = \bigcup_{l=1}^{t} \widetilde{W_l}$$

where the $\widetilde{W_l}$ are $\overline{k}$-irreducible and all have the same dimension. Then

$$(V \otimes_k \overline{k}) \prod_{\overline{k}} (W \otimes_k \overline{k}) \cong (V \prod_k W) \otimes_k \overline{k},$$

and

$$(V \prod_k W) \otimes_k \overline{k} = \bigcup_{j=1,l=1}^{s,t} \widetilde{V}_j \prod_k \widetilde{W}_l.$$

Thus, the argument is reduced to the case of k-irreducible varieties where the field is algebraically closed. Then, because the algebras involved are integral domains and because k is algebraically closed, it is clear that

$$\mathrm{tr.d}_k(k[V] \otimes_k k[W]) = \mathrm{tr.d}_k(k[V]) + \mathrm{tr.d}_k(k[W]). \qquad \square$$

Proposition 1.25 *Let V and W be affine k-varieties and let $\varphi\colon V \to W$ be a k-morphism. The following properties hold.*

(1) The fibred product $V \prod_W V$ is closed in $V \prod_k V$.

(2) The diagonal, Δ, is closed in $V \prod_W V$, and hence, in $V \prod_k V$. In fact, the map

$$\Delta_{V/W}\colon V \to V \prod_W V$$

(the diagonal map) is a closed immersion (i.e., it an isomorphism onto a closed subvariety, namely, the diagonal).

Proof. (1) We have

$$V \prod_k V = \mathrm{Spec}((k[V] \otimes_k k[V])_{\mathrm{red}}) = \mathrm{Spec}\, A,$$

and

$$V \prod_W V = \mathrm{Spec}((k[V] \otimes_{k[W]} k[V])_{\mathrm{red}}) = \mathrm{Spec}\, B.$$

The kernel of the map from A to B is the ideal $\mathfrak{I}$ generated by the elements of the form

$$i(w) \otimes 1 - 1 \otimes i(w),$$

where $i\colon k[W] \to k[V]$ is the k-algebra homomorphism corresponding to φ. Now, $B \cong A/\mathfrak{I}$, so that $\mathfrak{I}$ defines $V \prod_W V$ as a subvariety of $V \prod_k V$.

(2) Consider the map ρ of k-algebras

$$\rho\colon k[V] \otimes_{k[W]} k[V] \longrightarrow k[V]$$

defined via

$$\xi \otimes \eta \mapsto \xi\eta.$$

Observe that if $\sum_i \xi_i \otimes \eta_i \in \operatorname{Ker}\rho$, then

$$\begin{aligned}
\sum_i \xi_i \otimes \eta_i &= \sum_i \xi_i \otimes \eta_i - 1 \otimes \sum_i \xi_i\eta_i \\
&= \sum_i (\xi_i \otimes \eta_i - 1 \otimes \xi_i\eta_i) \\
&= \sum_i (\xi_i \otimes 1 - 1 \otimes \xi_i)(1 \otimes \eta_i),
\end{aligned}$$

which implies that the kernel of the map ρ is the ideal $\mathfrak{J}$ of $k[V] \otimes_{k[W]} k[V]$ generated by the elements $\xi \otimes_{k[W]} 1 - 1 \otimes_{k[W]} \xi$. Letting $C = k[V] \otimes_{k[W]} k[V]$, we have

$$\begin{aligned}
\operatorname{Hom}_{k-\mathrm{alg}}(C, \Omega) &= \operatorname{Hom}_{k-\mathrm{alg}}(C/\mathfrak{N}_C, \Omega) \\
&= (V \prod_W V)(\Omega),
\end{aligned}$$

since Ω is a field. Now, $\mathfrak{J}$ corresponds to the diagonal, as should be clear from the above (DX). Therefore, the diagonal is closed in $V \prod_W V$, since it is defined by an ideal, and the map ρ is a closed immersion. $\square$

Remark: If V and W are geometrically irreducible, then so is $V \prod_k W$ (DX).

We now consider fibres of k-morphisms. Given a k-morphism $\varphi\colon V \to W$ of affine k-varieties, for any $w \in W$ $(= W(\Omega))$, we have the set-theoretic fibre

$$\varphi^{-1}(w) = \{v \in V \mid \varphi(v) = w\}.$$

Then, V is the disjoint union

$$V = \bigcup_{w \in W} \varphi^{-1}(w).$$

If each fibre $\varphi^{-1}(w)$ is an affine variety, then V will be decomposed as a family of algebraic varieties indexed by another algebraic variety. When W and the fibres have a simpler structure than V, this yields a fruitful way of studying the structure of V. But, how can one view $\varphi^{-1}(w)$ as an affine variety? Here is the answer.

Given an affine k-variety V, for any $\xi \in V$, recall that

$$\mathfrak{P}(\xi) = \{f \in k[X_1, \ldots, X_q] \mid f(\xi) = 0\}.$$

Proposition 1.26 *Let $\varphi\colon V \to W$ be a morphism of affine k-varieties and let $w \in W$. Write*

$$\kappa(w) = \operatorname{Frac}(k[W]/\mathfrak{P}(w)),$$

and call it the residue field of W at w. Then, in a natural way, the fibre $\varphi^{-1}(w)$ is an affine $\kappa(w)$-variety. In fact,

$$\varphi^{-1}(w) = \operatorname{Spec}\left((k[V] \otimes_{k[W]} \kappa(w))_{\mathrm{red}}\right).$$

Proof. Consider
$$\mathrm{Spec}\left((k[V]\otimes_{k[W]}\kappa(w))_{\mathrm{red}}\right).$$
The set of points of this $\kappa(w)$-affine variety is
$$\mathrm{Hom}_{\kappa(w)-\mathrm{alg}}((k[V]\otimes_{k[W]}\kappa(w))_{\mathrm{red}},\Omega),$$
where Ω is a $\kappa(w)$-algebra because: The point $w\in W(\Omega)$ corresponds to a k-algebra homomorphism
$$h_w\in \mathrm{Hom}_{k-\mathrm{alg}}(k[W],\Omega),$$
and the kernel of h_w is $\mathfrak{P}(w)$. So, h_w induces a homomorphism (also denoted by h_w)
$$h_w\colon k[W]/\mathfrak{P}(w)\to\Omega,$$
which extends uniquely to a homomorphism
$$h_w\colon \kappa(w)\to\Omega,$$
and this makes Ω a $\kappa(w)$-algebra. The field Ω is also a $k[W]$-algebra via the homomorphism
$$k[W]\longrightarrow k[W]/\mathfrak{P}(w)\overset{h_w}{\hookrightarrow}\Omega.$$
Since Ω is a field, we have
$$\mathrm{Hom}_{\kappa(w)-\mathrm{alg}}((k[V]\otimes_{k[W]}\kappa(w))_{\mathrm{red}},\Omega)=\mathrm{Hom}_{k[W]-\mathrm{alg}}(k[V],\Omega).$$
If $\xi\in\mathrm{Hom}_{k[W]-\mathrm{alg}}(k[V],\Omega)$, then

(a) $\xi\colon k[V]\to\Omega$, and

(b) The diagram
$$\begin{tikzcd} k[V] \arrow[rr, "\xi"] \arrow[dr, "\widetilde{\varphi}"'] & & \Omega \\ & k[W] \arrow[ur, "h_w"'] & \end{tikzcd}$$
commutes.

However, (a) says that $\xi\in V(\Omega)$, and (b) says that $\xi\in\varphi^{-1}(w)$. Therefore, as sets,
$$\varphi^{-1}(w)=\mathrm{Spec}\left((k[V]\otimes_{k[W]}\kappa(w))_{\mathrm{red}}\right).$$
The rest is obvious. $\square$

Corollary 1.27 *If w and w' are k-conjugate points of W, then $\varphi^{-1}(w)$ and $\varphi^{-1}(w')$ are k-conjugate affine varieties.*

Proof. The k-algebra $k[V]$ is integral over $k[W]$ and finitely generated as a $k[W]$-algebra. A standard result of commutative algebra implies that $k[V]$ is a finite algebra over $k[W]$ (i.e., $k[V]$ is a finitely generated $k[V]$-module, see Atiyah and Macdonald [2], Corollary 5.2, Chapter 5, Zariski and Samuel [60], Theorem 1, Chapter V, or Bourbaki [7], Chapter V). $\square$

Corollary 1.28 *Let $\varphi\colon V \to W$ be an integral morphism of affine k-varieties. The following properties hold.*

(1) The affine variety V is a finite W-variety i.e, $k[V]$ is a finite $k[W]$-algebra.

(2) The fibres $\varphi^{-1}(w)$ are finite sets for all $w \in W$, i.e., $\dim_{\kappa(w)}(\varphi^{-1}(w)) = 0$.

Proof. By the proof of $k[V]$ is a finite $k[W]$-algebra and Corollary 1.27, $k[V] \otimes_{k[W]} \kappa(w)$ is a finite $\kappa(w)$-module. Then,

$$\mathrm{tr.d}_{\kappa(w)}\,(k[V] \otimes_{k[W]} \kappa(w)) = 0,$$

which means that the dimension of the fibre is 0. However, affine varieties of dimension 0 are finite sets. $\square$

A morphism $\varphi\colon V \to W$ satisfying condition (2) of Corollary 1.28 is called a *quasi-finite* morphism.

Note that quasi-finite does **not** imply finite. For example, if $V = \mathrm{Spec} k[X,Y](XY-1) \subseteq \mathbb{A}^2$, $W = \mathrm{Spec}\, k[X] = \mathbb{A}^1$, and $\varphi\colon V \to W$, the first projection from $\mathbb{A}^2$ to $\mathbb{A}^1$, then we observe that the map is not integral (since the element Y is not integral over the ring $k[X]$).

Definition 1.16 Given a topological space (for example, an affine variety) V, we say that a set Z is *locally closed in V* if

$$Z = U \cap W$$

where U is open and W is closed.

Observe that open and closed sets in a variety are locally closed. Let $Z_i = U_i \cap W_i$, $i = 1, 2$. Then,

$$Z_1 \cap Z_2 = U_1 \cap U_2 \cap W_1 \cap W_2,$$

so that $Z_1 \cap Z_2$ is locally closed. Thus, any finite intersection of locally closed sets is locally closed.

If $Z = U \cap W$, then $Z^c = U^c \cup W^c$, where U^c is closed and W^c is open. It follows that the Boolean algebra generated by the open and closed sets is just the set of finite unions of locally closed sets. Finite unions of locally closed sets are called *constructible sets*. We have the following important theorem.

Theorem 1.29 *Let V and W be affine k-varieties and let $\varphi\colon V \to W$ be a k-morphism. Then:*

(1) If $\varphi(V)$ is dense in W, there is some nonempty k-open set U in W so that

$$U \subseteq \varphi(V) \subseteq W.$$

(2) (Chevalley) The image of φ is a constructible set in W.

Proof. (1) Assume that $\varphi(V)$ is k-dense in W. Let V' be any k-irreducible component of V. Then, $\varphi(V')$ is k-irreducible in W, and thus, $\widetilde{W} = \overline{\varphi(V')}$ is again k-irreducible and closed in W. Let

$$W = \bigcup_{j=1}^{t} W_j,$$

where the W_j are the irredundant components of W. Then

$$\widetilde{W} = \bigcup_{j=1}^{t} \widetilde{W} \cap W_j,$$

and since $\widetilde{W}$ is k-irreducible, there is some j, $1 \leq j \leq t$, such that $\widetilde{W} = \widetilde{W} \cap W_j$, i.e.,

$$\widetilde{W} \subseteq W_j.$$

But, if

$$V = \bigcup_{i=1}^{s} V_i$$

is an irredundant decomposition of V, we showed that for every i, $1 \leq i \leq s$, there is some $j = j(i)$ so that

$$\varphi(V_i) \subseteq W_{j(i)}.$$

However,

$$\varphi(V) = \varphi\left(\bigcup_{i=1}^{s} V_i\right) = \bigcup_{i=1}^{s} \varphi(V_i).$$

Therefore,

$$W = \overline{\varphi(V)} = \overline{\bigcup_{i=1}^{s} \varphi(V_i)} = \bigcup_{i=1}^{s} \overline{\varphi(V_i)} \subseteq \bigcup_{i=1}^{s} \overline{W_{j(i)}} = \bigcup_{i=1}^{s} W_{j(i)} \subseteq \bigcup_{j=1}^{t} W_j = W,$$

and the inclusions are all equalities. Since the decompositions are irredundant, the $W_{j(i)}$ run over all the W_j's and, by denseness, $\varphi(V_i)$ is dense in $W_{j(i)}$.

Assume that the theorem (1) holds when V is k-irreducible (so is W, since $W = \overline{\varphi(V)}$). Then, for every i, there is some k-open subset $U_i \subseteq W_{j(i)}$ so that

$$U_i \subseteq \varphi(V_i) \subseteq W_{j(i)}.$$

If $C_i = W_{j(i)} - U_i$, then C_i is closed in $W_{j(i)}$, which implies that C_i is closed in W. The image $\varphi(V)$ misses at most

$$C = \bigcup_{i=1}^{s} C_i,$$

which is closed. Therefore, $U = C^c$ is a nonempty k-open contained in $\varphi(V)$.

Therefore, we may assume that V is k-irreducible. As $\varphi(V)$ is k-dense in W, we know from part (2) of Proposition 1.18 that $k[W] \hookrightarrow k[V]$ is an inclusion. Letting $r = \mathrm{tr.d}_{k[W]}k[V]$, we pick some transcendence base $\xi_1, \ldots, \xi_r$ ($\xi_j \in k[V]$) over $k[W]$, so that $k[V]$ is algebraic over $k[W][\xi_1, \ldots, \xi_r]$. Since

$$k[W][\xi_1, \ldots, \xi_r] \cong k[W] \otimes_k k[\xi_1, \ldots, \xi_r],$$

the map

$$k[W] \hookrightarrow k[W][\xi_1, \ldots, \xi_r] \hookrightarrow k[V] \tag{$*$}$$

is just the map

$$\widetilde{\varphi}\colon k[W] \hookrightarrow k[W] \otimes_k k[\xi_1, \ldots, \xi_r] \hookrightarrow k[V].$$

Reading the above geometrically, we get the map

$$\varphi\colon V \xrightarrow{\varphi_1} W \prod_k \mathbb{A}^r \xrightarrow{pr_1} W.$$

Since each $\eta \in k[V]$ is algebraic over $k[W \prod_k \mathbb{A}^r]$, we have equations

$$a_0(\xi_1, \ldots, \xi_r)\eta^s + a_1(\xi_1, \ldots, \xi_r)\eta^{s-1} + \cdots + a_s(\xi_1, \ldots, \xi_r) = 0,$$

where the coefficients $a_j(\xi_1, \ldots, \xi_r)$ are functions over W, and thus, depend on $w \in W$, but we omit w for simplicity of notation. If we multiply by $a_0(\xi_1, \ldots, \xi_r)^{s-1}$ and let $\zeta = a_0(\xi_1, \ldots, \xi_r)\eta$, we get

$$\zeta^s + b_1(\xi_1, \ldots, \xi_r)\zeta^{s-1} + \cdots + b_s(\xi_1, \ldots, \xi_r) = 0.$$

Therefore, for every $\eta \in k[V]$, there is some $\alpha \in k[W \prod_k \mathbb{A}^r]$ so that

$$\zeta = \alpha\eta$$

is integral over $k[W \prod_k \mathbb{A}^r]$. Since $k[V]$ is finitely generated, there exist $\eta_1, \ldots, \eta_t$ so that

$$k[V] = k[W \prod_k \mathbb{A}^r][\eta_1, \ldots, \eta_t],$$

and each η_j comes with its corresponding α_j and $\alpha_j\eta_j$ is integral over $k[W \prod_k \mathbb{A}^r]$. Let

$$b = \prod_{j=1}^{t} \alpha_j(\xi_1, \ldots, \xi_s) \in k[W \prod_k \mathbb{A}^r].$$

Consider the ring $k[W \prod_k \mathbb{A}^r]_b$ and the corresponding affine variety

$$\operatorname{Spec} k[W \prod_k \mathbb{A}^r]_b = U_1,$$

also denoted by $(W \prod_k \mathbb{A}^r)_b$. We have $U_1 \subseteq W \prod_k \mathbb{A}^r$, and on U_1, b and all the α_j's are invertible. Let us look at $\widehat{\varphi_1}(b) \in k[V]$, where $\widehat{\varphi_1}\colon k[W \prod_k \mathbb{A}^r] \to k[V]$ is the k-algebra homomorphism associated with the k-morphism $\varphi_1\colon V \to W \prod_k \mathbb{A}^r$. Then, we get

$$V_{\widehat{\varphi_1}(b)} \xrightarrow{\varphi_1} (W \prod_k \mathbb{A}^r)_b = U_1.$$

Since each α_j is invertible, on $V_{\widehat{\varphi_1}(b)}$, each η_j is integral over $k[U_1]$. But $V_{\widehat{\varphi_1}(b)}$ is generated by the η_j's; so $V_{\widehat{\varphi_1}(b)}$ is integral over $(W \prod_k \mathbb{A}^r)_b$. And therefore, the image of the morphism

$$V_{\widehat{\varphi_1}(b)} \xrightarrow{\varphi_1} U_1 \tag{$\dagger$}$$

is closed (see Theorem 1.19). Consequently, ($\dagger$) is a surjection of varieties, and we find

$$U_1 = (W \prod_k \mathbb{A}^r)_b = \varphi_1(V_{\widehat{\varphi_1}(b)}) \subseteq \varphi_1(V).$$

Even though U_1 is a nonempty open, we still need to show that there is some nonempty open $U \subseteq W$ such that $U \subseteq pr_1(U_1)$. For then, we will have $U \subseteq \varphi(V)$. Now, $b \in k[W \prod_k \mathbb{A}^r]$ means that b can be expressed by a formula of the form

$$b = \sum_{(\beta)} b_{(\beta)}(w)\xi^{(\beta)},$$

where (β) denotes the multi-index $(\beta) = (\beta_1, \ldots, \beta_r)$, $x^{(\beta)} = \xi_1^{\beta_1} \cdots \xi_r^{\beta_r}$, and $b_{(\beta)} \in k[W]$. Let

$$U = \{w \in W \mid \exists(\beta),\, b_{(\beta)}(w) \neq 0\}.$$

The set U is a k-open set in W. If $w \in U$, since b is a polynomial in the ξ_j's which is not identically null, there is some (β) such that $b_{(\beta)}(w) \neq 0$. Now, Ω and $\overline{k}$ are infinite, so there are some elements $t_1, \ldots, t_r \in \Omega$ (or $\overline{k}$) such that $b(w, t_1, \ldots, t_r) \neq 0$. However, $(w, t_1, \ldots, t_r) \in W \prod_k \mathbb{A}^r$ and $b(w, t_1, \ldots, t_r) \neq 0$, so that $(w, t_1, \ldots, t_r) \in U_1$ and $pr_1(w, t_1, \ldots, t_r) = w$. Therefore, $U \subseteq pr_1(U_1)$, which concludes the proof of (1).

(2) By a familiar argument (using irredundant decompositions), we may assume that the affine varieties are k-irreducible. We proceed by induction on $r = \dim(V)$.

When $r = 0$, the fact that V is irreducible implies that V is a finite set of k-conjugate points, and since the image of a finite set is finite, $\varphi(V)$ is constructible.

Assume that the claim holds for $r - 1$. As we said earlier, we may restrict our attention to the case where V is k-irreducible. Let $\widetilde{W} = \overline{\mathrm{Im}\ \varphi}$. If $\mathrm{Im}\ \varphi$ is constructible in $\widetilde{W}$, then

$$\mathrm{Im}\ \varphi = U_1 \cap \widetilde{W_1} \cup \ldots \cup U_n \cap \widetilde{W_n},$$

where U_j is open in $\widetilde{W}$, and $\widetilde{W_j}$ is closed in $\widetilde{W}$, which implies that $\widetilde{W_j}$ is closed in W. By definition of the relative topology, there are some open sets U_j' in W so that

$$\widetilde{W} \cap U_j' = U_j.$$

Then, we have

$$\begin{aligned} \mathrm{Im}\ \varphi &= (\widetilde{W} \cap U_1') \cap \widetilde{W_1} \cup \cdots \cup (\widetilde{W} \cap U_n') \cap \widetilde{W_n} \\ &= \widetilde{W} \cap (U_1' \cap \widetilde{W_1} \cup \cdots \cup U_n' \cap \widetilde{W_n}) \\ &= U_1' \cap \widetilde{W_1} \cup \cdots \cup U_n' \cap \widetilde{W_n}, \end{aligned}$$

a constructible set in W. As a consequence, we may assume that $W = \widetilde{W}$, i.e., that $\mathrm{Im}\ \varphi$ is dense in W. By (1), there is some nonempty open subset U of W such that $U \subseteq \varphi(V)$. Let $T = \varphi^{-1}(U)$. This is an open subset of V and moreover, $\varphi(T) = U$. Let $Z = V - T$. The set Z is k-closed in V, and thus

$$\dim Z < \dim V,$$

and by induction, Chevalley's result holds for Z. But then,

$$\varphi(V) = \varphi(Z) \cup \varphi(T) = \varphi(Z) \cup U,$$

and since $\varphi(Z)$ is constructible and U is open, $\varphi(Z) \cup U$ is also constructible. $\square$

We will be able to promote many of our results by introducing a generalization of affine varieties, called abstract varieties.

Definition 1.17 An *abstract k-variety* is a topological space X together with a collection of locally defined functions on X to Ω so that the following condition holds: For every $x \in X$, there is some open subset U with $x \in U$ so that U and the induced set of locally defined functions from X is k-isomorphic to $\mathrm{Spec}\, A$ for some affine k-algebra A. Given two abstract k-varieties X, Y, a *k-morphism* is a topological map $\varphi\colon X \to Y$ such that for every locally defined function g on Y, $\varphi^*(g)$ $(= g \circ \varphi)$ is a locally defined function on X.

Of course, abstract varieties form a category having the category of affine k-varieties as a full subcategory. But, many results true for affine varieties will fail for abstract varieties if they are "too big." Here, the concept of bigness has to do with the fact that an abstract variety might not be a finite union of affine varieties. Therefore, the concept of quasi-compactness introduced in the discussion immediately following Definition 1.2 should prove useful. We will generally assume it, and, in the interest of brevity, we tend to omit "abstract" in "abstract varieties." The following terminology will also be needed.

Definition 1.18 Given two k-varieties X, Y, a k-morphism $\varphi\colon X \to Y$ is an *affine morphism* if for every $y \in Y$, there is some open subset U with $y \in U$, where U is affine and $\varphi^{-1}(U)$ is affine. We say that φ is an *integral morphism* if φ is an affine morphism and $\varphi^{-1}(U)$ is integral over U (U as above, depends on $y \in Y$.) A morphism φ is a *finite-type morphism* if for every $y \in Y$, there is some open subset U with $y \in U$, where U is affine and $\varphi^{-1}(U)$ is quasi-compact, and lastly, φ is a *finite morphism* if it is an affine morphism and if $k[\varphi^{-1}(U)]$ is a finite $k[U]$-algebra (U as above, depends on $y \in Y$.)

Note that if φ is integral, then φ is a closed map and the fibres are finite (reduce to Y affine, statement local on Y).

Remarks:

(1) Each quasi-compact k-variety is a finite union (unique if irredundant) of irreducible k-varieties.

(2) An open subset U of a k-variety X is a k-variety. Indeed, $X = \bigcup_\alpha X_\alpha$ where each X_α is affine. So,
$$U = \bigcup_\alpha U \cap X_\alpha,$$
where each $U_\alpha = U \cap X_\alpha$ is an open subset of the affine k-variety X_α. Thus, we are reduced to case where X is an affine k-variety. In this case, since the $(X_\alpha)_{b_\alpha}$ form a basis of the topology (where $b_\alpha \in k[X_\alpha]$) and since an affine k-variety is quasi-compact, we have
$$U = \bigcup_{j=1}^{t} (X_\alpha)_{b_\alpha^j},$$
which shows that U is a finite union of affine k-varieties.

(3) If $Y \subseteq X$ and Y is closed, with X a k-variety, then Y is a k-variety. This is because $X = \bigcup_\alpha X_\alpha$ where each X_α is affine, and so
$$Y = \bigcup_\alpha Y \cap X_\alpha$$
where $Y \cap X_\alpha$ is k-closed in the affine k-variety X_α, and thus, of the form $\operatorname{Spec} A_\alpha/\mathfrak{A}_\alpha$ where $X_\alpha = \operatorname{Spec} A_\alpha$. As a consequence, Y is also a k-variety.

(4) If $Y \subseteq X$ and Y is locally closed, with X a k-variety, then Y is a k-variety (DX).

(5) An open subset U of an affine (or projective) k-variety X is a k-variety. However, such a k-variety U is neither affine nor projective in general. Such varieties are called *quasi-affine* or *quasi-projective* varieties, respectively, to distinguish them from open subsets of arbitrary varieties.

(6) If V is a k-irreducible variety, any nonempty open subset U of V is k-dense in V.

We can now generalize the previous theorem as follows. Call a morphism $\varphi\colon X \to Y$ of k-varieties a *dominating* or *dominant* morphism if $\varphi(X)$ is dense in Y.

Theorem 1.30 *Let X and Y be k-varieties with X quasi-compact, and let $\varphi\colon X \to Y$ be a k-morphism. Then:*

(1) If X is irreducible and φ is dominating there is some nonempty open set U in X so that

$$U \subseteq \varphi(X) \subseteq Y.$$

(2) (Chevalley) The image of φ is a constructible set in Y.

Proof. (1) We have $Y = \bigcup_\alpha Y_\alpha$ for some affine open sets Y_α. Let $X_\alpha = \varphi^{-1}(Y_\alpha)$. Since X is quasi-compact and each X_α is a union of affine open subsets, each X_α is quasi-compact. But, $\varphi(X)$ is dense in Y, so each $\varphi(X_\alpha)$ is dense in Y_α. Thus, we may assume that $Y = Y_\alpha$ is affine. Each affine open, Z, of X is dense (since X is irreducible), and thus, $\varphi(Z)$ is dense in Y. Consequently, we are reduced to the case where X is affine; and here, Theorem 1.29 applies.

(2) Since X is quasi-compact, $X = \bigcup_{j=1}^t X_j$, where each X_j is an affine open, and we may assume that X is affine. By the same argument as before, we may assume that $\varphi(X)$ is dense in Y. We get (2) by applying (1). $\square$

1.5 Further Readings

The material covered in this chapter belongs to the repertoire of classical algebraic geometry, basically as laid out by Oscar Zariski and André Weil, and rests on commutative algebra for its foundations. Both Zariski and Weil, independently, grew increasingly uncomfortable with the lack of rigor found in the otherwise beautiful work of the Italian school of algebraic geometry of the beginning of the twentieth century. One can only admire their tremendous accomplishments, providing perfectly rigorous foundations for algebraic geometry, mostly by developing the appropriate tools of commutative algebra. The next bold step, already implicitly anticipated by Weil, was taken by Jean-Pierre Serre, with the introduction of sheaves and cohomology. The next leap, of course, was taken by Alexander Grothendieck, with the introduction of schemes.

Readers will find other presentations of the material of this chapter (some more complete, some less) in the references listed below: Shafarevich [53], Chapter 1, Section 1–3, and Dieudonné [13], Chapter 1–2, are the closest in spirit; Hartshorne [33], Chapter 1, Section 1; Mumford [43], Chapter 1, Section 1–3; Mumford [42], Chapter 1, Section 1; Fulton [17], Chapter 1, 2, 6; Perrin [45], Chapter 1; Kendig [37]; Kempf [36], Chapter 1–3. An excellent tutorial on algebraic geometry can also be found in Danilov's article in [11], and Volume I of Ueno [56] is worth consulting. Finally, Smith et al [55] give a very elementary but delightful introduction to algebraic geometry.

Chapter 2

Dimension, Local Theory, Projective Geometry

2.1 Dimension Theory

In this section, we will finally show that if X is an irreducible k-variety and Y is a maximal closed irreducible subvariety in X, then $\dim(Y) = \dim(X) - 1$. Our proof will use a fundamental result due to Emmy Noether, the *normalization theorem*. As a preview, consider the affine variety $V \subseteq \mathbb{A}^2$ (an hyperbola)

$$V = \mathrm{Spec}(k[X,Y]/(XY-1)).$$

As we know, the restriction of the first projection, $pr_1\colon \mathbb{A}^2 \to \mathbb{A}^1$, to V is not an integral morphism. However, if we rotate the axes by $\pi/4$, we get a surjective integral map.

This is a general fact. Indeed, Noether's normalization theorem says that every irreducible affine k-variety of dimension r in $\mathbb{A}^n$ is a (finite) branched covering of $\mathbb{A}^r$.

Theorem 2.1 *Suppose V is an irreducible affine k-variety in $\mathbb{A}^n$, and let $\dim_k(V) = r$. Then, there is some change of coordinates in $\mathbb{A}^n$ so that the projection of $\mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^{n-r}$ onto $\mathbb{A}^r$ yields a surjective integral morphism*

$$V \hookrightarrow \mathbb{A}^n \xrightarrow{pr} \mathbb{A}^r.$$

If k is infinite, we may arrange the change of coordinates $(x_1,\ldots,x_n) \mapsto (y_1,\ldots,y_r)$ to be linear and if $k(V) = k[x_1,\ldots,x_n]$ is separably generated over k, then the $y_i \in k[x_1,\ldots,x_n]$ may be chosen so that $k[x_1,\ldots,x_n]$ is separably generated over $k[y_1,\ldots,y_r]$.

Proof. Since $k[V] = k[X_1,\ldots,X_n]/\mathfrak{A}$ for some radical ideal $\mathfrak{A}$, we have $k[V] = k[x_1,\ldots,x_n]$, the homomorphic image of the polynomial ring $k[X_1,\ldots,X_n]$. If $r = n$, then $V \cong \mathbb{A}^n$, and we are done. Thus, we may asume that $r < n$. We prove the theorem by induction on $n > r$.

The case $n = 1$, $r = 0$, is trivial. Assume that the theorem holds up to $n-1$. We need to show that there exist $y_2, \ldots, y_n \in k[x_1, \ldots, x_n]$ so that

$$k[y_2, \ldots, y_n] \hookrightarrow k[x_1, \ldots, x_n]$$

is an integral extension of rings, separable in the separable case (i.e., when the transcendence base is separable). Then, we use the induction hypothesis applied to $k[y_2, \ldots, y_n]$ and that integrality and separability is preserved under composition, to obtain the desired theorem.

Since $r < n$, we may assume that x_1 is algebraically dependent on $x_2, \ldots, x_n$. Therefore, there is a nontrivial equation

$$\sum_{(\alpha)} c_{(\alpha)} x^{(\alpha)} = 0,$$

where, as before, $(\alpha) = (\alpha_1, \ldots, \alpha_n)$. Choose some integers $m_2, \ldots, m_n$, to be determined later, and set

$$y_j = x_j - x_1^{m_j},$$

for $j = 2, \ldots, n$. Then, $x_j = y_j + x_1^{m_j}$, and we get

$$\sum_{(\alpha)} c_{(\alpha)} x_1^{\alpha_1} (y_2 + x_1^{m_2})^{\alpha_2} \cdots (y_n + x_1^{m_n})^{\alpha_n} = 0,$$

which can be written as

$$\sum_{(\alpha)} c_{(\alpha)} x_1^{(\alpha)\cdot(m)} + G(x_1, y_2, \ldots, y_n) = 0, \tag{$*$}$$

where

$$(m) = (1, m_2, \ldots, m_n),$$

and G involves the y_j's and x_1 at lower degree than the maximum of the $(\alpha)\cdot(m)$'s. If we show that the $(\alpha)\cdot(m)$'s are all distinct, then $(*)$ is an integral dependence of x_1 on $y_2, \ldots, y_n$. Since each x_j ($x_j = y_j + x_1^{m_j}$, $2 \leq j \leq n$) is also integral over $y_2, \ldots, y_n$, this implies that $k[x_1, \ldots, x_n]$ is integral over $k[y_2, \ldots, y_n]$. Now, the (α)'s are distinct, so we can consider the differences

$$(\delta^{(\lambda)}) = (\delta_1^{(\lambda)}, \ldots, \delta_n^{(\lambda)}) = (\alpha) - (\alpha')$$

of any two distinct (α)'s for all possible choices of the (α)'s (except that we do not include $(\alpha') - (\alpha)$ if we have included $(\alpha) - (\alpha')$). Assume that there are t such $\delta^{(\lambda)}$'s. Let $T_2, \ldots, T_n$ be some independent indeterminates, and consider

$$H(T_2, \ldots, T_n) = \prod_{j=1}^{t} (\delta^{(j)} \cdot \overrightarrow{T}),$$

where $\overrightarrow{T} = (1, T_2, \ldots, T_n)$. We have

$$H(T_2, \ldots, T_n) = \prod_{j=1}^{t} (\delta_1^{(j)} + \delta_2^{(j)} T_2 + \cdots + \delta_n^{(j)} T_n).$$

Since all the $\delta^{(j)}$'s are nonzero, H is not the null polynomial. Furthermore, the coefficients of H are integers. But then, it is well-known that there are some non-negative integers $m_2, \ldots, m_n$ such that

$$H(m_2, \ldots, m_n) \neq 0.$$

These are the required integers!

Remark: We can also find the non-negative integers $m_2, \ldots, m_n$ as follows. Let d be a non-negative integer larger than any of the components of a vector (α) such that $c_{(\alpha)} \neq 0$. Then, take

$$(m_2, \ldots, m_n) = (d, d^2, \ldots, d^{n-1}).$$

Let us now consider the case where k is infinite. Again, x_1 depends algebraically on $x_2, \ldots, x_n$, and in the separable case, we pick a separating transcendence base (by MacLane's theorem). Write the minimal polynomial for x_1 over $k(x_2, \ldots, x_n)$ as

$$P(U, x_2, \ldots, x_n) = 0.$$

We can assume that the coefficients of $P(U, x_2, \ldots, x_n)$ are in $k[x_2, \ldots, x_n]$, so that the polynomial $P(U, x_2, \ldots, x_n)$ is the result of substituting $U, x_2, \ldots, x_n$ for $X_1, X_2, \ldots, X_n$ in some non-zero polynomial $P(X_1, \ldots, X_n)$ with coefficients in k. Perform the linear change of variables

$$y_j = x_j - a_j x_1, \tag{$\dagger$}$$

for $j = 2, \ldots, n$, and where $a_2, \ldots, a_n \in k$ will be determined later. Since $x_j = y_j + a_j x_1$, it is sufficient to prove that x_1 is integral (and separable in the separable case) over $k[y_2, \ldots, y_n]$. The minimal equation $P(x_1, x) = 0$ (abbreviating $P(x_1, x_2, \ldots, x_n)$ by $P(x_1, x)$) becomes

$$P(x_1, y_2 + a_2 x_1, \ldots, y_n + a_n x_1) = 0,$$

which can be written as

$$P(x_1, y) = x_1^q f(1, a_2, \ldots, a_n) + Q(x_1, y_2, \ldots, y_n) = 0, \tag{$**$}$$

where $f(X_1, X_2, \ldots, X_n)$ is the highest degree form of $P(X_1, \ldots, X_n)$ and q its degree, and Q contains terms of degree lower than q in x_1. If we can find some a_j's such that $f(1, a_2, \ldots, a_n) \neq 0$, then we have an integral dependence of x_1 on $y_2, \ldots, y_n$; thus, the x_j's are integrally dependent on $y_2, \ldots, y_n$, and we finish by induction. In the separable case, we need the minimal polynomial for x_1 to have a simple root, i.e.,

$$\frac{dP}{dx_1}(x_1, y) \neq 0.$$

We have

$$\frac{dP}{dx_1}(x_1, y) = \frac{\partial P}{\partial x_1}(x_1, x) + a_2 \frac{\partial P}{\partial x_2}(x_1, x) + \cdots + a_n \frac{\partial P}{\partial x_n}(x_1, x).$$

But this is a linear form in the a_j's which is not identically zero, since it takes for $a_2 = \cdots = a_n = 0$ the value

$$\frac{\partial P}{\partial x_1}(x_1, x) \neq 0,$$

x_1 being separable over $k(x_2, \ldots, x_n)$. Thus, the equation

$$\frac{\partial P}{\partial x_1}(x_1, x) + a_2 \frac{\partial P}{\partial x_2}(x_1, x) + \cdots + a_n \frac{\partial P}{\partial x_n}(x_1, x) = 0$$

defines an affine hyperplane, i.e., the translate of a (linear) hyperplane. But then,

$$\frac{dP}{dx_1}(x_1, x) \neq 0$$

on the complement of a hyperplane, that is, an infinite open subset of $\mathbb{A}^{n-1}$, since k is infinite. On this infinite set where $\frac{dP}{dx_1}(x_1, x) \neq 0$, we can find $a_2, \ldots, a_n$ so that $f(1, a_2, \ldots, a_n) \neq 0$, which concludes the proof. $\square$

To use Noether's theorem, we need the following definitions.

Definition 2.1 A k-variety V is *separated* if it has an affine open covering $V = \bigcup_\alpha V_\alpha$ so that

(a) $V_\alpha \cap V_\beta$ is affine.

(b) $k[V_\alpha \cap V_\beta] = (k[V_\alpha] \otimes_{k[V]} k[V_\beta])_{\mathrm{red}}$, where $k[V]$ denotes the set of global holomorphic functions on V, i.e., $k[V] = \mathrm{Hom}_k(V, \mathbb{A}^1)$.

We will show later that the conditions of Definition 2.1 are equivalent to the fact that the diagonal Δ is closed in $V \prod_k V$. If we had used the product topology on $V \prod_k V$, this would be equivalent to V being Hausdorff. However, the Zariski topology (or k-topology) in $V \prod_k V$ is *not* the product topology, and thus, this does not imply that V is Hausdorff. Nevertheless, separatedness is the algebro-geometric substitute of being Hausdorff. Note that every k-affine variety is separated.

Definition 2.2 An irreducible k-variety V is *normal* if for every $v \in V$, there is some irreducible affine k-open subset U containing v so that $k[U]$ is integrally closed in $k(U)$ (where $k(U)$ denotes the fraction field of $k[U]$).

For example, $\mathbb{A}^n$ is normal, and when projective space is defined, it will be clear that it is normal.

Note that if V is a k-irreducible variety (not necessarily affine), the integer, $\dim V$, makes sense. For, $V_\alpha = \bigcup_\alpha V_\alpha$, where each V_α is an open irreducible affine subvariety. But each $V_\alpha \cap V_\beta$ is nonempty open and dense in both V_α and V_β, by irreducibility. Hence,

$$\dim(V_\alpha) = \dim(V_\alpha \cap V_\beta) = \dim(V_\beta);$$

so, $\dim(V)$ makes sense.

Proposition 2.2 *Let V, W be k-irreducible and separated k-varieties, with W normal. If $\dim(V) = \dim(W)$ and $\varphi\colon V \to W$ is a finite surjective morphism, then φ establishes a surjective map from the collection of closed k-irreducible varieties of V to those of W. In this map, maximal irreducible subvarieties of V map to maximal irreducible subvarieties of W, inclusion relations are preserved, dimensions are preserved, and no subvariety of V, except V itself, maps onto W.*

Proof. Let W_α be an affine open in W, then so is $V_\alpha = \varphi^{-1}(W_\alpha)$ in V, because φ is affine, since it is a finite morphism. If Z is an irreducible closed variety in V, then $Z_\alpha = Z \cap V_\alpha$ is irreducible in V_α since Z_α is dense in Z. Thus, we may assume that V and W are affine. Let $A = k[W]$ and $B = k[V]$. Since φ is finite and surjective, we see that A is contained in B and B is a finite A-algebra. Both A, B are integral domains, both are Noetherian, A is integrally closed, and no nonzero element of A is a zero divisor in B. These are the conditions for applying the Cohen-Seidenberg theorems I, II, and III. By Cohen-Seidenberg I (Zariski and Samuel [60], Theorem 3, Chapter V, Section 2, or Atiyah and Macdonald [2], Chapter 5), there is a surjective correspondence

$$\mathfrak{P} \mapsto \mathfrak{P} \cap A$$

between prime ideals of B and prime ideals of A, and thus, there is a surjective correspondence between irreducible subvarieties of V and their images in W.

Consider a maximal irreducible variety Z in V. Then, its corresponding ideal is a minimal prime ideal $\mathfrak{P}$. Let $\mathfrak{p} = \mathfrak{P} \cap A$, the ideal corresponding to $\varphi(Z)$. If $\varphi(Z)$ is not a maximal irreducible variety in W, then $\mathfrak{p}$ is not a minimal prime, and thus, there is some prime ideal $\mathfrak{q}$ of A such that

$$\mathfrak{q} \not\subseteq \mathfrak{p},$$

where the inclusion is strict. By Cohen-Seidenberg III (Zariski and Samuel [60], Theorem 6, Chapter V, Section 3, or Atiyah and Macdonald [2], Chapter 5), there is some prime ideal $\mathfrak{Q}$ in B such that

$$\mathfrak{Q} \not\subseteq \mathfrak{P}$$

and $\mathfrak{q} = \mathfrak{Q} \cap A$, contradicting the fact that $\mathfrak{P}$ is minimal. Thus, φ takes maximal irreducible varieties to maximal irreducible varieties.

Finally, by Cohen-Seidenberg II (Zariski and Samuel [60], Corollary to Theorem 3, Chapter V, Section 2, or Atiyah and Macdonald [2], Chapter 5), inclusions are preserved, and since φ is finite, dimension is preserved. The rest is clear. $\square$

We can finally prove the fundamental fact on dimension.

Proposition 2.3 *Let V be a separated k-irreducible variety and W a separated maximal k-irreducible subvariety of V. Then,*

$$\dim_k(W) = \dim_k(V) - 1.$$

Proof. We may assume that V and W are affine (using open covers, as usual). By Noether's normalization theorem (Theorem 2.1), there is a finite surjective morphism $\varphi\colon V \to \mathbb{A}^r$, where $r = \dim_k(V)$. However, $\mathbb{A}^r$ is normal, and by Proposition 2.2, we may assume that $V = \mathbb{A}^r$. Let W be a maximal irreducible k-variety in $\mathbb{A}^r$. It corresponds to a minimal prime ideal $\mathfrak{P}$ of $k[T_1, \ldots, T_r]$, which is a UFD. As a consequence, since $\mathfrak{P}$ is a minimal prime, it is equal to some principal ideal, i.e., $\mathfrak{P} = (g)$, where g is not a unit. Without loss of generality, we may assume that g involves T_r.

Now, the images $t_1, \ldots, t_{r-1}$ of $T_1, \ldots, T_{r-1}$ in $k[T_1, \ldots, T_r]/\mathfrak{P}$ are algebraically independent over k. Otherwise, there would be some polynomial $f \in k[T_1, \ldots, T_{r-1}]$ such that

$$f(t_1, \ldots, t_{r-1}) = 0.$$

But then, $f(T_1, \ldots, T_{r-1}) \in \mathfrak{P} = (g)$. Thus,

$$f(T_1, \ldots, T_{r-1}) = \alpha(T_1, \ldots, T_r) g(T_1, \ldots, T_r),$$

contradicting the algebraic independence of $T_1, \ldots, T_r$. Therefore, $\dim_k(W) \geq r - 1$, but since we also know that $\dim_k(W) \leq r - 1$, we get $\dim_k(W) = r - 1$. $\square$

Definition 2.3 Let V and W be separated k-irreducible varieties with $W \subseteq V$. We define the *codimension*, $\mathrm{codim}(W; V)$, of V in W by

$$\mathrm{codim}(W; V) = \dim_k(V) - \dim_k(W).$$

Given a chain of irreducible varieties

$$W = V_h \not\subseteq V_{h-1} \not\subseteq \cdots \not\subseteq V_0 = V,$$

where the inclusions are *strict*, we define the *height of W in V* to be the length h of a maximal such chain.

We have $\dim(V_h) = \dim(W)$, and by Proposition 2.3,

$$\dim(V_{h-1}) = \dim(W) + 1,$$

and thus, we get

$$\dim(V_0) = \dim(W) + h = \dim(V),$$

so that

$$h = \mathrm{codim}(W; V)$$

is the height of W in V.

Corollary 2.4 *(Combinatorial interpretation of dimension) If V is a separated k-irreducible variety, then* $\dim(V)$ *is the maximum of the chain length of all chains of the form*

$$V_h \not\subseteq V_{h-1} \not\subseteq \cdots \not\subseteq V_0 = V,$$

where V_h is any finite set of k-conjugate points. The codimension, $\mathrm{codim}(W; V)$*, is equal to the height of W in V.*

Certain subvarieties of a k-variety are particularly simple. Among these are the hypersurfaces, which are defined by:

Definition 2.4 Let V be a k-variety and f a nonconstant global holomorphic function on V. The *hypersurface cut out by f* is the subvariety

$$\{\xi \in V \mid f(\xi) = 0\}.$$

Corollary 2.5 *Let V be an affine k-irreducible variety and let W be a closed subvariety of V. The following statements are equivalent:*

(1) W is a maximal k-irreducible subvariety of V.

(2) W is a k-irreducible subvariety of V and $\dim(W) = \dim(V) - 1$.

(3) If $k[V]$ is a UFD, then (1) and (2) are both equivalent to the fact that W is an irreducible hypersurface in V.

Proof. The equivalence of (1) and (2) follows from Corollary 2.4. Assume that $k[V]$ is a UFD, and let W be a maximal k-irreducible subvariety of V. Then, W corresponds to a prime ideal $\mathfrak{P}$ of $k[V]$. Since $k[V]$ is a UFD, $\mathfrak{P}$ is a principal ideal, so that $\mathfrak{P} = (f)$. Then

$$W = \mathrm{Spec}\, k[V]/\mathfrak{P} = \mathrm{Spec}\, k[V]/(f) = \{\xi \in V \mid f(\xi) = 0\},$$

the hypersurface cut out by f.

Conversely, assume that W is the irreducible hypersurface cut out by f. Let Z be a maximal k-irreducible subvariety of V such that $W \subseteq Z$. By definition, $\mathfrak{I}(W) = (f)$, so that

(f) is prime, which implies that f is irreducible in $k[V]$. The variety Z corresponds to a minimal prime ideal $\mathfrak{P}$ of $k[V]$, and since $k[V]$ is a UFD, the ideal $\mathfrak{I}(Z) = \mathfrak{P} = (g)$. Since (g) is prime, g is irreducible in $k[V]$. But $W \subseteq Z$, so we get

$$\mathfrak{I}(Z) \subseteq \mathfrak{I}(W).$$

As a consequence, $g = \alpha f$, but as f and g are both irreducible, α must be a unit and

$$\mathfrak{I}(Z) = \mathfrak{I}(W),$$

which implies that $Z = W$. $\square$

In linear algebra, we know how the concept of dimension behaves with respect to intersection of subspaces. The intersection of two subspaces is never empty as 0 is common to both; this needs not happen with varieties. So, in investigating how our notion of dimension behaves with respect to intersections, some hypotheses of nontriviality must be assumed. The principal theorem is the *intersection dimension theorem*. The proof of this theorem uses another important theorem known as the *hypersurface section theorem*.

Theorem 2.6 *(Intersection Dimension theorem in $\mathbb{A}^n$) Let V and W be k-irreducible closed subvarieties of $\mathbb{A}^n$, with $\dim(V) = r$, $\dim(W) = s$, and assume that $V \cap W \neq \emptyset$. Then, each k-irreducible component of $V \cap W$ has dimension at least $r + s - n$.*

Proof. We may assume that k is algebraically closed, since dimension is stable under base extension. Consider the embedding

$$V \prod_k W \hookrightarrow \mathbb{A}^n \prod_k \mathbb{A}^n = \mathbb{A}^{2n},$$

and further consider $(V \prod_k W) \cap \Delta$. We have the commutative diagram

$$\begin{tikzcd}
\mathbb{A}^n \arrow[r, "\Delta"] & \mathbb{A}^{2n} \\
V \cap W \arrow[u] \arrow[r, "\Delta"'] & (V \prod_k W) \cap \Delta \arrow[u]
\end{tikzcd}$$

where the lower map is an isomorphism (DX). We need to prove the theorem for $(V \prod_k W) \cap \Delta$ in $\mathbb{A}^{2n}$. But

$$\Delta = \bigcap_{j=1}^{n} H_j,$$

where H_j is the hyperplane

$$H_j = \{(\xi, \eta) \mid \xi_j = \eta_j\}.$$

Hence, our theorem comes down to the following important statement (by applying it *seriatum* to the various intersections $(V \prod_k W) \cap H_1 \cap H_2 \cdots \cap H_j$; adding one j at a time). $\square$

Theorem 2.7 *(Hypersurface section theorem) If V is an irreducible subvariety of $\mathbb{A}^n$ and H is a hypersurface of $\mathbb{A}^n$, with V not contained in H, then every nonempty irreducible component of $V \cap H$ (hypersurface section) has codimension* 1 *in V.*

Proof. This proof makes use of a major theorem, Krull's principal ideal's theorem. There is a more elementary (but longer) proof using Noether's normalization. However, such a proof does not apply to a more general setting (schemes). This is why we make use of this rather "heavy" theorem. The k-algebra $k[V]$ is a homomorphic image of $k[T_1, \ldots, T_n]$, and H is given by the equation $f(T_1, \ldots, T_n) = 0$. Since V is not contained in H, the restriction, $\overline{f} = f \restriction V$, of f to V, i.e., the image of f in $k[V]$, is not identically zero. Furthermore,

$$k[V \cap H] = k[V]/(\overline{f}).$$

The irreducible components of $V \cap H$ correspond to the isolated prime ideals in $k[V]$ of the principal ideal $(\overline{f})$. By Krull's principal ideal theorem (Zariski and Samuel [60], Theorem 29, Chapter IV, Section 14), the isolated primes of $(\overline{f})$ are minimal primes in $k[V]$. Thus, these ideals correspond to maximal irreducible subvarieties of V. But we know that the dimension of these irreducible components is $\dim(V) - 1$, i.e., of codimension 1. $\square$

Corollary 2.8 *In an affine variety, each hypersurface is equidimensional (of codimension one).*

If V and W are contained in some affine variety Z not $\mathbb{A}^q$, the intersection dimension theorem (Theorem 2.6) may be **false**. Indeed, consider the following example.

Example 2.1 Let Z be the quadric cone in $\mathbb{A}^4$ given by

$$x_1x_2 - x_3x_4 = 0.$$

The cone Z has dimension 3 (it is a hypersurface). Let V be the plane

$$x_1 = x_3 = 0,$$

and W the plane

$$x_2 = x_4 = 0.$$

Observe that $V, W \subseteq Z$. Since V and W have dimension 2 and $V \cap W \neq \emptyset$, the intersection dimension theorem would yield $\dim(V \cap W) \geq 2 + 2 - 3 = 1$. However $V \cap W = \{(0, 0, 0, 0)\}$, the origin, whose dimension is zero!

What is the problem? The answer is that near 0, $\Delta \cap Z$ is not the locus of three equations, rather of four equations.

Again, in linear algebra, when we have a linear map of vector spaces, we can say what the dimension of the fibre of each point in the image is. The corresponding theorem in our case is necessarily more complicated, but generically, it proves to be the same statement.

Theorem 2.9 *(Fibre dimension theorem) Let V and W be irreducible k-varieties and let $\varphi\colon V \to W$ be a surjective morphism. Write $n = \dim(V)$ and $m = \dim(W)$. Then:*

(1) For all $w \in W$,

$$\dim_{\kappa(w)}\varphi^{-1}(w) \geq n - m.$$

(2) If w is k-generic, then

$$\dim_{\kappa(w)}\varphi^{-1}(w) = n - m.$$

(3) There is a nonempty open $U \subseteq W$ so that $\dim_{\kappa(w)}\varphi^{-1}(w) = n - m$, for all $w \in U$.

Proof. Statements (1), (2), and (3) are local on W; so, we may assume that W is affine. Say $W \subseteq \mathbb{A}^M$, with $M \geq m$. Since dimension is invariant under base extension, we may assume that $w \in W(k)$ and k is algebraically closed.

(1) Pick $\xi \in W$, $\xi \neq w$. There is a hyperplane $H \subseteq \mathbb{A}^M$ such that $w \in H$ and $\xi \notin H$. Thus, W is not contained in H. [In fact, if $L = 0$ is a linear form defining H, L^d $(d \geq 1)$ is a form of degree d defining a hypersurface of degree d, call it H'; W is not contained in H', but $w \in H'$.] By the hypersurface section theorem (Theorem 2.7), the dimension of any irreducible component of $W \cap H$ is $\dim(W) - 1$. Pick, $\xi_1, \ldots \xi_s$ in each of the components of $W \cap H$. Then, there is a hyperplane $\widetilde{H}$ so that $\xi_j \notin \widetilde{H}$ for all j, $1 \leq j \leq s$, but $w \in \widetilde{H}$. Then, by Theorem 2.7 again, the dimension of any component of $W \cap H \cap \widetilde{H}$ is $\dim(W) - 2$. Using this process, we get some hyperplanes $H_1, H_2, \ldots, H_m$ such that

$$w \in \bigcap_{j=1}^{m} H_j,$$

and if we write

$$W_j = W_{j-1} \cap H_j,$$

with $W_1 = W \cap H_1$, we get a chain

$$W \supset W_1 \supset W_2 \supset \cdots \supset W_m.$$

Here, $w \in W_m$, and

$$\dim(W_j) = \dim(W) - j.$$

Thus, the linear forms $L_1, \ldots, L_m$ associated with the H_j's define W_m in W and

$$\dim(W_m) = 0.$$

Consequently, W_m is a finite set of k-points:

$$W_m = \{w_1 = w, w_2, \ldots, w_t\}.$$

Let

$$U_0 = W - \{w_2, \ldots, w_t\},$$

it is a k-open dense subset of W. We can replace W by U_0, and thus, we may assume that $W_m = \{w\}$. We have $\varphi\colon V \to W$, and so, each $\widehat{\varphi}(L_j)$ is a function on V (where $\widehat{\varphi}\colon k[W] \to k[V]$). But $\varphi^{-1}(w)$ is the locus in V cut out by $\widehat{\varphi}(L_1), \ldots, \widehat{\varphi}(L_m)$; so, by the hypersurface intersection theorem (Theorem 2.7) (we may assume that $V \subseteq \mathbb{A}^N$), we get

$$\dim \varphi^{-1}(w) \geq \dim(V) - m \geq n - m.$$

Observe that (3) implies (2), since every generic point is in any nonempty open.

(3) Again, we may assume that W is affine. For any nonempty affine open, V_0, in V, as V_0 is dense in V, we see that $\varphi(V_0)$ is dense in W. Moreover, any nonempty intersection of V_0 with a fibre is dense in the fibre. Thus, we may assume that V and W are affine and that $\varphi(V)$ is dense in W. We can also assume that k is algebraically closed (since dimension is stable under base extension). We know that $k[W] \subseteq k[V]$, since $\varphi(V)$ is dense in W, and

$$\mathrm{tr.d}_{k[W]}\, k[V] = n - m.$$

We have $V \hookrightarrow \mathbb{A}^N$, with $k[V] = k[v_1, \ldots, v_N]$, and $W \hookrightarrow \mathbb{A}^M$, with $k[W] = k[w_1, \ldots, w_M]$, for some suitable M, N. We may also assume that $v_1, \ldots, v_{n-m}$ form a transcendence base of $k[V]$ over $k[W]$. Then, each v_j, $j = n - m + 1, \ldots, N$ is algebraic over $k[W][v_1, \ldots, v_{n-m}]$, and there are polynomials $G_j(T_1, \ldots, T_{n-m}, T)$ (coefficients in $k[W]$) so that

$$G_j(v_1, \ldots, v_{n-m}, v_j) = 0.$$

Pick $g_j(T_1, \ldots, T_{n-m})$ as the coefficient of highest degree of G_j in T. The set

$$\{w \in W \mid g_j(w) = 0\} = W_j$$

is a k-closed subset of W. Let

$$U = W - \bigcup_{j=n-m+1}^{N} W_j.$$

The open U is nonempty, since W is irreducible. On U, the polynomial G_j is not identically zero as a polynomial in $T_1, \ldots, T_{n-m}, T$, yet

$$G_j(v_1, \ldots, v_{n-m}, v_j) = 0.$$

Thus, v_j is algebraically dependent on $v_1, \ldots, v_{n-m}$ over $k[U]$. Letting $\widetilde{v_j}$ denote the restriction of v_j to $\varphi^{-1}(w)$ (i.e., the image of v_j in $k[V] \otimes_{k[W]} \kappa(w)$), where $w \in U$, we see that $\widetilde{v_j}$ is also algebraically dependent on $\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}$. Now,

$$\varphi^{-1}(w) = \mathrm{Spec}\, \kappa(w)[\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}],$$

which implies that

$$\mathrm{tr.d}_{\kappa(w)}\, \kappa(w)[\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}] \leq n - m.$$

However, by (1),

$$\dim(\varphi^{-1}(w)) \geq n - m,$$

and so,

$$\dim(\varphi^{-1}(w)) = n - m. \qquad \square$$

Corollary 2.10 *Assume that we are in the same situation as in the fibre dimension theorem (Theorem 2.9). Let*

$$W_l = \{w \in W \mid \dim(\varphi^{-1}(w)) \geq l\}.$$

Then, W_l is k-closed in W, i.e., the function

$$w \mapsto \dim(\varphi^{-1}(w))$$

is upper semi-continuous on W. Hence, W possesses a stratification

$$W = U_0 \cup U_1 \cup \cdots \cup U_n,$$

where $U_j = W_j - W_{j+1}$ is locally closed and $\dim(\varphi^{-1}(w)) = j$ for all $w \in U_j$.

Proof. The proof is by induction on $\dim(W)$. The case where $\dim(W) = 0$ is easy. Given W, Theorem 2.9 part (3) implies that there is some open set $U \subseteq W$ and some W_l ($l \geq 1$ and l minimum) so that

$$W_l \subseteq Z = W - U.$$

Also, Z is closed and we have some irredundant decomposition

$$Z = \bigcup_{j=1}^{t} Z_j,$$

where Z_j is irreducible and strictly contained in W. Then, $\dim(Z_j) < \dim(W)$, and we can apply the induction hypothesis to the maps $\varphi_j \colon \varphi^{-1}(Z_j) \to Z_j$, the details are left as an exercise (DX). $\square$

Note that the dimension of the fibres may jump, as shown by the following example (which is nothing but the "blowing-up" at a point in $\mathbb{A}^2$).

Example 2.2 Let $W = \mathbb{A}^2$, and consider $\mathbb{A}^2 \prod \mathbb{P}^1$. We use w_1, w_2 as cooordinates on W, and ξ_1, ξ_2 as homogeneous coordinates on $\mathbb{P}^1$. Write V for the subvariety of $\mathbb{A}^2 \prod \mathbb{P}^1$ given by the equation

$$w_1\xi_2 = w_2\xi_1.$$

This equation is homogeneous in ξ_1, ξ_2, and it defines a closed subvariety of $\mathbb{A}^2 \prod \mathbb{P}^1$. We get a morphism $\varphi \colon V \to W$ *via*

$$\varphi \colon V \hookrightarrow \mathbb{A}^2 \prod \mathbb{P}^1 \xrightarrow{pr_1} W = \mathbb{A}^2.$$

If $w = (w_1, w_2) \neq (0, 0)$, then the fibre over w is $\{(\xi_1 \colon \xi_2) \mid w_1\xi_2 = w_2\xi_1\}$.

1. If $w_1 \neq 0$, then $\xi_2 = (w_2/w_1)\xi_1$.
2. If $w_2 \neq 0$, then $\xi_1 = (w_1/w_2)\xi_2$.

In both cases, we get a single point, and the dimension of the fibre $\varphi^{-1}(w)$ is zero for $w \in \mathbb{A} - \{(0,0)\}$.

When $w = (0,0)$, the fibre is the whole $\mathbb{P}^1$. Thus, the dimension of the fibre at the origin jumps to 1.

In algebraic geometry, we have the analog of the notion of compactness for Hausdorff topologies, but here working for the Zariski topology. This is the notion of properness.

Definition 2.5 An abstract k-variety, V, is *proper* if V is separated, quasi-compact, and if for every k-variety W, the second projection map $pr_2 \colon V \prod_k W \to W$ is a closed map.

Remarks:

(1) As we said, the notion of properness of a k-variety is the algebraic substitute for compactness. An older terminology is the term *complete variety*. As an illustration of the similarity of properness and compactness, we have the following property (well known for continuous maps on compact spaces): If V is proper and W is separated, then for any morphism $\varphi \colon V \to W$, the map φ is a closed map.

Proof. Consider the graph morphism

$$\Gamma_\varphi \colon V \to V \prod_k W,$$

given by

$$\Gamma_\varphi(v) = (v, \varphi(v)).$$

Note that the image of Γ_φ is closed in $V \prod_k W$ because W is separated. Indeed, consider the morphism

$$(\varphi, \mathrm{id}) \colon V \prod_k W \to W \prod_k W$$

given by

$$(\varphi, \mathrm{id})(u, w) = (\varphi(u), w).$$

It is obvious that $\Gamma_\varphi = (\varphi, \mathrm{id})^{-1}(\Delta)$, where Δ is the diagonal in $W \prod_k W$. Thus, it is enough to prove that Δ is closed in $W \prod_k W$, but this follows from the fact that W is separated.

Also, pr_1 restricted to the image of Γ_φ is the inverse of the morphism Γ_φ. Thus, Γ_φ is an isomorphism and V is isomorphic to a closed subvariety of $V \prod_k W$. Now, φ is the composition

$$V \xrightarrow{\Gamma_\varphi} V \prod_k W \xrightarrow{pr_2} W,$$

and since Im Γ_φ is closed, the properness of V (pr_2 is closed) shows that φ is closed.

(2) When we introduce projective varietes (see Section 2.5), we will show that every projective variety is proper. However, there are proper varieties which are not projective, although this is a harder fact to demonstrate.

One of the pleasant consequences of properness and the holomorphic nature of morphisms is the following theorem about irreducibility:

Theorem 2.11 *(Irreducibility criterion) Let V be a proper k-variety, W a separated k-variety and $\varphi\colon V \to W$ a surjective morphism. Assume that*

(1) W is k-irreducible.

(2) $\varphi^{-1}(w)$ is $\kappa(w)$-irreducible for every $w \in W$.

(3) $\dim_{\kappa(w)}(\varphi^{-1}(w)) = n$, a constant for all $w \in W$.

Then, V is k-irreducible.

Proof. Let $V = \bigcup_{j=1}^q V_j$ be an irredundant decomposition of V into k-irreducible components. Consider V_j. It is closed in V, and thus, $\varphi(V_j)$ is closed in W, because V is proper. Since $\varphi\colon V \to W$ is surjective,

$$W = \bigcup_{j=1}^q \varphi(V_j).$$

But W is k-irreducible; so, it follows (after renumbering, if needed) that $\varphi(V_j) = W$ for $j = 1, \ldots, s$, and $\varphi(V_j)$ is strictly contained in W for $j = s+1, \ldots, q$. Thus,

$$\bigcup_{j=s+1}^q \varphi(V_j)$$

is a k-closed subset of W strictly contained in W, and

$$\widetilde{W} = W - \bigcup_{j=s+1}^q \varphi(V_j)$$

is a k-open dense subset of W. Let $\widetilde{V} = \varphi^{-1}(\widetilde{W})$, write $\widetilde{V}_j = \widetilde{V} \cap V_j$, and let φ_j be the restriction of φ to $\widetilde{V}_j$. Note,

$$\varphi_j(\widetilde{V}_j) = \varphi(\widetilde{V}_j) = \widetilde{W},$$

because, given any $w \in \widetilde{W}$, there exists $v \in V_j$ with $\varphi(v) = w$. Since $\varphi(v) \in \widetilde{W}$, the element v is in $\widetilde{V}$. Therefore, $v \in \widetilde{V} \cap V_j$; hence, $v \in \widetilde{V}_j$, as required. Write

$$\mu_j = \min\{\dim(\varphi_j^{-1}(w)) \ \mid \ w \in \widetilde{W}\}.$$

By the fibre dimension theorem (Theorem 2.9), there is some nonempty open subset $U_j \subseteq \widetilde{W}$ so that if $w \in U_j$, then $\dim(\varphi_j^{-1}(w)) = \mu_j$. Thus, as $U = \bigcap_{j=1}^{s} U_j$ is a dense k-open subset of $\widetilde{W}$, we have a nontrivial dense open, U, so that if $w \in U$, then $\dim(\varphi_j^{-1}(w)) = \mu_j$, for $j = 1, \ldots, s$. Pick $w_0 \in U$. Then

$$\varphi^{-1}(w_0) = \bigcup_{j=1}^{s} \varphi_j^{-1}(w_0).$$

However, $\varphi^{-1}(w_0)$ is $\kappa(w_0)$-irreducible, and thus, there is some j such that

$$\varphi^{-1}(w_0) = \varphi_j^{-1}(w_0).$$

We may assume that $j = 1$. Since the dimension of the fibres is constant, we get

$$\mu_1 = n.$$

By the fibre dimension theorem, $\dim \varphi_1^{-1}(w) \geq \dim \varphi_1^{-1}(w_0) = n$, for all $w \in W$. Now,

$$\varphi^{-1}(w) = \bigcup_{j=1}^{s} \varphi_j^{-1}(w), \tag{$*$}$$

and since $\dim \varphi_1^{-1}(w) \leq \dim \varphi^{-1}(w) = n$, we must have $\dim \varphi_1^{-1}(w) = n$ for all $w \in W$ and $(*)$ together with the irreducibility of $\varphi^{-1}(w)$ imply that $\varphi^{-1}(w) = \varphi_1^{-1}(w)$, for all $w \in W$. It follows that

$$V = \bigcup_{w \in W} \varphi^{-1}(w) = \bigcup_{w \in W} \varphi_1^{-1}(w) = V_1$$

and since V_1 is irreducible, so is V. $\square$

2.2 Local Theory, Zariski Tangent Space

Let V be a k-variety and p a point in V. Consider the locally defined holomorphic functions on V (to Ω) near p. We can define an equivalence relation $\sim$ on such functions so that, given F defined on some open U and G defined on some open V, with $p \in U \cap V$,

$$F \sim G$$

iff there is some open $W \subseteq U \cap V$ such that $F \restriction W = G \restriction W$. The *germ of F at p* is the equivalence class of F.

Definition 2.6 If V is a k-variety and p is a point in V, we let $\mathcal{O}_{V,p}$ be the *ring of germs of locally defined holomorphic functions at* p *on* V.

Actually, $\mathcal{O}_{V,p}$ is a k-algebra, although not necessarily finitely generated. Now, $V = \bigcup_\alpha V_\alpha$, where each V_α is an affine open; so, there is some α such that $p \in V_\alpha$, we have $V_\alpha = \mathrm{Spec}\, A_\alpha$ for an affine algebra A_α, and

$$\mathfrak{I}_\alpha(p) = \{f \in A_\alpha \mid f(p) = 0\}$$

is a prime ideal, $\mathfrak{p}_\alpha$, of A_α. Given a germ $[F]$ at p, we may assume that a representative of $[F]$ is defined on some open V_α. In fact, there exist smaller opens in V_α, for instance, $\{q \in V_\alpha \mid h(q) \neq 0\}$ where $h \in A_\alpha$, and where we can write

$$F = \frac{g}{h^m}$$

in $(A_\alpha)_h$. By replacing h^m by h, we may assume that

$$F = \frac{g}{h}$$

in $(A_\alpha)_h$. Similarly, we have

$$G = \frac{\widetilde{g}}{\widetilde{h}}$$

in $(A_\alpha)_{\widetilde{h}}$. However, using $h\widetilde{h}$ as denominator, we can assume that

$$F = \frac{g}{h} \quad \text{and} \quad G = \frac{\widetilde{g}}{h}.$$

Then, $F \sim G$ means that there is some $l \in (A_\alpha)_h$ so that

$$\frac{g}{h} = \frac{\widetilde{g}}{h}, \quad \text{in } (A_\alpha)_l.$$

That is, g/h and $\widetilde{g}/h$ have the same image in $(A_\alpha)_l$. As a consequence,

$$\mathcal{O}_{V,p} = \varinjlim_{l(p)\neq 0} (A_\alpha)_l = (A_\alpha)_{\mathfrak{p}_\alpha}.$$

In particular, $\mathcal{O}_{V,p}$ is a local ring. We have a map

$$A_\alpha \longrightarrow \mathcal{O}_{V,p} = (A_\alpha)_{\mathfrak{p}_\alpha},$$

and $\mathfrak{p}_\alpha$ is the pullback of the maximal ideal of $\mathcal{O}_{V,p}$, which we denote $\mathfrak{m}_p$.

If p is also in V_β, then then the open set $V_\alpha \cap V_\beta$ is covered by affine opens, W; so, p belongs to some affine open W such that $W \subseteq V_\alpha \cap V_\beta$. Then, there is some open $(V_\alpha)_h \subseteq W$

around p, and the computation of $\mathcal{O}_{V,p}$ as above gives the same computation for V_α, but the limit is taken over a cofinal family. We obtain

$$\mathcal{O}_{V,p} = (A_\alpha)_{\mathfrak{p}_\alpha} = k[W]_{\mathfrak{p}_\alpha} \quad \text{and} \quad \mathcal{O}_{V,p} = (A_\beta)_{\mathfrak{p}_\beta} = k[W]_{\mathfrak{p}_\beta},$$

and $\mathcal{O}_{V,p}$, computed as a direct limit, is independent of the affine open used. We see that

$$\mathcal{O}_{V,p}/\mathfrak{m}_p = \mathrm{Frac}(A_\alpha/\mathfrak{p}_\alpha) = \kappa(p).$$

Let's summarize all this in the following proposition:

Proposition 2.12 *At $p \in V$, we have a local ring, $\mathcal{O}_{V,p}$, consisting of the germs of locally defined holomorphic functions on V at p. If V_α is an affine open in V containing p, then $\mathcal{O}_{V,p}$ can be computed as $(A_\alpha)_{\mathfrak{p}_\alpha}$, where*

(1) $V_\alpha = \mathrm{Spec}\, A_\alpha$.

(2) $\mathfrak{p}_\alpha = \mathfrak{I}_\alpha(p) = \{f \in A_\alpha \mid f(p) = 0\}$.

Furthermore, under the map $A_\alpha \longrightarrow \mathcal{O}_{V,p}$, the maximal ideal $\mathfrak{m}_p$ (of $\mathcal{O}_{V,p}$) pulls back to $\mathfrak{p}_\alpha$, and the residue field of the local ring is just $\kappa(p)$.

Now consider V, an affine variety in $\mathbb{A}^n$. Write $\mathfrak{A}$ for the ideal $\mathfrak{I}(V) \subseteq k[X_1, \ldots, X_n]$. Consider $p \in V(L)$, where L is some field between k and Ω. Then, look at the collection of linear equations

$$\sum_{j=1}^{n} \left(\left(\frac{\partial f}{\partial x_j} \right)_p \right) (x_j - p_j) = 0, \tag{$*$}$$

where $p = (p_1, \ldots, p_n)$ and $f \in \mathfrak{A}$. If we write $\lambda_j = x_j - p_j$, these equations define a linear space over L. By Hilbert's basis theorem, $\mathfrak{A}$ is finitely generated, say $\mathfrak{A} = (f_1, \ldots, f_t)$; so,

$$f = \sum_{i=1}^{t} h_i f_i,$$

where $h_i \in k[X_1, \ldots, X_n]$. We get

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{t} \left(h_i \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial h_i}{\partial x_j} \right),$$

and, since $f_i(p) = 0$,

$$\left(\frac{\partial f}{\partial x_j} \right)_p = \sum_{i=1}^{t} h_i(p) \left(\frac{\partial f_i}{\partial x_j} \right)_p.$$

Equation $(*)$ becomes

$$\sum_{j=1}^{n}\sum_{i=1}^{t}\left(h_i(p)\left(\frac{\partial f_i}{\partial x_j}\right)_p\right)(x_j - p_j) = 0,$$

which yields

$$\sum_{i=1}^{t} h_i(p)\left(\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)_p (x_j - p_j)\right) = 0.$$

Hence, the vector space defined by $(*)$ is also defined by

$$\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)_p (x_j - p_j) = 0, \quad \text{for } i = 1, \ldots, t. \tag{$**$}$$

Definition 2.7 The L-linear space at $p \in V$ defined by $(**)$ is called the *Zariski tangent space at p on V*. It is denoted by $T_{V,p}(L)$.

Note that Definition 2.7 is an extrinsic definition. It depends on the embedding of V in $\mathbb{A}^n$. It is possible to give an intrinsic definition. For this, we review k-derivations.

Definition 2.8 A *k-derivation of $k[V]$ with values in L centered at p* consists of the following data:

(1) A k-linear map $D\colon k[V] \to L$. (values in L)

(2) $D(fg) = f(p)Dg + g(p)Df$ (Leibnitz rule) (centered at p)

(3) $D(\lambda) = 0$ for all $\lambda \in k$. (k-derivation)

The set of such derivations is denoted by $\mathrm{Der}_k(k[V], L; p)$.

The composition

$$k[X_1, \ldots, X_n] \longrightarrow k[V] \xrightarrow{D} L$$

is again a k-derivation (on the polynomial ring) centered at p with values in L. Note that a k-derivation on the polynomial ring (call it D again) factors as above iff $D \restriction \mathfrak{A} = 0$. This shows that

$$\mathrm{Der}_k(k[V], L; p) = \{D \in \mathrm{Der}_k(k[\mathbb{A}^n], L; p) \mid D \restriction \mathfrak{A} = 0\}.$$

However, a k-derivation $D \in \mathrm{Der}_k(k[\mathbb{A}^n], L; p)$ is determined by its values $D(X_j) = \lambda_j$ at the variables X_j. Clearly (DX),

$$D(f(X_1, \ldots, X_n)) = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\right)_p D(X_j).$$

But, observe that for any $(\lambda_1, \ldots, \lambda_n)$, the restriction of D to $\mathfrak{A}$ vanishes iff

$$\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)_p \lambda_j = 0, \quad \text{for every } f \in \mathfrak{A},$$

that is, iff

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_p \lambda_j = 0, \quad \text{for every } i = 1, \ldots, m,$$

where $f_1, \ldots, f_m$ generate the ideal $\mathfrak{A}$. Letting $\xi_j = \lambda_j + p_j \in L$, we have a bijection between $\mathrm{Der}_k(k[V], L; p)$ and

$$\left\{ (\xi_1, \ldots, \xi_n) \in L^n \;\middle|\; \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_p \lambda_j = 0, \;\; 1 \leq i \leq m \right\}.$$

It is given by the map

$$D \mapsto (\xi_1, \ldots, \xi_n),$$

with $\xi_j = D(X_j) + p_j$. This gives the isomorphism

$$T_{V,p}(L) \cong \mathrm{Der}_k(k[V], L; p).$$

We conclude that $T_{V,p}(L)$ is independent of the embedding of V into $\mathbb{A}^n$, up to isomorphism. Now, $\mathcal{O}_{V,p} = k[V]_{\mathfrak{p}}$, the localization of $k[V]$ at the prime ideal $\mathfrak{p} = \mathfrak{I}(p)$; so,

$$\mathcal{O}_{V,p} = \left\{ \left[\frac{f}{g}\right] \;\middle|\; f, g \in k[V],\; g \notin \mathfrak{p} \right\} = \left\{ \left[\frac{f}{g}\right] \;\middle|\; f, g \in k[V],\; g(p) \neq 0 \right\}.$$

Any k-derivation $D \in \mathrm{Der}_k(k[V], L; p)$ is uniquely extendable to $\mathcal{O}_{V,p}$ *via*

$$D\left(\frac{f}{g}\right) = \frac{g(p)Df - f(p)Dg}{g(p)^2}.$$

Therefore,

$$\mathrm{Der}_k(k[V], L; p) = \mathrm{Der}_k(\mathcal{O}_{V,p}, L; p).$$

The local ring $\mathcal{O}_{V,p}$ determines the point p, too. To see this, recall that any $p \in V(L)$ corresponds to a unique k-algebra morphism $\varphi_p \in \mathrm{Hom}_{k-\mathrm{alg}}(k[V], L)$, where $\varphi_p(f) = f(p)$. So, if $g \in k[V]$ with $g(p) \neq 0$, viewing g as a polynomial, we have $g \in k[\mathbb{A}^n] - \mathfrak{p}$, and this implies that

$$\varphi_p\left(\frac{f}{g}\right) = \frac{f(p)}{g(p)}.$$

This means that φ_p extends uniquely to $\mathcal{O}_{V,p}$ and kills $\mathfrak{m}_p$. Therefore, φ_p corresponds to a k-injection from $\kappa(p)$ to L, and so, $\mathcal{O}_{V,p}$ determines p. In summary, we have the following proposition:

Proposition 2.13 *If V is an irreducible affine k-variety, the Zariski tangent space, $T_{V,p}(L)$, at $p \in V(L)$ is canonically isomorphic to either* $\mathrm{Der}_k(k[V], L; p)$*, or* $\mathrm{Der}_k(\mathcal{O}_{V,p}, L; p)$*, and $V(L)$ corresponds to pairs $(\mathfrak{p}, \varphi)$, where*

(a) $\mathfrak{p}$ is a prime ideal of $k[V]$ with residue field $\kappa(\mathfrak{p})$ ($\kappa(\mathfrak{p}) = \mathrm{Frac}(k[V]/\mathfrak{p})$).

(b) $\varphi\colon \kappa(\mathfrak{p}) \to L$ is a k-injection.

The correspondence is given as follows. The pair $(\mathfrak{p}, \varphi)$ gives rise to the homomorphism

$$k[V] \longrightarrow \mathrm{Frac}(k[V]/\mathfrak{p}) \xrightarrow{\varphi} L,$$

which is a point in $V(L)$, and the point p in $V(L)$ gives the homomorphism

$$k[V] \longrightarrow \mathrm{Frac}(k[V]/\mathfrak{I}(p)) \longrightarrow L,$$

that is, the homomorphism

$$k[V] \longrightarrow \mathrm{Frac}(\mathcal{O}_{V,p}/\mathfrak{m}_p) \xrightarrow{\varphi} L,$$

and hence, the pair $(\mathfrak{m}_p, \varphi)$.

Proposition 2.14 *Let V be an irreducible k-variety. The function*

$$p \mapsto \dim_{\kappa(p)} T_{V,p}(\kappa(p))$$

is upper-semicontinuous on V, i.e.,

$$S_l = \{\xi \in V \mid \dim_{\kappa(p)} T_{V,p}(\kappa(p)) \geq l\}$$

is k-closed in V, and furthermore, $S_{l+1} \subseteq S_l$.

Proof. We may assume that V is affine (DX); so, we have $V \subseteq \mathbb{A}^n$ for some n. Hence, $T_{V,p}(\kappa(p))$ is the $\kappa(p)$-vector space given by the set of $(\lambda_1, \ldots, \lambda_n) \in \kappa(p)^n$ such that

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_p \lambda_j = 0, \quad \text{for } i = 1, \ldots, m,$$

where $f_1, \ldots, f_m$ generate the ideal $\mathfrak{A} = \mathfrak{I}(V)$. Hence, $T_{V,p}(\kappa(p))$ is the kernel of the linear map from $\mathbb{A}^n$ to $\mathbb{A}^m$ given by the $m \times n$ matrix

$$\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right).$$

It follows that

$$\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = n - \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right).$$

Consequently, $\dim_{\kappa(p)} T_{V,p}(\kappa(p)) \geq l$ iff

$$\mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right) \leq n - l;$$

and this holds iff the $(n-l+1) \times (n-l+1)$ minors are all singular at p. But the latter is true when and only when the corresponding determinants vanish at p. These give additional equations on V at p in order that $p \in S_l$ and this implies that S_l is closed in V. Since the S_l manifestly form a nonincreasing chain as l increases, there is a largest l for which $S_l = V$. The set S_{l+1} is closed in V, and it complement $\{\xi \mid \dim_{\kappa(p)} T_{V,p}(\kappa(p)) = l\}$ is k-open. This gives us the *tangent space stratification* by locally closed sets

$$V = U_0 \cup U_1 \cup \cdots \cup U_t,$$

where $U_0 = \{\xi \mid \dim_{\kappa(p)} T_{V,p}(\kappa(p)) = l\}$ is open, and $U_i = \{\xi \mid \dim_{\kappa(p)} T_{V,p}(\kappa(p)) = l + i\}$. We have U_1 open in $V - U_0 = S_{l+1}$, etc. $\square$

Remark: Given a hypersurface S in $\mathbb{A}^n$ defined by the equation $f = 0$, when are we in the "bad" closed set S_{l+1} which is the complement of U_0? This happens when and only when

$$\mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right) \leq 1 \quad \text{and} \quad p \in S,$$

that is , when $f(p) = 0$ and

$$\left(\frac{\partial f_i}{\partial x_j}\right)_p = 0, \quad \text{for } j = 1, \ldots, n.$$

Example 2.3 Assume that k has characteristic 0, and let $V \subseteq \mathbb{A}^2$ be the hypersurface defined by

$$Y^2 - X^3 = 0.$$

We have $\partial f/\partial X = 3X^2$, $\partial f/\partial Y = 2Y$, and the only bad point is $O = (0, 0)$. At this point, $\dim_k T_{V,O}(\Omega) = 2$. As a real curve, this cubic looks like the picture displayed in Figure 2.1. The singularity at the origin is called a cusp.

Example 2.4 Let $V \subseteq \mathbb{A}^2$ be the hypersurface defined by

$$Y^2 + X^3 - X^2 = 0.$$

Again, the only bad point is $O = (0, 0)$, where we have $\dim_k T_{V,O}(\Omega) = 2$. As a real curve, this cubic looks like the picture displayed in Figure 2.2. Its singularity at the origin is called a double point (a node).

The equation of the "tangent cone" is $Y^2 - X^2 = 0$.

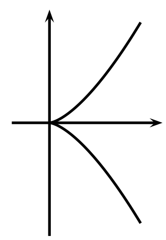

Figure 2.1: A Cuspidal Cubic

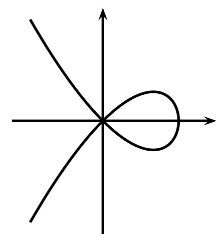

Figure 2.2: A Nodal Cubic

Example 2.5 Let $V \subseteq \mathbb{A}^3$ be the cone defined by

$$X^2 + Y^2 = Z^2.$$

The only bad point is $O = (0,0,0)$. We have $\dim_k T_{V,O}(\Omega) = 3$, and not 2.

Separability and derivations are intimately connected. Since tangent spaces are defined by derivations, it will be no surprise that some hypotheses of separability will enter into the theorem of about tangent space. The connection with separability is that separability of the field extension K/k (in the wide sense) implies that

$$\mathrm{tr.d}_k\, K = \dim_K \mathrm{Der}_k(K, K).$$

Proposition 2.15 *Let V be k-variety V such that V has an open affine covering $V = \bigcup_\alpha V_\alpha$, for which each function field $k(V_\alpha)$ is separably generated over k. Then, there is a nonempty k-open set $U \subseteq V$ so that*

$$\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = \dim_k V$$

for all $p \in U$.

Proof. In the decomposition $V = \bigcup_\alpha V_\alpha$, we may restrict attention to those V_α for which $\dim(V_\alpha) = \dim(V)$. If U_α works in each such V_α, then $\bigcup_\beta U_\beta$ works, where β ranges over those α's. Therefore, we may assume that V is affine. Let

$$V = V_1 \cup \cdots \cup V_t$$

be an irredundant decomposition into irreducible components. At least one of the V_j's has dimension $\dim(V)$. Say it is $j = 1$. Look at $V_1 \cap V_j$, $j = 2, \ldots, t$. Each $V_1 \cap V_j$ is a closed set, and so

$$W = V - \bigcup_{j=2}^{t} V_1 \cap V_j$$

is k-open. Also, $W \cap V_1$ is k-open in V_1 because it is the complement of all the closed sets $V_1 \cap V_j$ with $j \geq 2$. Take any open subset, U, of $V - \bigcup_{j=2}^{t} V_j$ for which U is a good open in V_1, that is, where $\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = \dim_k V_1$ whenever $p \in U$. Then, $U \cap W$ also has the right property. Hence, we may assume that V is affine *and* irreducible, so, $V \subseteq \mathbb{A}^n$. If so, recall that $T_{V,p}(\kappa(p))$ is the vector space consisting of all $(\lambda_1, \ldots, \lambda_n) \in \kappa(p)^n$ so that

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_p \lambda_j = 0,$$

where $\mathfrak{I}(V) = (f_1, \ldots, f_m)$. Since we assume that $k(V)$ is separably generated over k, we have

$$\mathrm{tr.d}_k\, K = \dim_K \mathrm{Der}_k(K, K),$$

where $K = k(V) = \mathrm{Frac}(k[V])$. Then,

$$\begin{aligned}
\dim_k(V) &= \mathrm{tr.d}_k\, k(V) \\
&= \dim_{k(V)} \mathrm{Der}_k(k(V), k(V)) \\
&= \dim_{k(V)} \mathrm{Der}_k(k[V], k(V)) \\
&= \dim_{k(V)} \{D \in \mathrm{Der}_k(k[\mathbb{A}^n], k(V); q) \mid D = 0 \text{ on } \mathfrak{I}(V), \text{ and } q \text{ generic}\} \\
&= \dim_{k(V)} \left\{ \lambda_1, \ldots, \lambda_n) \in k(V)^n \;\middle|\; \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_q \lambda_j = 0,\ 1 \leq i \leq m \right\} \\
&= n - \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_q \right).
\end{aligned}$$

Thus, we must show that there is some nonempty open subset $U \subseteq V$ so that

$$\mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p \right) = \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)_q \right), \quad \text{for all } p \in U,$$

and for q generic in V. Now, if q is k-generic, then $\kappa(q) = k(V)$ and the rank of the matrix

$$\left(\left(\frac{\partial f_i}{\partial x_j}\right)_q \right)$$

is just its rank as a matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)$$

whose entries are in the field $k(V)$. Under specialization, the rank can drop, but there is an open where the rank is constant (again, the argument by minors). This completes the proof.

A second argument for the rank part goes as follows. From our previous work, there is some open set $U \subseteq V$ so that the rank at q is equal to the rank at p, for all $p \in U$. Assume that the rank of

$$\left(\frac{\partial f_i}{\partial x_j}\right)$$

as a matrix with entries in $k(V)$ is r. By linear algebra, this means that there are matrices A, B (with entries in $k(V)$) so that

$$A\left(\frac{\partial f_i}{\partial x_j}\right)B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\alpha(X_1, \ldots, X_n)$ and $\beta(X_1, \ldots, X_n)$ be the common denominators of entries in A and B, respectively. So, $A = (1/\alpha)\widetilde{A}$ and $B = (1/\beta)\widetilde{B}$, and the entries in $\widetilde{A}$ and $\widetilde{B}$ are in $k[V]$. Let U be the open set where the polynomial $\alpha\beta\det(\widetilde{A})\det(\widetilde{B})$ is nonzero. Then, as

$$\frac{1}{\alpha\beta}\widetilde{A}\left(\frac{\partial f_i}{\partial x_j}\right)\widetilde{B} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{in } k(V),$$

applying the specialization corresponding to p, we get

$$\frac{1}{\alpha(p)\beta(p)}\widetilde{A}(p)\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right)\widetilde{B}(p) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\left(\left(\frac{\partial f_i}{\partial x_j}\right)_p\right)$$

has rank r. □

Now, if V is irreducible, we must have a big open subset U_0 of V where $\dim T_{V,p}(\kappa(p))$ is equal to the minimum it takes on V. Also, we have an open $\widetilde{U_0}$ where $\dim T_{V,p}(\kappa(p)) = \dim(V)$. Since these opens are dense, we find

$$U_0 \cap \widetilde{U_0} \neq \emptyset.$$

Therefore, we must have

$$U_0 = \widetilde{U_0},$$

and the minimum value taken by the dimension of the Zariski tangent space is just $\dim(V)$. In summary, the set

$$U_0 = \{p \in V \mid \dim T_{V,p}(\kappa(p)) = \dim(V)\} = \min_{q \in V} \dim T_{V,q}(\kappa(p))$$

is a k-open dense subset of V.

Definition 2.9 If V is an irreducible variety, a point $p \in V$ is *nonsingular* if

$$\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = \dim_k(V).$$

Otherwise, we say that p is *singular*. If V is quasi-compact, then $V = \bigcup_{i=1}^t V_i$ for some irredundant decomposition into irreducible components, and we say that $p \in V$ is *nonsingular* if $p \notin V_i \cap V_j$ (for all i, j, with $i \neq j$) and p is nonsingular in the component to which it belongs. Otherwise, we say that p is *singular*. The singular locus of V is denoted by $\mathrm{Sing}(V)$.

From previous observations, the singular locus, $\mathrm{Sing}(V)$, of V is a k-closed set. This leads to the Zariski stratification. Let U_0 be the set of nonsingular points in V, write $V_1 = \mathrm{Sing}(V) = V - U_0$, and let U_1 be the set of nonsingular points in V_1. We can set $V_2 = V_1 - U_1$, and so on. Then, we obtain the *Zariski-stratification of* V into disjoint locally closed strata

$$V = U_0 \cup U_1 \cup \cdots \cup U_t,$$

where each U_i is a nonsingular variety and U_0 is the open subset of nonsingular points in V.

Example 2.6 In this example (see Figure 2.3), $\mathrm{Sing}(V)$ consists of a line with a bad point on it (the origin). V_1 is that line, and $V_2 = \mathrm{Sing}(V_1)$ is the bad point.

Example 2.7 In this example (see Figure 2.4), $\mathrm{Sing}(V)$ consists of three points. Observe that V is reducible and consists of components of dimension 1 and 2.

Example 2.8 This example shows that troubles may arise in characteristic $p > 0$. Let $k = (\mathbb{Z}/p\mathbb{Z})(T)$, the field of rational functions over $\mathbb{Z}/p\mathbb{Z}$. We let V be the variety in $\mathbb{A}^2_k$ defined by

$$TX^p + Y^{2p} - 1 = 0.$$

Letting $f = TX^p + Y^{2p} - 1$, we get

$$\frac{\partial f}{\partial X} = pTX^{p-1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial Y} = 2pY^{2p-1} = 0.$$

Thus, $\dim T_{V,p}(\Omega) = 2$ for all points p! However, $\dim_k(V) = 1$.

Why do we use the function field $(\mathbb{Z}/p\mathbb{Z})(T)$? If T does not appear in the equation (e.g., $T = 1$), we have

$$X^p + Y^{2p} - 1 = (X + Y^2 - 1)^p,$$

the ideal $\mathfrak{I}(V)$ (over $\mathbb{Z}/p\mathbb{Z}$) is generated by $X + Y^2 - 1$, and so there is no problem. This example (with T appearing) shows that it is necessary to assume that each $k[V_\alpha]$ is separably generated.

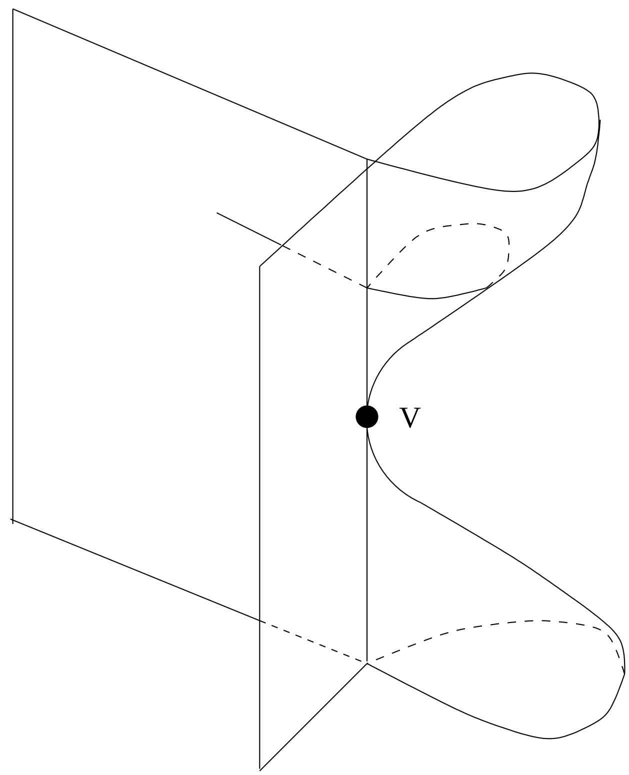

Figure 2.3: Example of A Surface with Singularites

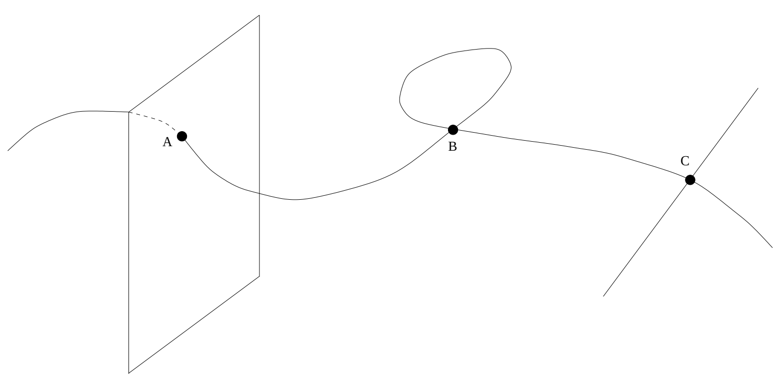

Figure 2.4: Example of A Variety with Singularites

Let us take a closer look at the tangent space $T_{V,p}(L)$.

Pick, p, a point of an irreducible variety V. We have

$$\kappa(p) = \mathrm{Frac}(k[V]/\mathfrak{I}(p)) = k\left(\overline{\{p\}}\right).$$

We want $\kappa(p)$ separable over k, in the wide sense (i.e., $\kappa(p)$ is not necessarily algebraic over k); we have

$$T_{V,p}(\Omega) = \mathrm{Der}_k(\mathcal{O}_{V,p}, \Omega; p).$$

Recall from commutative algebra that

$$\varprojlim_r \mathcal{O}_{V,p}/\mathfrak{m}_p^r$$

is the completion of $\mathcal{O}_{V,p}$, denoted by $\widehat{\mathcal{O}}_{V,p}$. Since $\mathcal{O}_{V,p} = k[V]_{\mathfrak{I}(p)}$ is a Noetherian ring, by Krull's intersection theorem (Zariski and Samuel [60], Corollary 1 Chapter IV, Section 7),

$$\bigcap_r \mathfrak{m}_p^r = ((0)).$$

Thus, we have an injection $i\colon \mathcal{O}_{V,p} \to \widehat{\mathcal{O}}_{V,p}$ and the $\mathfrak{m}_p$-adic topology is Hausdorff. Also, the characteristic of $\kappa(p)$, the residue field of $\mathcal{O}_{V,p}$, is equal to the characteristic of k. By the structure theorem of I.S. Cohen (1945) (Zariski and Samuel [61], Theorem 27, Chapter VIII, Section 12), $\widehat{\mathcal{O}}_{V,p}$ contains a unique field k-isomorphic to $\kappa(p)$. We can write

$$\widehat{\mathcal{O}}_{V,p} = \kappa(p) \coprod \widehat{\mathfrak{m}}_p,$$

as a module over $\kappa(p)$. The multiplication in the ring $\widehat{\mathcal{O}}_{V,p}$ is given by

$$(\lambda, m)(\lambda', m') = (\lambda\lambda', (\lambda m' + \lambda' m + mm')).$$

We also have the isomorphism

$$\widehat{\mathfrak{m}}_p^r/\widehat{\mathfrak{m}}_p^{r+1} \cong \mathfrak{m}_p^r/\mathfrak{m}_p^{r+1}.$$

Given a derivation $D \in \mathrm{Der}_k(\mathcal{O}_{V,p}, \Omega; p)$, the restriction $D \restriction \mathfrak{m}_p$ of D to $\mathfrak{m}_p$ has the property that $D \restriction \mathfrak{m}_p^2 = 0$. Indeed,

$$D\left(\sum_i a_i b_i\right) = \sum_i D(a_i b_i) = a_i(p) D b_i + b_i(p) D a_i.$$

Since $a_i, b_i \in \mathfrak{m}_p$, we have $a_i(p) = b_i(p) = 0$, and so, $D\left(\sum_i a_i b_i\right) = 0$, which proves that $D \restriction \mathfrak{m}_p^2 = 0$. As a consequence, D is a linear map from $\mathfrak{m}_p/\mathfrak{m}_p^2$ to Ω. However, since

$$\widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2 \cong \mathfrak{m}_p/\mathfrak{m}_p^2,$$

D is a linear map from $\widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2$ to Ω. Thus, by the Cohen splitting, to know D on $\widehat{\mathcal{O}}_{V,p}$, we need to know D on $\kappa(p)$.

(1) Assume at first, that $\kappa(p)$ is separable algebraic over k. Since $D \restriction k = 0$, we get $D \restriction \kappa(p) = 0$.

Conversely, given a linear map $L\colon \widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2 \to \Omega$ over $\kappa(p)$, how do we make a derivation D inducing L?

Define D on $\widehat{\mathfrak{m}}_p$ *via*

$$\widehat{\mathfrak{m}}_p \longrightarrow \widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2 \xrightarrow{L} \Omega,$$

and define $D = 0$ on $\kappa(p)$. Hence, we can define

$$D(\lambda, m) = L(m \ (\mathrm{mod}\, \widehat{\mathfrak{m}}_p^2)).$$

We need to check that it is a derivation. Letting $\xi = (\lambda, m)$ and $\eta = (\lambda', m')$, we have

$$\begin{aligned} D(\xi\eta) &= D(\lambda\lambda', (\lambda m' + \lambda' m + mm')) \\ &= L(\lambda m' + \lambda' m + mm' \ (\mathrm{mod}\, \widehat{\mathfrak{m}}_p^2)) \\ &= L(\lambda m' + \lambda' m \ (\mathrm{mod}\, \widehat{\mathfrak{m}}_p^2)) \\ &= \lambda L(m') + \lambda' L(m) \\ &= \xi(p)D(\eta) + \eta(p)D(\xi). \end{aligned}$$

As a summary,

$$\begin{aligned} \mathrm{Der}_k(\mathcal{O}_{V,p}, \Omega; p) &= \mathrm{Der}_k(\widehat{\mathcal{O}}_{V,p}, \Omega; p) \\ &= \text{the set of } \kappa(p)\text{-linear maps } \widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2 \longrightarrow \Omega \\ &= \text{the set of } \kappa(p)\text{-linear maps } \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \Omega \end{aligned}$$

(because $\kappa(p)$ is separable algebraic over k).

Of course, $p \in V(\Omega)$; but a more canonical choice is $p \in V(\kappa(p))$. If we use this choice of field, we find that

$$T_{V,p}(\kappa(p)) = \mathrm{Der}_k(\mathcal{O}_{V,p}, \kappa(p); p) = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D,$$

the dual of the $\kappa(p)$-vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$. When p is thought of as a point of $V(\Omega)$, then

$$T_{V,p}(\Omega) = \mathrm{Der}_k(\mathcal{O}_{V,p}, \Omega; p).$$

(2) Assume now that p is not a closed point of V, but continue assuming that $\kappa(p)$ is separable over k. Now, a derivation trivial on k does *not* imply that it is trivial on $\kappa(p)$. Hence, we need $\mathrm{Der}_k(\kappa(p), \kappa(p))$. We get

$$\begin{aligned} T_{V,p}(\kappa(p)) &= \mathrm{Der}_k(\kappa(p), \kappa(p)) \coprod \mathrm{Der}_{\kappa(p)}(\widehat{\mathcal{O}}_{V,p}, \kappa(p)) \\ &= \mathrm{Der}_k(\kappa(p), \kappa(p)) \coprod \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D. \end{aligned}$$

If we compute the dimensions over $\kappa(p)$, since $\kappa(p)$ is separable over k, we get

$$\dim_{\kappa(p)} \mathrm{Der}_k(\kappa(p), \kappa(p)) = \mathrm{tr.d}_k\, \kappa(p) = \dim_k \overline{\{p\}},$$

and thus,

$$\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = \dim_k \overline{\{p\}} + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right). \tag{$*$}$$

We can us this computation to compare the dimension of $T_{V,p}(\kappa(p))$ as $\kappa(p)$-vector space with the dimension of V as k-variety. Consider a maximal chain of prime ideals

$$\mathfrak{m}_p \supset \mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \cdots \supset \mathfrak{p}_d = (0);$$

by definition, d is the *height* of $\mathfrak{m}_p$. This chain is in one-to-one correspondence with the chain of ideals of $k[V]$:

$$\mathfrak{I} = \mathfrak{I}(p) \supset \mathfrak{P}_1 \supset \mathfrak{P}_2 \supset \cdots \supset \mathfrak{P}_d = (0).$$

Geometrically, this is a chain of varieties

$$\overline{\{p\}} \subset V_1 \subset \cdots \subset V_d = V.$$

So, *the height of* $\mathfrak{m}_p$ *is equal to the codimension of* $\overline{\{p\}}$ *in* V. But, the (Krull) dimension of $\mathcal{O}_{V,p}$ is equal to the height of $\mathfrak{m}_p$, and thus

$$\dim_k(V) = \dim_k \overline{\{p\}} + \dim \mathcal{O}_{V,p}. \tag{$**$}$$

If $\kappa(p)$ is separable over k, $(*)$ and $(**)$ show

$$\dim_{\kappa(p)} T_{V,p}(\kappa(p)) = \dim_k(V) \quad \text{iff} \quad \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) = \dim_{\kappa(p)} \mathcal{O}_{V,p}.$$

Putting things together, we find that the following properties hold: Given $p \in V$,

$$\begin{aligned} T_{V,p}(\kappa(p)) &= \mathrm{Der}_k(\kappa(p), \kappa(p)) \coprod \mathrm{Der}_{\kappa(p)}(\widehat{\mathcal{O}}_{V,p}, \kappa(p)) \\ &= \mathrm{Der}_k(\kappa(p), \kappa(p)) \coprod \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D, \end{aligned}$$

and,

$$\begin{aligned} \dim_{\kappa(p)} T_{V,p}(\kappa(p)) &= \dim_{\kappa(p)} \mathrm{Der}_k(\kappa(p), \kappa(p)) + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \\ &\geq \dim_k \overline{\{p\}} + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right), \end{aligned}$$

where equality holds if $\kappa(p)$ is separable over k,

$$\dim_k(V) = \dim_k \overline{\{p\}} + \dim \mathcal{O}_{V,p} \leq \dim_k \overline{\{p\}} + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right).$$

Hence,

$$\begin{aligned} \dim_{\kappa(p)} T_{V,p}(\kappa(p)) &= \dim_{\kappa(p)} \mathrm{Der}_k(\kappa(p), \kappa(p)) + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \\ &\geq \dim_k \overline{\{p\}} + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \\ &\geq \dim_k \overline{\{p\}} + \dim \mathcal{O}_{V,p} = \dim_k V, \end{aligned}$$

where equality implies that

(1) $\dim_k \overline{\{p\}} = \dim_{\kappa(p)} \mathrm{Der}_k(\kappa(p), \kappa(p))$;

(2) $\dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) = \dim \mathcal{O}_{V,p}$.

Of course, condition (1) will be taken care of by our separability assumption. Local rings satisfying condition (2) are called *regular local rings*. Thus, if p is a nonsingular point on V, then $\mathcal{O}_{V,p}$ is regular; we can summarize all this in the following proposition due to Zariski.

Proposition 2.16 *Let V be an irreducible k-variety and let $p \in V$. The following properties hold:*

(1) If p is a closed point and $\kappa(p)$ is separable over k, then

$$T_{V,p}(\kappa(p)) = \mathrm{Der}_{\kappa(p)}(\widehat{\mathcal{O}}_{V,p}, \kappa(p)) = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D,$$

as $\kappa(p)$-vector space.

(2) If p is not necessarily closed, then

$$T_{V,p}(\kappa(p))/\mathrm{Der}_k(\kappa(p), \kappa(p)) \cong \mathrm{Der}_{\kappa(p)}(\widehat{\mathcal{O}}_{V,p}, \kappa(p)) = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D.$$

(3) If $\kappa(p)$ is separable over k, then

$$\dim_k T_{V,p}(\kappa(p)) = \dim_k \overline{\{p\}} + \dim \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right).$$

(4) We always have

$$\begin{aligned} \dim_{\kappa(p)} T_{V,p}(\kappa(p)) &\geq \dim_k \overline{\{p\}} + \dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \\ &\geq \dim_k \overline{\{p\}} + \dim \mathcal{O}_{V,p} = \dim_k V. \end{aligned}$$

(5) If p is nonsingular, then $\mathcal{O}_{V,p}$ is a regular local ring. If $\kappa(p)$ is separable over k and $\mathcal{O}_{V,p}$ is a regular ring, then p is nonsingular.

(6) Separable generation is automatic if k is perfect, e.g., (a) k has characteristic 0, (b) k is algebraically closed, (c) k is a finite field (Zariski and Samuel [60], Theorem 31, Chapter II, Section 13).

Since $\kappa(p)$ is the field canonically associated to p, and because we want $T_{V,p}$ to be a relative invariant depending on V and $\kappa(p)$, we make the following improved definition of the Zariski tangent space $T_{V,p}$ to V at p.

Definition 2.10 Let V be an irreducible k-variety and let $p \in V$. The *Zariski tangent space, $T_{V,p}$, to V at p* is the $\kappa(p)$-vector space $\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^D$ (where $\mathfrak{m}_p$ is the maximal ideal in $\mathcal{O}_{V,p}$). The *Zariski cotangent space to V at p* is the $\kappa(p)$-vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Remark: We have

$$\text{old } T_{V,p}(\kappa(p))/\mathrm{Der}_k(\kappa(p),\kappa(p)) = \text{ new } T_{V,p}.$$

If p is closed and $\kappa(p)$ is separable over k, then

$$\text{old } T_{V,p}(\kappa(p)) = \text{ new } T_{V,p}.$$

It is interesting to observe that the tangent space at a point can be recovered from the points of a variety with values in a special ring; namely, the *ring of dual numbers over the integers*. This is the ring $\mathbb{Z}[T]/(T^2)$, and it is usually denoted Λ. Define

$$\Lambda_{\text{at } p} = \Lambda \otimes_{\mathbb{Z}} \kappa(p) = \kappa(p)[T]/(T^2).$$

If V is an affine irreducible variety, we can form $V(\Lambda_{\text{at } p})$ for any point $p \in V$. By definition

$$V(\Lambda_{\text{at } p}) = \mathrm{Hom}_{k-\mathrm{alg}}(k[V], \Lambda_{\text{at } p}).$$

We have $k[V] \cong k[X_1, \ldots, X_n]/\mathfrak{I}(V)$, and thus, $k[V] = k[x_1, \ldots, x_n]$ (where x_j is the image of X_j). If $\varphi \in V(\Lambda_{\text{at } p})$, then

$$\varphi(x_j) = \alpha(x_j) + \beta(x_j)\epsilon,$$

where $\epsilon = T \bmod (T^2)$. Since φ is a homomorphism, $\varphi(x_j x_k) = \varphi(x_j)\varphi(x_k)$ implies that

$$\begin{aligned} \alpha(x_j x_k) &= \alpha(x_j)\alpha(x_k) \\ \beta(x_j x_k) &= \alpha(x_j)\beta(x_k) + \beta(x_j)\alpha(x_k). \end{aligned}$$

Also, since φ is k-linear, so are α and β. Thus,

(1) $\alpha \in \mathrm{Hom}_{k-\mathrm{alg}}(k[V], \kappa(p))$;

(2) β is a "derivation" $D \colon k[V] \to \kappa(p)$.

In (1), we will always take α to be the homomorphism given by p itself, so that in (2), we get that β is a k-derivation of $k[V]$ with values in $\kappa(p)$ centered at p.

Conversely, (1) and (2) as just modified give a point of V with values in $\Lambda_{\text{at } p}$, centered at p. We'll get (1) automatically if we extend α, β to $\mathcal{O}_{V,p} = k[V]_{\mathfrak{I}(p)}$ and demand that α is just our map *res*: $\mathcal{O}_{V,p} \to \kappa(p)$. If we use the notation $V(\Lambda_{\text{at } p}; p)$ to mean those homomorphisms in which α is just the point p, we get

$$T_{V,p}(\kappa(p)) = \mathrm{Der}_k(\mathcal{O}_{V,p}, \kappa(p); p) \cong V(\Lambda_{\text{at } p}; p).$$

Hence, points of V in a sufficiently general ring give us tangent vectors of V at p.

In a similar manner, we can define the *jet space to V at p* to be the space

$$\mathrm{Jet}_p(V) = \coprod_n \left(\mathfrak{m}_p^n/\mathfrak{m}_p^{n+1}\right)^D,$$

and the *co-jet space to* V *at* p as the space

$$\text{co-Jet}_p(V) = \coprod_n \mathfrak{m}_p^n/\mathfrak{m}_p^{n+1} = \text{gr}(\mathcal{O}_{V,p}) = \text{gr}(\widehat{\mathcal{O}}_{V,p}).$$

Remark: The ring $\mathcal{O}_{V,p}$ is a regular local ring iff $\text{co-Jet}_p(V)$ is a polynomial ring over $\kappa(p)$ in $\dim_{\kappa(p)} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)$ variables. By definition, a point p is a *regular point* of V if $\mathcal{O}_{V,p}$ is a regular local ring. We write V_{reg} for the set of regular points of V. Then,

$$V_{\text{nonsing}} \subseteq V_{\text{reg}}.$$

Equality holds when k has characteristic 0.

2.3 Local Structure of a Variety

As in differential geometry the local structure of an algebraic variety has a great deal to do with the tangent space analysis and jet space analysis at a point. Moreover, one needs the completion of the local ring $\mathcal{O}_{V,p}$, and hence, one is led into an analysis and study of the power series ring centered at p. To set up the notation, we let A be a commutative ring and denote the ring of formal power series in the variables $X_1, \ldots, X_n$ by $A[[X_1, \ldots, X_n]]$. We have the following facts.

(1) For any $f \in A[[X_1, \ldots, X_n]]$, f is a unit iff $f(0, \ldots, 0)$ is a unit in A.

(2) A is a local ring iff $A[[X_1, \ldots, X_n]]$ is a local ring.

(3) A is Noetherian iff $A[[X_1, \ldots, X_n]]$ is Noetherian.

(4) If $\mathcal{O}$ is a local ring, then in the $\mathfrak{m}$-adic topology, $\widehat{\mathcal{O}}$ is Hausdorff iff $\bigcap_{j=0}^{\infty} \mathfrak{m}^j = (0)$, and the latter holds when $\mathcal{O}$ is Noetherian.

The fundamental results in this case are all essentially easy corollaries of the following lemma:

Lemma 2.17 *Let* $\mathcal{O}$ *be a complete Hausdorff local domain with respect to the* $\mathfrak{m}$*-adic topology, and let* $f \in \mathcal{O}[[X]]$*. Assume that*

(a) $f(0) \in \mathfrak{m}$*.*

(b) $\left(\frac{df}{dX}\right)(0)$ *is a unit of* $\mathcal{O}$*.*

Then, there exist unique elements $\alpha \in \mathfrak{m}$ *and* $u(X) \in \mathcal{O}[[X]]$*, so that*

(1) $u(X)$ *is a unit of* $\mathcal{O}[[X]]$*.*

(2) $f(X) = u(X)(X - \alpha)$*.*

Proof. We get $u(X)$ and α by successive approximations as follows. Refer to equation (2) by (†) in what follows. We compute the unknown coefficients of $u(X)$ and the element α by successive approximations. Write $u(X) = \sum_{j=0}^{\infty} u_j X^j$ and $f(X) = \sum_{j=0}^{\infty} a_j X^j$; reduce the coefficients modulo $\mathfrak{m}$ in (†); then, since $\alpha \in \mathfrak{m}$, (†) becomes

$$\overline{f(X)} = X\overline{u(X)},$$

which implies that

$$\sum_{j=0}^{\infty} \overline{a_j} X^j = \sum_{j=0}^{\infty} \overline{u_j} X^{j+1}.$$

Since $\overline{a_0} = 0$, we have $a_0 \in \mathfrak{m}$ and $\overline{u_j} = \overline{a_{j+1}}$. Thus,

$$u_j = a_{j+1} \pmod{\mathfrak{m}}.$$

Note that

$$\overline{u_0} = \overline{a_1} = \overline{\frac{\partial f}{\partial X}(0)} \neq 0$$

in $\kappa = \mathcal{O}/\mathfrak{m}$, which implies that if $u(X)$ exists at all, then it is a unit. Write

$$u_j = a_{j+1} + \xi_j^{(1)},$$

where $\xi_j^{(1)} \in \mathfrak{m}$, $j \geq 0$. Remember that $\alpha \in \mathfrak{m}$; so, upon reducing (†) modulo $\mathfrak{m}^2$, we get

$$\overline{\overline{f(X)}} = \overline{\overline{u(X)}}(X - \overline{\overline{\alpha}}).$$

This implies that

$$\begin{aligned}
\sum_{j=0}^{\infty} \overline{\overline{a_j}} X^j &= \sum_{j=0}^{\infty} \overline{\overline{u_j}} X^j (X - \overline{\overline{\alpha}}) \\
&= \sum_{j=0}^{\infty} \overline{\overline{u_j}} X^{j+1} - \sum_{j=0}^{\infty} \overline{\overline{u_j}}\, \overline{\overline{\alpha}} X^j \\
&= \sum_{j=0}^{\infty} \left(\overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} \right) X^{j+1} - \sum_{j=0}^{\infty} \left(\overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} \right) \overline{\overline{\alpha}} X^j \\
&= \sum_{j=0}^{\infty} \overline{\overline{a}}_{j+1} X^{j+1} + \sum_{j=0}^{\infty} \overline{\overline{\xi_j^{(1)}}} X^{j+1} - \sum_{j=0}^{\infty} \overline{\overline{a}}_{j+1}\, \overline{\overline{\alpha}} X^j.
\end{aligned}$$

When $j = 0$, we get

$$\overline{\overline{a_0}} = -\overline{\overline{a_1}}\,\overline{\overline{\alpha}}.$$

Since a_1 is a unit, $\overline{\overline{\alpha}}$ exists. Now, looking at the coefficient of X^{j+1}, we get

$$\overline{\overline{a}}_{j+1} = \overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} - \overline{\overline{a}}_{j+2} \overline{\overline{\alpha}},$$

which implies that

$$\overline{\overline{\xi_j^{(1)}}} = \overline{\overline{a}}_{j+2}\overline{\overline{\alpha}},$$

and $\overline{\overline{\xi_j^{(1)}}}$ exists.

We now proceed by induction. Assume that we know the coefficients $u_j^{(t)} \in \mathcal{O}$ of the t-th approximation to $u(X)$ and that $u(X)$ using these coefficients $(\mathrm{mod}\, \mathfrak{m}^{t+1})$ works in (†), and further that the $u_l^{(t)}$'s are consistent for $l \leq t$. Also, assume $\alpha^{(t)} \in \mathfrak{m}$, that $\alpha^{(t)}$ $(\mathrm{mod}\, \mathfrak{m}^{t+1})$ works in (†), and that the $\alpha^{(l)}$ are consistent for $l \leq t$. Look at $u_j^{(t)} + \xi_j^{(t+1)}$, $\alpha^{(t)} + \eta^{(t+1)}$, where $\xi_j^{(t+1)}, \eta^{(t+1)} \in \mathfrak{m}^{t+1}$. We want to determine $\xi_j^{(t+1)}$ and $\eta^{(t+1)}$, so that (†) will work for these modulo $\mathfrak{m}^{t+2}$. For simplicity, write bar as a superscript to denote reduction modulo $\mathfrak{m}^{t+2}$. Then, reducing (†) modulo $\mathfrak{m}^{t+2}$, we get

$$\begin{aligned}
\sum_{j=0}^{\infty} \overline{a_j} X^j &= \sum_{j=0}^{\infty} \overline{u_j} X^j (X - \overline{\alpha}) \\
&= \sum_{j=0}^{\infty} \overline{u_j} X^{j+1} - \sum_{j=0}^{\infty} \overline{u_j}\,\overline{\alpha} X^j \\
&= \sum_{j=0}^{\infty} \left(\overline{u_j^{(t)}} + \overline{\xi_j^{(t+1)}}\right) X^{j+1} - \sum_{j=0}^{\infty} \left(\overline{u_j^{(t)}} + \overline{\xi_j^{(t+1)}}\right) \left(\overline{\alpha^{(t)}} + \overline{\eta^{(t+1)}}\right) X^j \\
&= \sum_{j=0}^{\infty} \overline{u_j^{(t)}} X^{j+1} + \sum_{j=0}^{\infty} \overline{\xi_j^{(t+1)}} X^{j+1} - \sum_{j=0}^{\infty} \overline{u_j^{(t)}}\,\overline{\alpha^{(t)}} X^j - \sum_{j=0}^{\infty} \overline{u_j^{(t)}}\,\overline{\eta^{(t+1)}} X^j.
\end{aligned}$$

For $j = 0$, we get

$$\overline{a_0} = -\overline{u_0^{(t)}}\,\overline{\alpha^{(t)}} - \overline{u_0^{(t)}}\,\overline{\eta^{(t+1)}}.$$

But $\overline{u_0^{(t)}}$ is a unit, and so, $\overline{\eta^{(t+1)}}$ exists. Now, look at the coefficient of X^{j+1}, we have

$$\overline{a_{j+1}} = \overline{u_j^{(t)}} + \overline{\xi_j^{(t+1)}} - \overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}} - \overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}.$$

But $\overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}}$ and $\overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}$ are now known and in $\mathfrak{m}^{t+1}$ modulo $\mathfrak{m}^{t+2}$, and thus,

$$\overline{\xi_j^{(t+1)}} = \overline{a_{j+1}} - \overline{u_j^{(t)}} + \overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}} + \overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}$$

exists and the induction step goes through. As a consequence

$$u(X) \in \varprojlim_t (\mathcal{O}/\mathfrak{m}^t)[[X]]$$

and

$$\alpha \in \varprojlim_t (\mathfrak{m}/\mathfrak{m}^t)[[X]]$$

exist; and so, $u(X) \in \widehat{\mathcal{O}}[[X]] = \mathcal{O}[[X]]$, and $\alpha \in \widehat{\mathfrak{m}} = \mathfrak{m}$.

We still have to prove the uniqueness of $u(X)$ and α. Assume that

$$f = u(X - \alpha) = \widetilde{u}(X - \widetilde{\alpha}).$$

Since $\widetilde{u}$ is a unit,

$$\widetilde{u}^{-1}u(X - \alpha) = X - \widetilde{\alpha}.$$

Thus, we may assume that $\widetilde{u} = 1$. Since $\alpha \in \mathfrak{m}$, we can plug α into the power series which defines u, and get convergence in the $\mathfrak{m}$-adic topology of $\mathcal{O}$. We get

$$u(\alpha)(\alpha - \alpha) = \alpha - \widetilde{\alpha},$$

so that $\alpha = \widetilde{\alpha}$. Then,

$$u(X - \alpha) = X - \widetilde{\alpha},$$

and since we assumed that $\mathcal{O}$ is a domain, so is $\mathcal{O}[[X]]$, and thus, $u = 1$. $\square$

The fundamental lemma just proved leads almost immediately to the *formal implicit function theorem*:

Theorem 2.18 *(First form of the implicit function theorem: Weierstrass preparation theorem) Given $f \in k[[Z_1, \ldots, Z_n]]$, if*

$$f(0, \ldots, 0) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial Z_1}(0) \neq 0,$$

then there exist unique power series $u(Z_1, \ldots, Z_n)$ and $g(Z_2, \ldots, Z_n)$ so that $u(Z_1, \ldots, Z_n)$ is a unit, $g(0, \ldots, 0) = 0$, and $f(Z_1, \ldots, Z_n)$ factors as

$$f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1 - g(Z_2, \ldots, Z_n)). \tag{$*$}$$

Moreover, every power series $h(Z_1, \ldots, Z_n)$ factors uniquely as

$$h(Z_1, \ldots, Z_n) = f(Z_1, \ldots, Z_n)q(Z_1, \ldots, Z_n) + r(Z_2, \ldots, Z_n).$$

Hence, there is a canonical isomorphism

$$k[[Z_1, \ldots, Z_n]]/(f) \cong k[[Z_2, \ldots, Z_n]],$$

so that the following diagram commutes

$$\begin{tikzcd}
k[[Z_1, \ldots, Z_n]] \arrow[rr] & & k[[Z_1, \ldots, Z_n]]/(f) \\
& k[[Z_2, \ldots, Z_n]] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. First, observe that equation $(*)$ (the Weierstrass preparation theorem) implies the second statement. For, assume $(*)$; then, u is a unit, so there is v such that $vu = 1$. Consequently,

$$vf = Z_1 - g(Z_2, \ldots, Z_n),$$

and the ideal (f) equals the ideal (vf), because v is a unit. So,

$$k[[Z_1, \ldots, Z_n]]/(f) = k[[Z_1, \ldots, Z_n]]/(vf),$$

and we get the residue ring by setting Z_1 equal to $g(Z_2, \ldots, Z_n)$. It follows that the canonical isomorphism

$$k[[Z_1, \ldots, Z_n]]/(f) \cong k[[Z_2, \ldots, Z_n]]$$

is given as follows: In $h(Z_1, \ldots, Z_n)$, replace every occurrence of Z_1 by $g(Z_2, \ldots, Z_n)$; we obtain

$$\overline{h}(Z_2, \ldots, Z_n) = h(g(Z_2, \ldots, Z_n), Z_2, \ldots, Z_n),$$

and the diagram obviously commutes. Write $r(Z_2, \ldots, Z_n)$ instead of $\overline{h}(Z_2, \ldots, Z_n)$. Then,

$$h(Z_1, \ldots, Z_n) - r(Z_2, \ldots, Z_n) = fq$$

for some $q(Z_1, \ldots, Z_n)$. We still have to show uniqueness. Assume that

$$h(Z_1, \ldots, Z_n) = fq + r = f\widetilde{q} + \widetilde{r}.$$

Since $g(0, \ldots, 0) = 0$, we have $g \in \mathfrak{m}$; thus, we can plug in $Z_1 = g(Z_2, \ldots, Z_n)$ and get $\mathfrak{m}$-adic convergence. By $(*)$, f goes to 0, and the commutative diagram shows $r \pmod f = r$ and $\widetilde{r} \pmod f = \widetilde{r}$. Hence, we get

$$r = \widetilde{r},$$

so that

$$fq - f\widetilde{q} = 0.$$

Now, $k[[Z_1, \ldots, Z_n]]$ is a domain, so $q = \widetilde{q}$.

Now let us prove $(*)$, the Weierstrass preparation theorem. We will apply the previous lemma to $\mathcal{O} = k[[Z_2, \ldots, Z_n]]$, because then, $\mathcal{O}[[Z_1]] = k[[Z_1, \ldots, Z_n]]$. Viewing f as an element of $\mathcal{O}[[Z_1]]$, we find that $f(0)$ is a power series in $Z_2, \ldots, Z_n$, and $f(0) \in \mathfrak{m}$ (the maximal ideal of $\mathcal{O}$), since $f(0, \ldots, 0) = 0$. Also $df/dZ_1 = \partial f/\partial Z_1$, and at $(0, \ldots, 0)$, this is not zero. Therefore, $\partial f/\partial Z_1(0)$ is a unit. Now, we can apply the fundamental lemma (Lemma 2.17). It says that there is some $g = \alpha \in \mathfrak{m}$ and some $u(Z_1, \ldots, Z_n)$ a unit, and we have

$$f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, z_n)(Z_1 - g(Z_2, \ldots, Z_n)).$$

Since $g \in \mathfrak{m}$, we have $g(0, \ldots, 0) = 0$. Uniqueness is obtained as in the lemma. $\square$

We can now apply induction to get the second version of the *formal implicit function theorem, or FIFT*.

Theorem 2.19 *(Second form of the implicit function theorem)*
Given $f_1, \ldots, f_r \in k[[Z_1, \ldots, Z_n]]$, if $f_j(0, \ldots, 0) = 0$ for $j = 1, \ldots, r$ and

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r$$

(so that $n \geq r$), then we can reorder the variables so that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r, \quad \text{where } 1 \leq i, j \leq r,$$

and there is a canonical isomorphism

$$k[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_r) \cong k[[Z_{r+1}, \ldots, Z_n]],$$

which makes the following diagram commute

$$\begin{tikzcd}
k[[Z_1, \ldots, Z_n]] \arrow[rr] & & k[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_r) \\
& k[[Z_{r+1}, \ldots, Z_n]] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. The proof of this statement is quite simple (using induction) from the previous theorem (DX). $\square$

An obvious question is what happens to these theorems in the convergent analytic case? Denote by $\mathbb{C}\{Z_1, \ldots, Z_n\}$ the subring of $\mathbb{C}[[Z_1, \ldots, Z_n]]$ consisting of convergent power series in the norm topology. For any $\xi \in \mathbb{C}^n$ and any $\epsilon > 0$, we define the *polydisc* $PD(\xi, \epsilon)$ by

$$PD(\xi, \epsilon) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i - \xi_i| < \epsilon, \text{ for every } i,\ 1 \leq i \leq n\}.$$

Here is the *convergent implicit function theorem* in the rank one case.

Theorem 2.20 *Let $f \in \mathbb{C}\{Z_1, \ldots, Z_n\}$ and suppose that $f(0, \ldots, 0) = 0$, but*

$$\frac{\partial f}{\partial Z_1}(0, \ldots, 0) \neq 0.$$

Then, there exists a unique power series $g(Z_2, \ldots, Z_n) \in \mathbb{C}\{Z_2 \ldots, Z_n\}$ and there is some $\epsilon > 0$, so that in the polydisc $PD(0, \epsilon)$, we have $f(\xi_1, \ldots, \xi_n) = 0$ if and only if $\xi_1 = g(\xi_2, \ldots, \xi_n)$.

Remark: To prove this, look at the formal version of the implicit function theorem and at the fundamental lemma (i.e., the construction of $u(Z_1, \ldots, Z_n)$ and $g(Z_2, \ldots, Z_n)$). Then show (tricky and messy!) that u and g converge in some ϵ-neighborhood of $(0, \ldots, 0)$. By the Weierstrass preparation theorem (now proved in the convergent case),

$$f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1 - g(Z_2, \ldots, Z_n)).$$

But, for ϵ small enough, $PD(0, \epsilon)$ is contained in this open and on the polydisc:

(1) u converges and is never 0.

(2) g converges and $g(0,\dots,0)=0$.

So, in $PD(0,\epsilon)$, we have $f=0$ iff $u(Z_1-g)=0$ iff $Z_1=g(Z_2,\dots,Z_n)$.

However, to avoid tricks and mess, observe that a polydisc is just a product of one-dimensional discs. Therefore, there exists a Cauchy multi-integral formula for f valid in $PD(\xi,\epsilon)$ (DX). We know that the implicit function theorem for holomorphic functions of one variable is an easy consequence of Cauchy's formula (Ahlfors [1], Chapter 4, Theorem 4 ff). Thus, we get that $Z_1=g(Z_2,\dots,Z_n)$ where g is a holomorphic function, and thus, a convergent power series in some suitable polydisc.

An easy induction yields the *convergent implicit function theorem*:

Theorem 2.21 *(Convergent implicit function theorem) Let $f_1,\dots,f_r\in\mathbb{C}\{Z_1,\dots,Z_n\}$. If $f_j(0,\dots,0)=0$ for $j=1,\dots,r$ and*

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right)=r$$

(so that $n\geq r$), then there is a permutation of the variables so that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right)=r,\quad \textit{where } 1\leq i,j\leq r$$

and there exist r unique power series $g_j(Z_{r+1},\dots,Z_n)\in\mathbb{C}\{Z_2\dots,Z_n\}$ $(1\leq j\leq r)$ and an $\epsilon>0$, so that in the polydisc $PD(0,\epsilon)$, we have

$$f_1(\xi)=\cdots=f_r(\xi)=0\quad \textit{iff}\quad \xi_j=g_j(\xi_{r+1},\dots,\xi_n),\quad \textit{for } j=1,\dots,r.$$

Moreover

$$\mathbb{C}\{Z_1,\dots,Z_n\}/(f_1,\dots,f_r)\cong\mathbb{C}\{Z_{r+1},\dots,Z_n\}.$$

When $r=n$, we have another form of the convergent implicit function theorem also called the *inverse function theorem.*

Theorem 2.22 *(Inverse function theorem) Let $f_1,\dots,f_n\in\mathbb{C}\{Z_1,\dots,Z_n\}$ and suppose that $f_j(0,\dots,0)=0$ for $j=1,\dots,n$, but*

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0,\dots,0)\right)=n.$$

Then, there exist n unique power series $g_j(W_1,\dots,W_n)\in\mathbb{C}\{W_1\dots,W_n\}$ $(1\leq j\leq n)$ and there are some open neighborhoods of $(0,\dots,0)$ (in the Z's and in the W's), call them U and V, so that the holomorphic maps

$$(Z_1,\dots,Z_n)\mapsto(W_1=f_1(Z_1,\dots,Z_n),\dots,W_n=f_n(Z_1,\dots,Z_n))\colon U\to V$$
$$(W_1,\dots,W_n)\mapsto(Z_1=g_1(W_1,\dots,W_n),\dots,Z_n=g_n(W_1,\dots,W_n))\colon V\to U$$

are inverse isomorphisms.

The reader should have no difficulty in supplying the proof. Use of the formal implicit function theorem and formal power series in general will give us the local structure of irreducible varieties. We need the

Definition 2.11 Let X be an affine variety, $X \subseteq \mathbb{A}^n$, and assume that $\dim X = d$. Then, X is a *complete intersection* if $\mathfrak{I}(X)$ has $n-d$ generators. If X is any k-variety and $\xi \in X$, then X is a *local complete intersection at* ξ if there is some affine open $X(\xi)$, with $\xi \in X(\xi) \subseteq X$, so that $X(\xi)$ can be embedded in $\mathbb{A}^n$ for some n, and $X(\xi)$ is a complete intersection. The variety X is a *local complete intersection* if it is a local complete intersection at ξ for all $\xi \in X$.

Theorem 2.23 *(Local complete intersection theorem) Let V be an irreducible k-variety and $p \in V$ a k-rational nonsingular point. Write* $\dim(V) = d$ *and assume that near p, the variety V has local embedding dimension n, which means that there is some affine open, $U \subseteq V$, with $p \in U$ such that U can be embedded into $\mathbb{A}^n$ as a k-closed subset (we may assume that n is minimal). Then, there exist polynomials $f_1, \ldots, f_r$ in n variables with $r = n - d$, so that k-locally on V near p, the variety V is cut out by $f_1, \ldots, f_r$. This means that there exists a possibly smaller k-open $W \subseteq U \subseteq V$ with $p \in W$ so that*

$$q \in W \quad \textit{if and only if} \quad f_1(q) = \cdots = f_r(q) = 0.$$

The local complete intersection theorem will be obtained from the following affine form of the theorem.

Theorem 2.24 *(Affine local complete intersection theorem) Let $V \subseteq \mathbb{A}^n$ be an affine irreducible k-variety of dimension* $\dim(V) = d$*, and assume that $V = V(\mathfrak{p})$. If $p \in V$ is nonsingular k-rational point, then there exist $f_1, \ldots, f_r \in \mathfrak{p}$, with $r = n - d$, so that*

$$\mathfrak{p} = \left\{ g \in k[Z_1, \ldots, Z_n] \;\middle|\; g = \sum_{i=1}^{r} \frac{h_i(Z_1, \ldots, Z_n)}{l(Z_1, \ldots, Z_n)} f_i(Z_1, \ldots, Z_n)\,, \quad \textit{and} \quad l(p) \neq 0 \right\}, \qquad (\dagger)$$

where h_i and $l \in k[Z_1, \ldots, Z_n]$. The f_i's having the above property are exactly those $f_i \in \mathfrak{p}$ whose differentials df_i cut out the tangent space $T_{V,p}$ (i.e., these differentials are linearly independent).

What are we saying? Intuitively, near p (in the k-topology), the behavior of V should be controlled by $\mathcal{O}_{V,p}$. Write $\mathfrak{A}$ for the ideal $(f_1, \ldots, f_r)$ (this is contained in $\mathfrak{p}$), and consider the diagram

$$\begin{tikzcd}
k[Z_1, \ldots, Z_n]]r] \arrow[d] & k[Z_1, \ldots, Z_n]_{\mathfrak{I}(p)} = \mathcal{O}_{\mathbb{A}^n,p} \arrow[d] \\
k[V] \arrow[r] & k[V]_{\overline{\mathfrak{I}}(p)} = \mathcal{O}_{V,p}.
\end{tikzcd}$$

The kernel of the left vertical map is $\mathfrak{p}$ and the kernel of the right vertical map is

$$\mathfrak{p}^e = \mathfrak{p}\mathcal{O}_{\mathbb{A}^n,p}.$$

Since,

$$\mathfrak{A}^e = (f_1, \ldots, f_r)\mathcal{O}_{\mathbb{A}^n,p},$$

the righthand side of (†) is exactly $\mathfrak{A}^{ec}$ (that is, $\mathfrak{A}$ extended and then contracted). We know that $\mathfrak{p}^{ec} = \mathfrak{p}$, and thus, we are saying that

$$\mathfrak{p} = \mathfrak{A}^{ec}.$$

But then, $\mathfrak{p}^e = \mathfrak{A}^e$, which means that (†) says: If $f_1, \ldots, f_r$ generate $\mathfrak{p}^e$ (i.e., $\mathfrak{p}$ in $\mathcal{O}_{\mathbb{A}^n,p}$), then they cut out V near p.

Proof of the local complete intersection theorem (Theorem 2.23). We show that the affine local complete intersection theorem (Theorem 2.24) implies the general one (Theorem 2.23). There is some open affine set, say U, with $p \in U$. By working with U instead of V, we may assume that V is affine. Let $V = V(\mathfrak{p})$, and let $\mathfrak{A} = (f_1, \ldots, f_r)$, in $k[\mathbb{A}^n]$. Suppose that g_1, $\ldots$, g_t are some generators for $\mathfrak{p}$. By the affine local complete intersection theorem (Theorem 2.24), there are some $l_1, \ldots, l_t$ with $l_j(p) \neq 0$, so that

$$g_j = \sum_{i=1}^{r} \frac{h_{ij}}{l_j} f_i, \quad \text{for } j = 1, \ldots, t.$$

Let $l = \prod_{j=1}^{t} l_j$ and let W be the k-open where l does not vanish. We have $p \in W$, and we also have

$$k[V \cap W] = k[V]_{(\overline{l})} = k[X_1, \ldots, X_n]_{(l)}/\mathfrak{p}.$$

But

$$l_j g_j = \sum_{i=1}^{r} h_{ij} f_i,$$

and on $V \cap W$, the l_j's are units. Therefore, $\mathfrak{p}_{(l)} = \mathfrak{A}_{(l)}$, that is,

$$\mathfrak{p}k[X_1, \ldots, X_n]_{(l)} = \mathfrak{A}k[X_1, \ldots, X_n]_{(l)}.$$

Thus, on $V \cap W$, we have $\mathfrak{p} = \mathfrak{A}$ in the above sense, and so, $V \cap W$ is the variety given by the f_j's. The affine version of the theorem implies that $r = n - d$. $\square$

Remarks:

(1) The set $Y = V - V \cap W$ is k-closed, $Y \subseteq V$, and $p \notin Y$.

(2) The local complete intersection theorem says that X is a local complete intersection at every nonsingular point; so, $X - \operatorname{Sing} X$ is a local complete intersection.

We now turn to the proof of the affine theorem.

Proof of the affine local complete intersection theorem (Theorem 2.24). Let the righthand side of (†) be $\mathfrak{P}$. Given any $g \in \mathfrak{P}$, there is some l so that

$$lg = \sum_{i=1}^{r} h_i f_i.$$

Since $f_i \in \mathfrak{p}$, we have $lg \in \mathfrak{p}$. But $l(p) \neq 0$, so $l \notin \mathfrak{p}$; and since $\mathfrak{p}$ is prime, we must have $g \in \mathfrak{p}$. Thus, we have

$$\mathfrak{P} \subseteq \mathfrak{p}.$$

By translation, we can move p to the origin, and we may assume that $p = 0$. Now, the proof of our theorem rests on the following proposition:

Proposition 2.25 *(Zariski) Let $f_1, \ldots, f_r \in k[X_1, \ldots, X_n]$ be polynomials with $f_1(0, \ldots, 0) = \cdots = f_r(0, \ldots, 0) = 0$, and linearly independent linear terms at $(0, \ldots, 0)$. Then, the ideal*

$$\mathfrak{P} = \left\{ g \in k[X_1, \ldots, X_n] \;\middle|\; g = \sum_{i=1}^{r} \frac{h_i(X_1, \ldots, X_n)}{l(X_1, \ldots, X_n)} f_i(X_1, \ldots, X_n)\,, \quad \textit{and} \quad l(0, \ldots, 0) \neq 0 \right\}$$

is a prime ideal and $V(\mathfrak{P})$ has dimension $n-r$. Moreover, $(0, \ldots, 0) \in V(\mathfrak{P})$ is a nonsingular point and $V(f_1, \ldots, f_r) = V(\mathfrak{P}) \cup Y$, where Y is k-closed and $(0, \ldots, 0) \notin Y$.

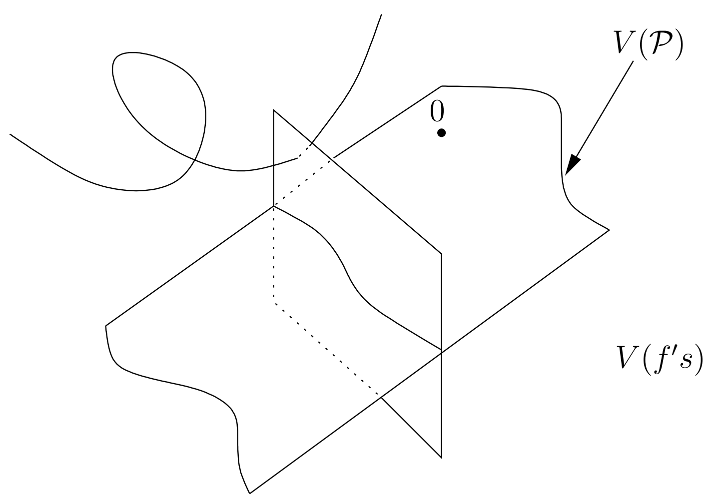

Figure 2.5: Illustration of Proposition 2.25

If we assume Zariski's Proposition 2.25, we can finish the proof of the affine local complete intersection theorem (Theorem 2.24): Since $p = (0, \ldots, 0)$ is nonsingular, we find

$\dim T_{V,0} = d$, the differentials of $f_1, \ldots, f_r$ are linearly independent if and only if they cut out $T_{V,0}$. Then, $V(\mathfrak{P})$ has dimension $n - r = d$. By Proposition 2.25, $\mathfrak{P}$ is prime, and we have already proved $\mathfrak{P} \subseteq \mathfrak{p}$. However,

$$\dim V(\mathfrak{P}) = \dim V(\mathfrak{p});$$

so, we get $V(\mathfrak{P}) = V(\mathfrak{p})$, and thus, $\mathfrak{P} = \mathfrak{p}$. This proves the affine local complete intersection theorem. $\square$

It remains to prove Zariski's proposition.

Proof of Proposition 2.25. We have the three rings

$$\begin{aligned} R &= k[X_1, \ldots, X_n], \\ R' &= k[X_1, \ldots, X_n]_{(X_1, \ldots, X_n)} = \mathcal{O}_{\mathbb{A}^n,0}, \quad \text{and} \\ R'' &= k[[X_1, \ldots, X_n]]. \end{aligned}$$

If $l \in \mathcal{O}_{\mathbb{A}^n,0} \cap k[X_1, \ldots, X_n]$ and $l(0) \neq 0$, then

$$l(X_1, \ldots, X_n) = l(0) \left(1 + \sum_{j=1}^{n} a_j(X_1, \ldots, X_n) X_j \right),$$

where $a_j(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$. But then,

$$\frac{1}{1 + \sum_{j=1}^{n} a_j(X_1, \ldots, X_n) X_j} = \sum_{r=0}^{\infty} (-1)^r \left(\sum_{j=1}^{n} a_j(X_1, \ldots, X_n) X_j \right)^r,$$

which belongs to $k[[X_1, \ldots, X_n]]$. Hence, we have inclusions

$$R \hookrightarrow R' \hookrightarrow R''.$$

Let $\mathfrak{P}' = (f_1, \ldots, f_r)R'$ and write $\mathfrak{P}'' = (f_1, \ldots, f_r)R''$. By definition, $\mathfrak{P} = \mathfrak{P}' \cap R$. If we can show that $\mathfrak{P}'$ is a prime ideal, then $\mathfrak{P}$ will be prime, too.

Claim: $\mathfrak{P}' = \mathfrak{P}'' \cap R'$.

Let $g \in \mathfrak{P}'' \cap R'$. Then,

$$g = \sum_{i=1}^{r} h_i f_i,$$

with $g \in R'$, by assumption, and with $h_i \in R''$. We can define the notion of "vanishing to order t of a power series," and with "obvious notation," we can write

$$h_i = \widetilde{h_i} + O(X^t),$$

where $\deg \widetilde{h}_i < t$. Because $f_i(0, \ldots, 0) = 0$ for each i, we find that

$$g = \sum_{i=1}^{r} \widetilde{h}_i f_i + O(X^{t+1}),$$

and thus,

$$g \in \mathfrak{P}' + (X_1, \ldots, X_n)^{t+1} R', \quad \text{for all } t.$$

As a consequence,

$$g \in \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + (X_1, \ldots, X_n)^{t+1} R'\right);$$

so,

$$\mathfrak{P}'' \cap R' \subseteq \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + (X_1, \ldots, X_n)^{t+1} R'\right).$$

But R' is a Noetherian local ring, and by Krull's intersection theorem (Zariski and Samuel [60], Theorem 12$'$, Chapter IV, Section 7), $\mathfrak{P}'$ is closed in the $\mathfrak{M}$-adic topology of R' (where, $\mathfrak{M} = (X_1, \ldots, X_n)R'$). Consequently,

$$\mathfrak{P}' = \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + \mathfrak{M}^{t+1}\right),$$

and we have proved

$$\mathfrak{P}'' \cap R' \subseteq \mathfrak{P}'.$$

Since we already know that $\mathfrak{P}' \subseteq \mathfrak{P}'' \cap R'$, we get our claim. Thus, if we knew $\mathfrak{P}''$ were prime, then so would be $\mathfrak{P}'$. Now, the linear terms of $f_1, \ldots, f_r$ at $(0, \ldots, 0)$ are linearly independent, thus,

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial X_j}(0)\right) = r,$$

and we can apply the formal implicit function theorem (Theorem 2.19). As a result, we get the isomorphism

$$R''/\mathfrak{P}'' \cong k[[X_{r+1}, \ldots, X_n]].$$

However, since $k[[X_{r+1}, \ldots, X_n]]$ is an integral domain, $\mathfrak{P}''$ must be a prime ideal. Hence, our chain of arguments proved that $\mathfrak{P}$ is a prime ideal. To calculate the dimension of $V(\mathfrak{P})$, observe that

$$\mathfrak{P}'' \cap R = \mathfrak{P}'' \cap R' \cap R = \mathfrak{P}' \cap R = \mathfrak{P},$$

and we also have

$$k[X_1, \ldots, X_n]/\mathfrak{P} \hookrightarrow k[[X_1, \ldots, X_n]]/\mathfrak{P}'' \cong k[[X_{r+1}, \ldots, X_n]].$$

Therefore, $X_{r+1}, \ldots, X_n \pmod{\mathfrak{P}}$ are algebraically independent over k, which implies that $\dim V(\mathfrak{P}) \geq n - r$. Now, the linear terms of $f_1, \ldots, f_r$ cut out the linear space $T_{V,0}$, and by linear independence, this space has dimension $n - r$. Then,

$$n - r = \dim T_{V,0} \geq \dim V(\mathfrak{P}) \geq n - r,$$

so that $\dim V(\mathfrak{P}) = n - r$, and 0 is nonsingular.

If $g \in \mathfrak{P}$, there exists some l with $l(0) \neq 0$ such that

$$g = \sum_{i=1}^{r} \frac{h_i}{l} f_i,$$

which implies that $lg \in (f_1, \ldots, f_r)$. Applying this fact to each of the generators of $\mathfrak{P}$, say, $g_1, \ldots, g_t$, and letting $l = \prod_{i=1}^{t} l_i$, we have

$$l\mathfrak{P} \subseteq (f_1, \ldots, f_r) \subseteq \mathfrak{P}.$$

As a consequence,

$$V(\mathfrak{P}) \subseteq V(f_1, \ldots, f_r) \subseteq V(l\mathfrak{P}) = V(l) \cup V(\mathfrak{P}).$$

If we let $Y = V(l) \cap V(f_1, \ldots, f_r)$, we have

$$V(f_1, \ldots, f_r) = V(\mathfrak{P}) \cup Y.$$

Since $l(0) \neq 0$, we have $0 \notin Y$. $\square$

We would like to take a closer look at the completion, $\widehat{\mathcal{O}}_{V,\xi}$, of the local ring at some nonsingular point $\xi \in V$. Since everything is local, we may assume that $V \subseteq \mathbb{A}^n$ is affine, and $V = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $d = \dim V$, and let $\xi \in V$ be a k-rational point. We know that $\dim T_{V,\xi} = d$, and that $T_{V,\xi}$ is cut out by the r linearly independent differentials $df_1, \ldots, df_r$ at ξ for some $f_1, \ldots, f_r \in \mathfrak{p}$, where $r = n - d$. Also,

$$\mathcal{O}_{\mathbb{A}^n,\xi}/\mathfrak{p} = \mathcal{O}_{V,\xi}.$$

Pick $y_1, \ldots, y_d \in \mathfrak{m}_\xi$ so that $dy_1, \ldots, dy_d$ are linearly independent at ξ. This is equivalent to saying that the residue classes $\overline{y_1}, \ldots, \overline{y_d}$ form a basis of $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$. We have

$$k[X_1, \ldots, X_n]_{(X_1-\xi_1,\ldots,X_n-\xi_n)} = \mathcal{O}_{\mathbb{A}^n,\xi} \hookrightarrow k[[X_1 - \xi_1, \ldots, X_n - \xi_n]]$$

and

$$(k[X_1, \ldots, X_n]/\mathfrak{p})_{(X_1-\xi_1,\ldots,X_n-\xi_n)} = \mathcal{O}_{V,\xi} \hookrightarrow k[[X_1 - \xi_1, \ldots, X_n - \xi_n]]/(f_1, \ldots, f_r).$$

By the formal implicit function theorem,

$$k[[X_1 - \xi_1, \ldots, X_n - \xi_n]]/(f_1, \ldots, f_r) \cong k[[X_{n-d+1} - \xi_{n-d+1}, \ldots, X_n - \xi_n]].$$

However, by the same theorem, the last ring is isomorphic to $k[[y_1, \ldots, y_d]]$. Therefore,

$$\widehat{\mathcal{O}}_{V,\xi} \cong k[[y_1, \ldots, y_d]].$$

We can summarize the above as follows:

Theorem 2.26 *Given an irreducible variety V and a nonsingular point $p \in V$, pick $y_1, \ldots,$ y_s in $\mathfrak{m}_p$ (where $s = \dim V$ if p is a closed point, and $s = \dim V - \dim \overline{\{p\}}$ otherwise), so that $dy_1, \ldots, dy_s$ are linearly independent forms on $T_{V,p}$ (equivalently, $\overline{y_1}, \ldots, \overline{y_s}$ are linearly independent in $\mathfrak{m}_p/\mathfrak{m}_p^2$ over $\kappa(p)$). Then, every $f \in \widehat{\mathcal{O}}_{V,p}$ can be written uniquely as a power series with coefficients in $\kappa(p)$, in the y_j's, that is,*

$$f \in \kappa(p)[[y_1, \ldots, y_s]].$$

This theorem is the formal series equivalent of the well-known Taylor series theorem in complex analysis. Namely, $f \in \widehat{\mathcal{O}_{V,p}}$ is a formal holomorphic function (formal referring to the fact that we have completed the ring of holomorphic functions in the $\mathfrak{m}$-adic topology) and we have just shown that such an f is expressed as a formal power series. The coefficients of this power series are necessarily in the field of definition of p, that is, in $\kappa(p)$.

Having mentioned complex analysis, we may inquire into the connection of our algebraic theory with the theory of *complex analytic spaces*. To avoid confusion, we call the usual complex topology the *norm topology*. We proceed as follows:

(1) Let $U \subseteq \mathbb{C}^q$ be norm-open. Consider p power series $f_1, \ldots, f_p$, norm-convergent on U. Define the topological space $Z_U \subseteq U$ by

$$Z_U = \{(\xi_1, \ldots, \xi_q) \in U \mid f_j(\xi_1, \ldots, \xi_q) = 0, \quad \text{with} \quad 1 \leq j \leq p\}.$$

(2) Define a *local holomorphic function* on Z_U, at $\xi \in Z_U$, to be a holomorphic function on some open $\widetilde{U} \subseteq U$ such that $\xi \in \widetilde{U}$. Call two such functions f, g equal if and only if $f \restriction Z_U = g \restriction Z_U$.

The pair $(Z_U, \mathcal{O}_{Z_U})$, where $\mathcal{O}_{Z_U}$ denotes the collection of locally defined holomorphic functions, is called a *complex analytic space chunk*.

It is quite clear what morphisms should be, and we get a category which is the analog of the category of affine varieties. By gluing complex analytic space chunks together, we get the concept and category of *complex analytic spaces*.

A complex algebraic variety, X, determines a unique complex analytic space, X^{an}, as follows: First, assume that $X \subseteq \mathbb{A}^n$ and that $X = V(\mathfrak{A})$, where $\mathfrak{A} \subseteq k[X_1, \ldots, X_n]$ is a radical ideal. Take $U = \mathbb{C}^q$ and $f_1, \ldots, f_p$ some generators of $\mathfrak{A}$. Since these are polynomials, they may be considered as power series, and obviously converge. We get $Z = X(\mathbb{C})$, and take all norm locally defined complex holomorphic functions on Z. This gives a complex analytic space chunk X^{an}. If X is an abstract variety, it is obtained by polynomial map gluing; hence, by holomorphic map gluing. Thus, each algebraic variety, X, yields a complex analytic space, X^{an}. Clearly, we have a functor

$$X \mapsto X^{\mathrm{an}}.$$

This functor was studied by Jean-Pierre Serre in his famous paper [48], also know as GAGA.

Theorem 2.27 *Let X be an irreducible complex algebraic variety and let $p \in X(\mathbb{C})$ be a nonsingular point. If locally in the Zariski topology near p, the variety X may be embedded in $\mathbb{A}^n$, then there exist d of the coordinates (of $\mathbb{A}^n$), say $Z_{n-d+1}, \ldots, Z_n$, so that $dZ_{n-d+1}, \ldots, dZ_n$ are linearly independent forms on $T_{X,p}$, and there exists some $\epsilon > 0$, and we have $n-d$ converging power series $g_1(Z_{n-d+1}, \ldots, Z_n)$, ..., $g_{n-d}(Z_{n-d+1}, \ldots, Z_n)$, so that*

$$(Z_1, \ldots, Z_n) \in PD(p, \epsilon) \quad \textit{iff} \quad Z_i - p_i = g_i(Z_{n-d+1} - p_{n-d+1}, \ldots, Z_n - p_n),\ i = 1, \ldots, n-d.$$

Any choice of d of the coordinates $Z_1, \ldots, Z_n$ so that the corresponding dZ_i's are linearly independent on $T_{X,p}$ will serve, and the map

$$X \cap PD(p, \epsilon) \longrightarrow PD(0, \epsilon)$$

given by

$$(Z_1, \ldots, Z_n) \mapsto (Z_{n-d+1} - p_{n-d+1}, \ldots, Z_n - p_n)$$

is an analytic isomorphism. Hence, if we take $(X - \mathrm{Sing}\, X)^{\mathrm{an}}$, it has the natural structure of a complex analytic manifold. Furthermore, X^{an} is a complex analytic manifold if and only if X is a nonsingular variety.

Proof. Since p is nonsingular, by the local complete intersection theorem (Theorem 2.23), we can cut out X locally (in the Zariski topology) by $f_1, \ldots, f_{n-d}$ and then we know that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(p)\right)$$

is maximal. By the convergent implicit function theorem (Theorem 2.21), there is some $\epsilon > 0$ and there are some power series $g_1, \ldots, g_{n-d}$ so that on $PD(p, \epsilon)$, we have

$$f_i(Z_1, \ldots, Z_n) = 0 \quad \text{iff} \quad Z_i - p_i = g_i(Z_{n-d+1} - p_{n-d+1}, \ldots, Z_n - p_n) \quad \text{for } i = 1, \ldots, n-d. (*)$$

The lefthand side says exactly that

$$(Z_1, \ldots, Z_n) \in X \cap PD(p, \epsilon).$$

We get a map by projection on the last d coordinates

$$X \cap PD(p, \epsilon) \longrightarrow PD(0, \epsilon),$$

whose inverse is given by the righthand side of equation $(*)$; and thus, the map is an analytic isomorphism. By the formal implicit function theorem (Theorem 2.19),

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_{n-d}) \cong \mathbb{C}[[Z_{n-d+1}, \ldots, Z_n]].$$

Hence, $dZ_{n-d+1}, \ldots, dZ_n$ are linearly independent on $T_{X,p}$. If conversely, the last d coordinates have linearly independent differentials $dZ_{n-d+1}, \ldots, dZ_n$, then

$$\dim T_{X,p} \leq d.$$

But p is nonsingular, and thus, $dZ_{n-d+1}, \ldots, dZ_n$ form a basis of $T_{X,p}$. Now, $T_{\mathbb{A}^n,p}$ is cut out by $df_1, \ldots, df_{n-d}$, $dZ_{n-d+1}, \ldots, dZ_n$, where $f_1, \ldots, f_{n-d}$ cut out X locally (in the Zariski topology) at p, by the local complete intersection theorem. It follows that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(p)\right)$$

is maximal (that is, $n-d$) and we can repeat our previous arguments. The last statement of the theorem is just a recap of what has already been proved. $\square$

Remark: The three notions of dimension

(a) algebraic (by transcendence degree)

(b) combinatorial (by chains of subvarieties)

(c) differential geometric (by tangent space dimension)

are seen to be all the same.

A complex variety has two topologies: its norm topology, and its Zariski topology. Obviously, every Zariski open is a norm open; and equally obviously, the converse is false. However, we can make some comparison between the topologies, and this is what we turn to now. We will need to know what a projective variety is and refer the reader to Section 2.5.

Proposition 2.28 *(Topological comparison, projective case) If X is a projective complex variety and U is Zariski–dense and Zariski–open, then U is norm–dense.*

Proof. The proof will be given in Section 2.5. The projective case leads to the general case:

Proposition 2.29 *(Topological comparison, general case) If X is a any complex variety and U is Zariski–dense and Zariski–open, then U is norm–dense.*

Proof. As usual, the argument reduces to the affine case; for assume that the argument works in the affine case. Let $X = \bigcup_\alpha U_\alpha$ be a cover by Zariski-open affine varieties. Assume that X_0 is Zariski–dense and Zariski–open in X. Write $X_{0,\alpha}$ for $X_0 \cap U_\alpha$. For any S, let $\overline{S}$ be the norm-closure of S and $\widehat{S}$ be the Zariski-closure of S. Of course, $\overline{S} \subseteq \widehat{S}$. The set $X_{0,\alpha}$ is Zariski–open in U_α and clearly, $X_{0,\alpha}$ is Zariski–dense in U_α. By the affine case, $X_{0,\alpha}$ is norm–dense in U_α, that is,

$$\overline{X_{0,\alpha}} \supseteq U_\alpha.$$

But then,

$$\overline{X_0} \supseteq \overline{X_{0,\alpha}} \supseteq U_\alpha$$

for all α, and thus,

$$\overline{X_0} = X,$$

and X_0 is norm-dense in X.

Let us now assume that $X \subseteq \mathbb{A}^n$ is affine. We know that $\mathbb{A}^n \subseteq \mathbb{P}^n$ and that $\mathbb{A}^n$ is Zariski-open in $\mathbb{P}^n$. Let $\widetilde{X}$ be the Zariski–closure of X in $\mathbb{P}^n$. We know that X_0 is Zariski–open in X and in $\widetilde{X}$, and that X_0 is Zariski-dense in $\widetilde{X}$, and hence in X. By Proposition 2.28, the open X_0 is norm-dense in X. $\square$

2.4 Nonsingular Varieties: Further Local Structure

Recall that in the interesection dimension theorem (Theorem 2.6), we examined the dimensions of the irreducible components of the intersection of two varieties in $\mathbb{A}^q$ and showed by counter-example that our results was not necessarily true if $\mathbb{A}^q$ was replaced by an arbitrary variety. The trouble occurred at a singular point of the ambient variety (in our current language). If we restrict attention to a neighborhood of a nonsingular point of the ambient variety, the theorem remains true. Here is the exact statement:

Theorem 2.30 *(Intersection dimension theorem: General form) Let X, Y, Z be irreducible k-varieties, with $X, Y \subseteq Z$ and Z separated. If $p \in X \cap Y$ and p is a nonsingular point on Z, then we can write*

$$X \cap Y = \left(\bigcup_{\alpha} W_\alpha\right) \cup Q,$$

where the W_α's are the $\kappa(p)$-irreducible components of $X \cap Y$ passing through p and Q is the union of the other irreducible components, and further, we have

$$\dim W_\alpha \geq \dim X + \dim Y - \dim Z$$

for all α.

Proof. Since we can base-extend from k to $\kappa(p)$ and the dimension is preserved, by base-extending, we may asume that $\kappa(p)$ is k. By taking a big affine open subset around p, we may further assume that X, Y, Z are affine. Let $n = \dim Z$. As in the proof of Theorem 2.6, we have the isomorphism

$$X \cap Y \cong (X \prod Y) \cap \Delta_Z.$$

Since p is nonsingular on Z, by the local complete intersection theorem, we can pick $g_1, \ldots, g_n$ so that $dg_1, \ldots, dg_n$ are linearly independent and cut out $T_{Z,p}$.

Consider the functions $f_1, \ldots, f_n$ on $Z \prod Z$ given by

$$f_i(z_1, \ldots, z_n; w_1, \ldots, w_n) = g_i(z_1, \ldots, z_n) - g_i(w_1, \ldots, w_n).$$

Clearly, $f_i \restriction \Delta_Z = 0$, which implies that

$$\Delta_Z \subseteq V(f_1, \ldots, f_n).$$

The differentials $df_1, \ldots, df_n$ are linearly independent at $(p, p) \in \Delta_Z$, and by the local complete intersection theorem (Theorem 2.23), we get

$$V(f_1, \ldots, f_n) = \Delta_Z \cup R,$$

where R is the union of components not passing through p. However, since Z is separated, Δ_Z is a closed irreducible variety, and reverting to the original argument of Theorem 2.6, we get

$$\dim W_\alpha \geq \dim X + \dim Y - \dim Z. \qquad \square$$

To go further and understand the local structure of an irreducible variety near a nonsingular point on it, we need the following famous theorem first proved by Zariski (1947) in the case at hand [59]. However, the theorem is more general and holds for an arbitrary regular local ring as was proved by M. Auslander and D. Buchsbaum, and independently Jean-Pierre Serre (all in 1959).

Theorem 2.31 *Let X be an irreducible k-variety and let p be a closed point on X. If p is nonsingular, then $\mathcal{O}_{X,p}$ is a UFD.*

In order to prove Theorem 2.31, we need and will prove the following algebraic theorem:

Theorem 2.32 *If A is a local noetherian ring and if its completion $\widehat{A}$ is a UFD, then, A itself is a UFD.*

Proof of Theorem 2.31. Assume Theorem 2.32, then, as p is nonsingular,

$$\widehat{\mathcal{O}}_{X,p} \cong \kappa(p)[[Y_1, \ldots, Y_\delta]],$$

for some δ, and the latter ring is a UFD, by elementary algebra. Therefore, Theorem 2.32 implies Theorem 2.31. $\square$

Proof of Theorem 2.32. The proof proceeds in three steps.

Step 1. I claim that for every ideal $\mathfrak{A} \subseteq A$ we have

$$\mathfrak{A} = A \cap \mathfrak{A}\widehat{A}.$$

Clearly, $\mathfrak{A} \subseteq A \cap \mathfrak{A}\widehat{A}$. We need to prove that

$$A \cap \mathfrak{A}\widehat{A} \subseteq \mathfrak{A}.$$

Pick $f \in A \cap \mathfrak{A}\widehat{A}$, then, $f \in A$ and

$$f = \sum_{i=1}^{t} \alpha_i a_i,$$

and $\alpha_i \in \widehat{A}$ and $a_i \in \mathfrak{A}$. Write

$$\alpha_i = \alpha_i^{(n)} + O(\widehat{\mathfrak{m}}^{n+1}),$$

where $\alpha_i^{(n)} \in A$, and $\mathfrak{m}$ is the maximal ideal of A. Then,

$$f = \sum_i \alpha_i^{(n)} a_i + \sum_i O(\widehat{\mathfrak{m}}^{n+1}) a_i,$$

and $\sum_i \alpha_i^{(n)} a_i \in \mathfrak{A}$. So,

$$f \in \mathfrak{A} + \mathfrak{A}\widehat{\mathfrak{m}}^{n+1} = \mathfrak{A} + \mathfrak{A}\mathfrak{m}^{n+1}\widehat{A},$$

and this is true for all n. The piece of f in $\mathfrak{A}\mathfrak{m}^{n+1}\widehat{A}$ lies in A, and thus, in $\mathfrak{m}^{n+1}$. We find that $f \in \mathfrak{A} + \mathfrak{m}^{n+1}$ for all n, and we have

$$f \in \bigcap_{n \geq 0} (\mathfrak{A} + \mathfrak{m}^{n+1}) = \mathfrak{A},$$

by Krull's intersection theorem.

Step $1\frac{1}{2}$. I claim that

$$\mathrm{Frac}(A) \cap \widehat{A} = A.$$

This means that given $f/g \in \mathrm{Frac}(A)$ and $f/g \in \widehat{A}$, then $f/g \in A$. Equivalently, this means that if g divides f in $\widehat{A}$, then g divides f in A. Look at

$$\mathfrak{A} = gA.$$

If $f/g \in \widehat{A}$, then $f \in g\widehat{A}$, and since $f \in A$, we have

$$f \in A \cap g\widehat{A}.$$

But $g\widehat{A} = \mathfrak{A}\widehat{A}$, and by Step 1, we find that

$$gA = \mathfrak{A} = A \cap \mathfrak{A}\widehat{A},$$

so, $f \in gA$, as claimed.

We now come to the heart of the proof.

Step 2. Let $f, g \in A$ with f irreducible. I claim that either f divides g in A or $(f, g) = 1$ in $\widehat{A}$ (where (f, g) denotes the gcd of f and g).

Assuming this has been established, here is how we prove Theorem 2.32: Firstly, since A is noetherian, factorization into irreducible factors exists (but not necessarily uniquely). By elementary algebra, one knows that to prove uniqueness, it suffices to prove that if f is irreducible then f is prime. That is, if f is irreducible and f divides gh, then we must prove either f divides g or f divides h.

If f divides g, then we are done. Otherwise, $(f,g)=1$ in $\widehat{A}$, by Step 2. Now, f divides gh in $\widehat{A}$ and $\widehat{A}$ is a UFD, so that as $(f,g)=1$ in $\widehat{A}$ we find that f divides h in $\widehat{A}$. By Step $1\frac{1}{2}$, we get that f divides h in A, as desired.

Proof of Step 2. Let $f,g\in\widehat{A}$ and let d be the gcd of f and g in $\widehat{A}$. Thus,

$$f=dF,\quad\text{and}\quad g=dG,$$

where $d,F,G\in\widehat{A}$, and

$$(F,G)=1\quad\text{in}\quad\widehat{A}.$$

Let $\mathrm{ord}_{\widehat{\mathfrak{m}}}F=n_0$ (that is, n_0 is characterized by the fact that $F\in\widehat{\mathfrak{m}}^{n_0}$ but $F\notin\widehat{\mathfrak{m}}^{n_0+1}$). Either F is a unit or a nonunit in $\widehat{A}$. If F is a unit in $\widehat{A}$, then $n_0=0$, and $f=dF$ implies that $F^{-1}f=d$; then,

$$F^{-1}fG=g,$$

which implies that f divides g in $\widehat{A}$. By Step $1\frac{1}{2}$, we get that f divides g in A.

We now have to deal with the case where $\mathrm{ord}(F)=n_0>0$. We have

$$F=\lim_{n\mapsto\infty}F_n\quad\text{and}\quad G=\lim_{n\mapsto\infty}G_n,$$

in the $\mathfrak{m}$-adic topology, with F_n and $G_n\in A$, and $F-F_n$ and $G-G_n\in\widehat{\mathfrak{m}}^{n+1}$. Look at

$$\frac{g}{f}-\frac{G_n}{F_n}=\frac{gF_n-fG_n}{fF_n}.$$

Now,

$$\begin{aligned}
gF_n-fG_n &= g(F_n-F)+gF-fG_n\\
&= g(F_n-F)+dGF-fG_n\\
&= g(F_n-F)+fG-fG_n\\
&= g(F_n-F)+f(G-G_n).
\end{aligned}$$

The righthand side belongs to $(f,g)\widehat{\mathfrak{m}}^{n+1}$, which means that it belongs to $(f,g)\mathfrak{m}^{n+1}\widehat{A}$. However, the lefthand side is in A, and thus, the righthand side belongs to

$$A\cap(f,g)\mathfrak{m}^{n+1}\widehat{A}.$$

Letting $\mathfrak{A}=(f,g)\mathfrak{m}^{n+1}$, we can apply Step 1, and thus, the lefthand side belongs to $(f,g)\mathfrak{m}^{n+1}$. This means that there are some $\sigma_n,\tau_n\in\mathfrak{m}^{n+1}\subseteq A$ so that

$$gF_n-fG_n=f\sigma_n+g\tau_n;$$

It follows that

$$g(F_n-\tau_n)=f(G_n+\sigma_n);$$

so, if we let

$$\alpha_n=G_n+\sigma_n\quad\text{and}\quad\beta_n=F_n-\tau_n,$$

we have the following properties:

(1) $g\beta_n = f\alpha_n$, with $\alpha_n, \beta_n \in A$,

(2) $\alpha_n \equiv G_n \pmod{\mathfrak{m}^{n+1}}$ and $\beta_n \equiv F_n \pmod{\mathfrak{m}^{n+1}}$,

(3) $G_n \equiv G \pmod{\mathfrak{m}^{n+1}\widehat{A}}$ and $F_n \equiv F \pmod{\mathfrak{m}^{n+1}\widehat{A}}$.

Choose $n = n_0$. Since $\mathrm{ord}(F) = n_0 > 0$, we have $\mathrm{ord}(F_{n_0}) = n_0$, and thus, $\mathrm{ord}(\beta_{n_0}) = n_0$. Look at (1):

$$g\beta_{n_0} = f\alpha_{n_0},$$

so

$$dG\beta_{n_0} = dF\alpha_{n_0},$$

and, because $\widehat{A}$ is an integral domain,

$$G\beta_{n_0} = F\alpha_{n_0}.$$

However, $(F, G) = 1$ in $\widehat{A}$ and F divides $G\beta_{n_0}$. Hence, F divides β_{n_0}, so that there is some $H \in \widehat{A}$ with $\beta_{n_0} = FH$ and

$$\mathrm{ord}(\beta_{n_0}) = \mathrm{ord}(F) + \mathrm{ord}(H).$$

But $\mathrm{ord}(F) = n_0$, and consequently, $\mathrm{ord}(H) = 0$, and H is a unit. Since $\beta_{n_0} = FH$, we see that β_{n_0} divides F, and thus,

$$F = \beta_{n_0}\delta$$

for some $\delta \in \widehat{A}$. Again, $\mathrm{ord}(\delta) = 0$, and we conclude that δ is a unit. Then,

$$\beta_{n_0}\delta d = dF = f,$$

so that β_{n_0} divides f in $\widehat{A}$. By step $1\frac{1}{2}$, β_{n_0} divides f in A. But f is irreducible and β_{n_0} is not a unit, and so $\beta_{n_0}u = f$ where u *is* a unit. Thus, $\delta d = u$ is a unit, and since δ is a unit, so is d, as desired. $\square$

The unique factorization theorem just proved has important consequences for the local structure of a variety near a nonsingular point:

Theorem 2.33 *Let X be an irreducible k-variety and let $\xi \in X$ be a nonsingular k-point. Given $f \in \mathcal{O}_{X,\xi}$, with f is irreducible and $f(\xi) = 0$, the locally defined subvariety, $\{x \in X \mid f(x) = 0\}$, is an irreducible subvariety of codimension 1 in X. Conversely, if Y is a locally defined codimension 1 subvariety of X through ξ, then, there is some irreducible $f \in \mathcal{O}_{X,\xi}$ so that near enough ξ, we have*

$$Y = \{x \in X \mid f(x) = 0\} \quad \textit{and} \quad \mathfrak{I}(Y)\mathcal{O}_{X,\xi} = f\mathcal{O}_{X,\xi}.$$

Lastly, if f is any locally defined holomorphic function on X and ξ is a point (not necessarily nonsingular) so that $f(\xi) = 0$, then sufficiently locally near ξ, the zero locus, $\{x \in X \mid f(x) = 0\}$, is a finite union of irreducible components through ξ, each of codimension 1. If ξ is also nonsingular, then these irreducible branches at ξ correspond bijectively to the irreducible factors of f in $\mathcal{O}_{X,\xi}$.

Proof. Let $\xi \in X$ be a nonsingular k-point, and f be in $\mathcal{O}_{X,\xi}$, with f irreducible. The question is local on X, and we may assume that X is affine. Also,

$$\mathcal{O}_{X,\xi} = \varinjlim_{g \notin \mathfrak{I}(\xi)} A_g,$$

with $A = k[X]$. Thus, we may assume that $f = F/G$, with $G(\xi) \neq 0$ and with $F, G \in A$. Upon replacing X by X_G (where X_G is an open such that $\xi \in X_G$), we may assume that f is the image of some $F \in A = k[X]$. The variety X is irreducible and $X = V(\mathfrak{p})$, where $\mathfrak{p}$ is some prime ideal. Near ξ (i.e., on some open affine subset U_0 with $\xi \in U_0$), let

$$\mathfrak{P} = \{g \in k[Z_1, \ldots, Z_n] \mid lg \in \mathfrak{p} + (f), \quad \text{where} \quad l(\xi) \neq 0\}, \tag{$*$}$$

and let $\mathfrak{m}$ be the ideal of ξ on X. This means that $\mathfrak{m} = \{g \in k[X] \mid g(\xi) = 0\}$. We have

$$\mathfrak{p} \subseteq \mathfrak{P} \subseteq \mathfrak{m}.$$

Reading the above in A, we get $\overline{\mathfrak{P}} \subseteq \overline{\mathfrak{m}}$, and in $\mathcal{O}_{X,\xi}$, we find from $(*)$ that $\overline{\mathfrak{P}^e} = f\mathcal{O}_{X,\xi}$. Thus, $\overline{\mathfrak{P}^e}$ is a prime ideal, because f is irreducible and $\mathcal{O}_{X,\xi}$ is a UFD. Then, $\overline{\mathfrak{P}}$ is prime and $Y = \operatorname{Spec} A/\overline{\mathfrak{P}}$ is a variety locally defined by $f = 0$, and is irreducible. We have $Y \not\subseteq X$, since $f = 0$ on Y but not on X, and we find that

$$\dim(Y) \leq \dim(X) - 1.$$

We will prove equality by a tangent space argument.

Claim. There is some affine open $U \subseteq Y$ with $\xi \in U$ so that for all $u \in U$: $T_{Y,u}$ is cut out from $T_{X,u}$ by the equation $df = 0$.

Let $g_1, \ldots, g_t$ be generators for $\mathfrak{P}$. Thus, $dg_1 = \cdots = dg_t = 0$ cut out $T_{Y,u}$ near ξ, i.e., in some suitable open set U_0 with $\xi \in U_0$. By $(*)$, on U_0, there exist $l_1, \ldots, l_t$ so that

$$l_i g_i = p_i + \lambda_i f,$$

where $p_i \in \mathfrak{p}$, and the λ_i's are polynomials. Let $l = \prod l_i$, and take

$$U = U_0 \cap \{\eta \mid l(\eta) \neq 0\}.$$

The set U is open and affine. By differentiating, we get

$$l_i dg_i + (dl_i)g_i = dp_i + (d\lambda_i)f + \lambda_i df. \tag{$\dagger$}$$

On $U \subseteq Y \subseteq X$, we have

(1) $f = 0$ (in Y).

(2) $p_i = 0$ (in X).

(3) $g_i = 0$ (in Y).

(4) $l_i \neq 0$.

(5) $dp_i = 0$, as we are in $T_{X,u}$, with $u \in U$.

In view of (†), we get
$$l_i(u)dg_i(u) = \lambda_i(u)df(u).$$
Assume that $df(u) = 0$. Since $l_i(u) \neq 0$, we get $dg_i(u) = 0$, which implies that the equation $df(u) = 0$ cuts out a subspace of $T_{Y,u}$. Then, $T_{Y,u}$ contains the hyperplane $df = 0$ of $T_{X,u}$, which implies that
$$\dim(T_{Y,u}) \geq \dim(T_{X,u}) - 1.$$
Since $\dim(Y) = \dim(T_{Y,u})$ near ξ (but not necessarily at ξ) and $\dim(X) = \dim(T_{X,u})$, we get $\dim(Y) \geq \dim(X) - 1$, and thus, (by previous work),
$$\dim(Y) = \dim(X) - 1.$$

Conversely, assume that Y is locally defined near ξ, and is of codimension 1. Replacing X by this affine neighborhood, we may assume that $Y \subseteq X$, is globally defined, and of codimension 1. Also recall that ξ is assumed to be nonsingular. We have the ideal $\mathfrak{I}(Y)\mathcal{O}_{X,\xi}$ in $\mathcal{O}_{X,\xi}$, and we can write
$$\mathfrak{I}(Y)\mathcal{O}_{X,\xi} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$
where the $\mathfrak{p}_j$'s are minimal primes of $\mathcal{O}_{X,\xi}$, each of height 1. Since $\mathcal{O}_{X,\xi}$ is a UFD, every $\mathfrak{p}_i$ is principal, i.e., $\mathfrak{p}_i = f_i\mathcal{O}_{X,\xi}$, where f_i is irreducible. As
$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = \mathfrak{p}_1 \cdots \mathfrak{p}_t,$$
we get
$$\mathfrak{I}(Y)\mathcal{O}_{X,\xi} = f\mathcal{O}_{X,\xi},$$
where $f = f_1 \cdots f_t$. The above argument implies that $\mathfrak{I}(Y) = (F)$ in some A_G, where $A = k[X]$; $G(\xi) \neq 0$; $G \in A$. Thus, $\mathfrak{I}(Y)$ is locally principal. Observe also that if Y is irreducible, then $\mathfrak{I}(Y)$ is prime; so, $f = f_j$ for some j, i.e., f is irreducible.

Now, consider $f \in \mathcal{O}_{X,\xi}$, where ξ is not necessarily nonsingular, and look at the local variety through ξ defined by $f = 0$ (remember, $f(\xi) = 0$). The radical ideal $\mathfrak{A} = \mathfrak{I}(Y)$ (in $A = k[Z_1, \ldots, Z_n]/\mathfrak{p}$) defining Y has a decomposition
$$\mathfrak{A} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$
and since $\mathfrak{A} = \sqrt{\mathfrak{A}}$, the $\mathfrak{p}_j$'s are the minimal primes containing $\mathfrak{A}$ (the isolated primes of $\mathfrak{A}$). Let $g_1, \ldots, g_t$ be generators of $\mathfrak{A}$. The image of g_j in $\mathcal{O}_{X,\xi}$ has the form $\lambda_j f$ (remember, Y is locally principal by hypothesis). Since
$$\mathcal{O}_{X,\xi} = \varinjlim_{G \notin \mathfrak{I}(\xi)} A_G,$$

take G enough for $g_1, \ldots, g_t$, and then the open X_G so that $g_j = \widetilde{\lambda_j} F$, where $\mathfrak{I}(Y)$ in A_G is just (F), and $F/G = f$ in $\mathcal{O}_{X,\xi}$. Thus,

$$\mathfrak{A} = \mathfrak{I}(Y) = \bigcap_{j=1}^{s} \mathfrak{p}_j,$$

where in the above intersection, we find only the primes surviving in $\mathcal{O}_{X,\xi}$, i.e., those with $\mathfrak{p}_j \subseteq \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{I}(\xi)$. By Krull's principal ideal theorem (Zariski and Samuel [60], Theorem 29, Chapter IV, Section 14), these $\mathfrak{p}_j$'s are minimal ideals, and thus, the components of Y have codimension 1.

If ξ is actually nonsingular, then these surviving $\mathfrak{p}_j$'s are minimal in the UFD $\mathcal{O}_{X,\xi}$. Hence, locally enough, each $\mathfrak{p}_j$ is principal; say $\mathfrak{p}_j = (f_j)$. Then,

$$(f) = \mathfrak{A} = (f_1) \cap \cdots \cap (f_s) = (f_1 \cdots f_s);$$

so that

$$f = u f_1 \cdots f_s$$

where u is a unit. The irreducible branches of Y through ξ are the irreducible factors of the local equation $f = 0$ defining Y locally. $\square$

2.5 Projective Space, Projective Varieties and Graded Rings

We begin by defining $\mathbb{P}^n$. As a set, $\mathbb{P}^n(\Omega)$ is the collection of all hyperplanes through the origin in $\mathbb{A}^{n+1}(\Omega)$. To specify a hyperplane H in $\mathbb{A}^{n+1}(\Omega)$ is to give a linear form

$$a_0 X_0 + a_1 X_1 + \cdots + a_n X_n = 0,$$

where $a_j \in \Omega$. If we define the equivalence relation $\sim$ on $\mathbb{A}^{n+1}(\Omega) - \{0\}$ so that

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \quad \text{iff} \quad (\exists \lambda \in \Omega^*)(b_i = \lambda a_i),\ 0 \leq i \leq n,$$

then, denoting the equivalence class of $(a_0, \ldots, a_n)$ by $(a_0 \colon \cdots \colon a_n)$, and calling such classes *homogeneous coordinates*, we get a bijection

$$H \longleftrightarrow (a_0 \colon \cdots \colon a_n),$$

where $a_i \neq 0$, for some i, $0 \leq i \leq n$. We can also view $\mathbb{P}^n(\Omega)$ as the homogeneous set $(\mathbb{A}^{n+1}(\Omega) - \{0\})/\mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group variety. Here, the symbol $\mathbb{G}_m$, which is the multiplicative group variety, associates to each field K between k and Ω the group K^* under multiplication. The reader can check that $\mathbb{G}_m$ is the affine variety Spec $k[X, Y]/(XY - 1)$. Given a field, L, such that $k \subseteq L \subseteq \Omega$, we define $\mathbb{P}^n(L)$ as the set of hyperplanes that are defined by some linear form with coefficients in L.

Consider

$$U_i = \{(a_0\colon \cdots : a_n) \mid a_i \neq 0\} \subseteq \mathbb{P}^n.$$

We have

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i,$$

and $U_i \cong \mathbb{A}^n$, via the bijection

$$(a_0\colon \cdots a_i\colon \cdots : a_n) \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right).$$

If we pull back the functions from $\mathbb{A}^n$ and install them on U_i, then U_i is an affine k-variety isomorphic to $\mathbb{A}^n$. Thus, $\mathbb{P}^n$ is a k-variety, for it is not hard to check that the gluing conditions hold. In fact, given $(X_0\colon \cdots : X_n) \in \mathbb{P}^n$, define the symbols $\xi_j^{(i)}$ as follows:

$$\xi_j^{(i)} = \frac{X_j}{X_i}.$$

Note that the following equations hold:

(a) $\xi_i^{(i)} = 1$.

(b) $\xi_j^{(i)}\xi_i^{(j)} = 1$, on $U_i \cap U_j$.

(c) $\xi_k^{(j)}\xi_j^{(i)} = \xi_k^{(i)}$, on $U_i \cap U_j \cap U_k$.

Then, $U_i = \mathrm{Spec}[\xi_0^{(i)}, \ldots, \widehat{\xi_i^{(i)}}, \ldots, \xi_n^{(i)}]$, where the symbol $\widehat{\xi_i^{(i)}}$ means $\xi_i^{(i)}$ omitted, and we get

$$U_i \cap U_j = (U_i)_{\xi_j^{(i)}} = (U_j)_{\xi_i^{(j)}}.$$

To give a subvariety, V, of $\mathbb{P}^n$, we specify it locally:

(a) On U_i, we have a closed subvariety specified by $\mathrm{Spec}\, k[U_i]/\mathfrak{A}_i$, where $\sqrt{\mathfrak{A}_i} = \mathfrak{A}_i$.

(b) These subvarieties agree on $U_i \cap U_j$ for all $i \neq j$, which means that

$$(\mathfrak{A}_i)_{\xi_j^{(i)}} = (\mathfrak{A}_j)_{\xi_i^{(j)}}$$

for all $i \neq j$.

(c) On the triple overlap $U_i \cap U_j \cap U_k$, the gluing conditions hold.

Now, it is easy to see that the above conditions are equivalent to the fact that there exist homogeneous polynomials $F_1, \ldots, F_p$ in $X_0, \ldots, X_n$ so that if we denote by $f_l^{(i)}$ the polynomial F_l dehomogenized at X_i, then $\mathfrak{A}_i = (f_1^{(i)}, \ldots, f_p^{(i)})$, and conversely.

Recall that an ideal $\mathfrak{A} \subseteq k[X_0, \ldots, X_n]$ is *homogeneous* if whenever $f = f_0 + \cdots + f_m \in \mathfrak{A}$, where f_i is the homogeneous component of f of degree i, then $f_i \in \mathfrak{A}$ for $i = 0, \ldots, m$. Clearly, the above condition is equivalent to the fact that $\mathfrak{A}$ is generated by homogeneous polynomials. So, we find that there is a bijection between k-closed subvarieties of $\mathbb{P}^n$ and homogeneous ideals, $\mathfrak{A}$, of $k[X_0, \ldots, X_n]$ so that

(1) $\sqrt{\mathfrak{A}} = \mathfrak{A}$.

(2) $(X_0, \ldots, X_n)^h \not\subseteq \mathfrak{A}$ for all $h \geq 1$. (We say that $\mathfrak{A}$ is a *relevant ideal* when condition (2) holds.)

Given $V \subseteq \mathbb{P}^n$, we can define the graded ring $S = k[X_0, \ldots, X_n]/\mathfrak{A}$, where $\mathfrak{A} = \mathfrak{I}(V)$ is a homogeneous ideal satisfying (1) and (2). Conversely, given the ring $k[X_0, \ldots, X_n]/\mathfrak{A}$ and a homogeneous ideal, $\mathfrak{A}$, satifying (1) and (2), we get the k-closed subvariety $V(\mathfrak{A})$ in $\mathbb{P}^n$ by

$$V(\mathfrak{A}) = \{(\xi_0\colon \cdots : \xi_n) \in \mathbb{P}^n(\Omega) \mid f(\xi_0, \ldots, \xi_n) = 0, \quad \text{where } f \in \mathfrak{A} \text{ and } f \text{ is homogeneous}\}.$$

We denote this variety by $\mathrm{Proj}(S)$, where $S = k[X_0, \ldots, X_n]/\mathfrak{A}$.

Unlike the affine case, it frequently happens

$$\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$$

as k-varieties, and yet S is not isomorphic to S' as graded rings.

The d-uple embedding illustrates this point:

Let $M_0, \ldots, M_N$ be the monomials of degree d in the variables $X_0, \ldots, X_n$, ordered lexicographically. Since there are $\binom{d+n}{d}$ such monomials,

$$N = \binom{d+n}{d} - 1.$$

We define the d-uple embedding, which is the map $\Phi_d\colon \mathbb{P}^n \to \mathbb{P}^N$, as follows:

$$(\xi_0\colon \cdots : \xi_n) \mapsto (M_0(\xi_0, \ldots, \xi_n)\colon \cdots : M_N(\xi_0, \ldots, \xi_n)).$$

This map is well-defined since $\xi_i \neq 0$ for some i, and the monomial X_i^d is in the list, so $\xi_i^d \neq 0$; the map Φ_d is clearly injective. We need to show that it is a morphism. For this, consider the algebra k-homomorphism $\varphi_d\colon k[Z_0, \ldots, Z_N] \to k[X_0, \ldots, X_n]$ defined so that

$$\varphi_d(Z_j) = M_j(X_0, \ldots, X_n).$$

If we localize at Z_i, the effect on the righthand side is to localize at $M_i(X_0, \ldots, X_n)$. Thus, Φ_d is indeed a k-morphism, and moreover, if $\mathfrak{A} = \mathrm{Ker}\ \varphi_d$, then $\mathfrak{A}$ is a homogeneous ideal generated by quadratic equations. Indeed, we can use the notation $Z_{i_0,\ldots,i_n}$, where $i_0 + \cdots + i_n = d$ and $i_j \geq 0$, to denote the N homogeneous coordinates of $\mathbb{P}^N$; then the equations (these are called the *Plücker relations*) are the quadratic equations

$$Z_{i_0,\ldots,i_n} Z_{j_0,\ldots,j_n} = Z_{h_0,\ldots,h_n} Z_{k_0,\ldots,k_n},$$

and they hold whenever

$$i_0 + j_0 = h_0 + k_0, \cdots, i_n + j_n = h_n + k_n.$$

We get an injection

$$\overline{\varphi_d} \colon k[Z_0, \ldots, Z_n]/\mathfrak{A} \hookrightarrow k[X_0, \ldots, X_n],$$

and it is easy to check that

$$\mathrm{Proj}(k[Z_0, \ldots, Z_n]/\mathfrak{A}) = V(\mathfrak{A}) \hookrightarrow \mathbb{P}^N$$

and that $\Phi_d \colon \mathbb{P}^n \to \mathbb{P}^N$ embeds $\mathbb{P}^n$ as a closed subvariety of $\mathbb{P}^N$. Yet, $\overline{\varphi_d}$ is not an isomorphism, since it is not surjective.

The meaning of the d-uple embedding can be explained as follows: Let H be a hyperplane in $\mathbb{P}^N$ given by the equation

$$\sum_{j=0}^{N} \alpha_j Z_j = 0.$$

Applying φ_d, we get

$$\sum_{j=0}^{N} \alpha_j M_j(X_0, \ldots, X_n) = 0,$$

the equation of a hypersurface of degree d in $\mathbb{P}^n$. Thus, we get a map φ_d^* mapping hyperplanes in $\mathbb{P}^N$ to hypersurfaces in $\mathbb{P}^n$, and this is clearly a bijection. Hence, the d-uple embedding, $\Phi_d \colon \mathbb{P}^n \to \mathbb{P}^N$, gives a bijection

$$H \mapsto \varphi_d^*(H)$$

between hyperplanes in $\mathbb{P}^N$ and hypersurfaces of degree d in $\mathbb{P}^n$. In a sense, we have "straightened out" hypersurfaces in $\mathbb{P}^n$ to become hyper*planes* in $\mathbb{P}^N$.

Let $X = \Phi_d(\mathbb{P}^n) \hookrightarrow \mathbb{P}^N$. Now, consider a hyperplane, H, in $\mathbb{P}^N$ and look at $H \cap X$. Observe that

$$\begin{aligned} H \cap X &= \left\{ (\cdots : M_j(\xi_0, \ldots, \xi_n) \colon \cdots) \;\middle|\; \sum_{j=0}^{N} \alpha_j M_j(\xi_0, \ldots, \xi_n) = 0 \right\} \\ &= \{(\xi_0 \colon \cdots : \xi_n) \in \mathbb{P}^n \mid (\xi_0 \colon \cdots : \xi_n) \in \varphi_d^*(H)\}. \end{aligned}$$

Hence, under the d-uple map Φ_d, the hypersurface $\varphi_d^*(H)$ goes to the hyperplane section $H \cap X$, where X is the image of $\mathbb{P}^n$ in $\mathbb{P}^N$.

Example 2.9 First, consider the case $d = 2$ and $n = 1$, we have the map

$$(\xi_0 \colon \xi_1) \mapsto (\xi_0^2 \colon \xi_0\xi_1 \colon \xi_1^2) = (Z_0 \colon Z_1 \colon Z_2).$$

The equation of the image of $\mathbb{P}^1$ in $\mathbb{P}^2$ is

$$Z_1^2 = Z_0 Z_2.$$

This is a conic in $\mathbb{P}^2$. More generally, the image of $\mathbb{P}^1$ under Φ_d is a curve of degree d in $\mathbb{P}^d$. This curve is the *rational normal curve* in $\mathbb{P}^d$. The rational normal curve is *nondegenerate*, i.e., it is not contained in any hyperplane of $\mathbb{P}^d$ (DX).

Now, when $d = 3$ and $n = 1$, the map Φ_d is

$$(\xi_0 \colon \xi_1) \mapsto (\xi_0^3 \colon \xi_0^2\xi_1 \colon \xi_0\xi_1^2 \colon \xi_1^3) = (Z_0 \colon Z_1 \colon Z_2 \colon Z_3).$$

The Plücker equations are then:

$$\begin{aligned} Z_0Z_3 &= Z_1Z_2 \\ Z_1^2 &= Z_0Z_2 \\ Z_2^2 &= Z_1Z_3. \end{aligned}$$

These are the equations of the *twisted cubic*, as the rational normal curve of degree 3 is called.

Just as in the case of affine varieties, we can consider products of abstract and projective varieties. In the following, most details will be left to the reader:

Let X, Y, Z be k-varieties, and $\pi_X \colon X \to Z$ and $\pi_Y \colon Y \to Z$ be morphisms. Then, $X \prod_Z Y$ exists as a k-variety, i.e., the functor from k-varieties to sets,

$$T \mapsto \mathrm{Hom}\,(T, X) \times_{\mathrm{Hom}\,(T,Z)} \mathrm{Hom}\,(T, Y)$$

is representable. To show this, we cover X, Y, Z by affine open varieties X_α, Y_β, Z_γ. We can arrange this so that $\pi_X \restriction X_\alpha \colon X_\alpha \to Z_\gamma$ and $\pi_Y \restriction Y_\beta \colon Y_\beta \to Z_\gamma$ for all α, β and where γ depends on α and β. Then, perform the following steps.

(1) $X_\alpha \prod_{Z_\gamma} Y_\beta$ exists by previous work;

(2) Check that on the category of k-varieties (not just affine varieties), the variety $X_\alpha \prod_{Z_\gamma} Y_\beta$ represents the product functor (as above);

(3) Prove that

$$X_\alpha \prod_{Z_\gamma} Y_\beta \cong X_\alpha \prod_Z Y_\beta;$$

(4) And finally, prove that (gluing)

$$\bigcup_{\alpha,\beta}\left(X_\alpha \prod_Z Y_\beta\right) \cong X \prod_Z Y.$$

However, while the product of projective varieties is certainly an abstract variety, it is not completely obvious that it is a projective variety. This is true, and to prove it, we need to introduce the *Segre embedding*. The Segre embedding is a morphism

$$\Sigma\colon \mathbb{P}^r \prod \mathbb{P}^s \longrightarrow \mathbb{P}^{(r+1)(s+1)-1}.$$

Set-theoretically, the map Σ is given by

$$(x_0\colon \cdots : x_r; y_0\colon \cdots : y_s) \mapsto (x_0y_0\colon \cdots : x_iy_j\colon \cdots : x_ry_s),$$

where the x_iy_j are ordered lexicographically. Let $Z_0, \ldots, Z_{(r+1)(s+1)-1}$ be the coordinates in $\mathbb{P}^{(r+1)(s+1)-1}$. We can also denote these variables by $Z_{i\,j}$, where $0 \leq i \leq r$ and $0 \leq j \leq s$–and order them lexicographically. The algebra map

$$\sigma\colon S = k[Z_{0\,0}, \ldots, Z_{r\,s}] \longrightarrow k[X_1, \ldots, Z_r] \otimes k[Y_1, \ldots, Y_s],$$

given by

$$Z_{i\,j} \mapsto X_i \otimes Y_j,$$

gives by dehomogenizing at $Z_{i\,j}$ the map

$$\sigma_{ij}\colon S_{(Z_{i\,j})} \longrightarrow k[X_1, \ldots, Z_r]_{(X_i)} \otimes k[Y_1, \ldots, Y_s]_{(Y_j)}. \tag{$*$}$$

We see that Spec $S_{(Z_{i\,j})}$ defines an affine open in $\mathbb{P}^{(r+1)(s+1)-1}$, and the righthand side of $(*)$ defines the affine open $U_i \prod U_j$ in $\mathbb{P}^r \prod \mathbb{P}^s$. These affine opens glue and give our morphism (DX). To identify the image of the Segre morphism, take $\mathfrak{B} = \mathrm{Ker}\,\sigma$. This is a homogeneous radical ideal, and

$$\mathrm{Proj}(S/\mathfrak{B}) = V(\mathfrak{B}) \subseteq \mathbb{P}^{(r+1)(s+1)-1}$$

is just the image $\Sigma(\mathbb{P}^r \prod \mathbb{P}^s)$ in $\mathbb{P}^{(r+1)(s+1)-1}$ (DX). The ideal, $\mathfrak{B}$, is generated by the quadratic equations

$$Z_{i\,j}Z_{k\,l} = Z_{k\,j}Z_{i\,l}.$$

Example 2.10 Consider the Segre embedding $\Sigma\colon \mathbb{P}^1 \prod \mathbb{P}^1 \to \mathbb{P}^3$, given by

$$((x_0\colon x_1), (y_0\colon y_1)) \mapsto (x_0y_0\colon x_0y_1\colon x_1y_0\colon x_1y_1) = (Z_0\colon Z_1\colon Z_2\colon Z_3).$$

The quadratic Segre relation is the single equation:

$$Z_1Z_2 = Z_0Z_3.$$

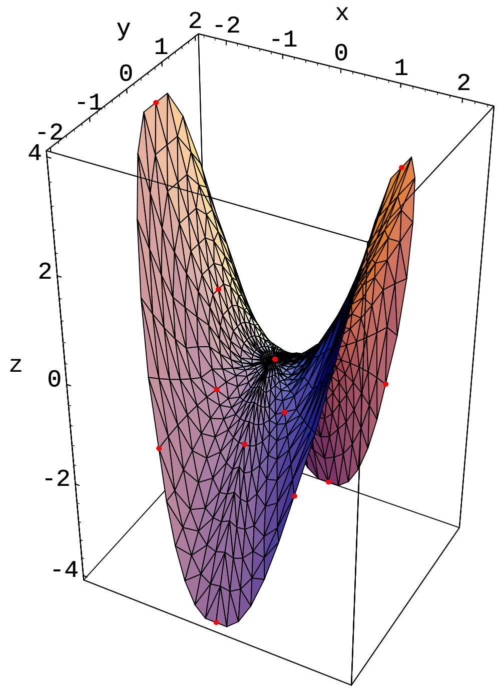

Figure 2.6: An hyperbolic paraboloid

Let Q denote the image variety $\Sigma(\mathbb{P}^1 \prod \mathbb{P}^1)$. This is a nondegenerate quadric surface in $\mathbb{P}^3$. There are two families of rulings (lines) in Q, they are $p \prod \mathbb{P}^1$ and $\mathbb{P}^1 \prod p$, where p is any point in $\mathbb{P}^1$. Let us look at Q over the field $\mathbb{R}$. If we take $Z_0 = 0$ to be the hyperplane at infinity, then letting

$$z_1 = \frac{Z_1}{Z_0}, \quad z_2 = \frac{Z_2}{Z_0}, \quad z_3 = \frac{Z_3}{Z_0},$$

we find the equation of Q to be

$$z_3 = z_1 z_2.$$

This is a hyperbolic paraboloid in $\mathbb{R}^3$, and it is displayed in Figure 2.6.

We can also define the *Veronese map*:

$$V\colon \mathbb{P}^n \longrightarrow \mathbb{P}^{(n+1)^2-1},$$

which is the composition

$$V = \Sigma \circ \Delta\colon \mathbb{P}^n \xrightarrow{\Delta} \mathbb{P}^n \prod \mathbb{P}^n \xrightarrow{\Sigma} \mathbb{P}^{(n+1)^2-1}.$$

The image of $\mathbb{P}^n$ is a rational variety, and it is closed in $\mathbb{P}^{(n+1)^2-1}$.

Remark: Look at $\mathbb{P}^n$ over $\mathbb{C}$. Let H be the hyperplane whose equation is $Z_0 = 0$. Then, we have

$$\mathbb{P}^n = (\mathbb{P}^n - H) \cup H = \mathbb{A}^n \cup H.$$

However, the hyperplane H is in bijection with $\mathbb{P}^{n-1}$; and so,

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

Repeating this argument, we get

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^1 \cup \{1\}.$$

This "cell decomposition" holds for $\mathbb{P}^n$ over any field. However, over $\mathbb{C}$, we can compute the cohomology of $\mathbb{P}^n$ because the boundaries of the cells have the right dimensions. We find that

$$H_l(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } l = 2s,\ 0 \le s \le n. \\ (0) & \text{if } l \text{ is odd.} \\ (0) & \text{if } l > 2n, \end{cases}$$

and the same for cohomology.

Remarks: (DX)

(1) $\mathbb{P}^n$ is geometrically irreducible.

(2) $\mathbb{P}^n$ is separated, and this is true of every projective variety.

(3) There exist nonprojective even nonalgebraic complex analytic manifolds. For example, the quotient of $\mathbb{C}^2 - \{0\}$ under the action given by $z \mapsto 2z$ is a complex analytic manifold *not* embeddable in any $\mathbb{P}^N$ for any $N \ge 0$.

Just as in the affine case, we can analyze the dimensions of irreducible components of an intersection, so in the projective case we can do the same. But here, the theorem is even simpler because nonemptiness of the intersection is guaranteed under suitable conditions on the dimensions of the intersecting varieties.

Theorem 2.34 *(Projective intersection dimension theorem) Let V, W be irreducible projective varieties, with $\dim(V) = r$ and $\dim(W) = s$. If $r + s \ge n$, then $V \cap W \neq \emptyset$, and every irreducible component of $V \cap W$ has dimension at least $r + s - n$.*

Proof. We reduce the proof to the affine case by considering the cones $C(V)$ and $C(W)$ in $\mathbb{A}^{n+1}$.[1] In this case, we apply the affine version of the theorem (Theorem 2.6) (DX). □

Projective varieties have the important property that the image of a projective variety under a morphism is always closed. To prove this, we will need a simple lemma:

Lemma 2.35 *If X, Y, Z are k-varieties and $\pi_X \colon X \to Z$ and $\pi_Y \colon Y \to Z$ are morphisms, and if π_X is a closed immersion, then $pr_2 \colon X \prod_Z Y \to Y$ is a closed immersion. (Base extension of a closed immersion is a closed immersion).*

[1] If $V = \operatorname{Proj} S$ where S is a graded ring, then $C(V)$ is just $\operatorname{Spec} S$.

Proof. The question is local on X, Y, Z; so we may assume that X, Y, Z are affine. Assume that $X = \mathrm{Spec}\, A$; $Y = \mathrm{Spec}\, B$; $Z = \mathrm{Spec}\, C$. Since $\pi_X \colon X \to Z$ is a closed immersion, we have $A = C/\mathfrak{C}$, for some radical ideal, $\mathfrak{C}$. Then,

$$\begin{aligned} X \prod_Z Y &= \mathrm{Spec}((A \otimes_C B)/\mathfrak{N}) \\ &= \mathrm{Spec}((C/\mathfrak{C} \otimes_C B)/\mathfrak{N}) \\ &= \mathrm{Spec}((B/\mathfrak{C}B)/\mathfrak{N}) \hookrightarrow \mathrm{Spec}\, B = Y. \quad \square \end{aligned}$$

Having proved the lemma, we can now prove the main theorem.

Theorem 2.36 *(Properness of projective varieties) A projective variety, V, is a proper variety. This means that for every variety W,*

$$pr_2 \colon V \prod W \longrightarrow W$$

is a closed map.

Proof. (1) We reduce the proof to the case where W is affine. Assume that the theorem holds when W is affine. Cover W with affine opens W_α, so that $W = \bigcup_\alpha W_\alpha$. Check that

$$V \prod W \cong \bigcup_\alpha \left(V \prod W_\alpha \right).$$

Let $C \subseteq V \prod W$ be a closed subvariety. If C_α denotes $C \cap (V \prod W_\alpha)$, then,

$$pr_2(C) \cap W_\alpha = pr_2(C_\alpha).$$

But, $pr_2(C_\alpha)$ is closed in W_α, which implies that $pr_2(C)$ is closed in W.

(2) We reduce the proof to the case where $V = \mathbb{P}^n$. Assume that the theorem holds for $\mathbb{P}^n$. Look at the closed immersion $V \hookrightarrow \mathbb{P}^n$. By Lemma 2.35,

$$V \prod W \hookrightarrow \mathbb{P}^n \prod W$$

is also a closed immersion. Hence, we have the commutative diagram

$$\begin{tikzcd} C \arrow[r, hook] & V \prod W \arrow[rr] \arrow[dr, "pr_2"'] & & \mathbb{P}^n \prod_n W \arrow[dl, "pr_2"] \\ & & W & \end{tikzcd},$$

and this shows that we may assume that $V = \mathbb{P}^n$.

(3) Lastly, we reduce the proof to the case: $W = \mathbb{A}^m$. Assume that the theorem holds for $W = \mathbb{A}^m$. By (1), we may assume that W is closed in $\mathbb{A}^m$, then we have the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, hook] & \mathbb{P}^n \prod W \arrow[r, hook] \arrow[d, "(pr_2)_W"'] & \mathbb{P}^n \prod \mathbb{A}^m \arrow[d, "(pr_2)_{\mathbb{A}^m}"] \\ & W \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

where the arrows in the top line are closed immersions, by Lemma 2.35. So,

$$(pr_2)_W(C) = (pr_2)_{\mathbb{A}^m}(C) \cap W,$$

and, since by hypothesis, $(pr_2)_{\mathbb{A}^m}(C)$ is closed, and W is closed, we find $(pr_2)_W(C)$ is also closed.

We are now reduced to the essential case: Which is to prove that $pr_2 \colon \mathbb{P}^n \prod \mathbb{A}^m \to \mathbb{A}^m$ is a closed map. Let C be a closed subvariety of $\mathbb{P}^n \prod \mathbb{A}^m$. Then, C is the common solution set of a system of equations of form

$$f_j(X_0, \ldots, X_n; Y_1, \ldots, Y_m) = 0, \quad \text{for } j = 1, \ldots, p, \tag{$\dagger$}$$

where f_j is homogeneous in the X_j's and we restrict to solutions for which $X_j \neq 0$ for some j, with $0 \leq j \leq n$. Pick $y \in \mathbb{A}^m$, and write $y = (y_1, \ldots, y_m)$; also write $(\dagger)(y)$ for the system $(\dagger)$ in which we have set $Y_j = y_j$ for $j = 1, \ldots, m$.

Plan of the proof: We will prove that $(pr_2(C))^c$ (the complement of $pr_2(C)$) is open. Observe that

$$\begin{aligned} y \in pr_2(C) \quad & \text{iff} \quad (\exists x)((x, y) \in C) \\ & \text{iff} \quad (\exists x)(x_j \neq 0 \text{ for some } j, \text{ and } (\dagger)(y) \text{ holds}). \end{aligned}$$

Thus,

$$y \in (pr_2(C))^c \quad \text{iff} \quad (\forall x)(\text{if } (\dagger)(y) \text{ holds, then } x_j = 0, \quad \text{for } 0 \leq j \leq n).$$

Let $\mathfrak{A}(y)$ be the ideal generated by the polynomials, $f_j(X_0, \ldots, X_n, y_1, \ldots, y_m)$, occurring in $(\dagger)(y)$. Hence,

$$y \in (pr_2(C))^c \quad \text{iff} \quad (\exists d \geq 0)((X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)).$$

Let

$$\mathcal{N}_d = \{y \in \mathbb{A}^m \mid (X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)\}.$$

Then,

$$(pr_2(C))^c = \bigcup_{d=1}^{\infty} \mathcal{N}_d.$$

Now,

$$\mathcal{N}_d \subseteq \mathcal{N}_{d+1},$$

and so,

$$(pr_2(C))^c = \bigcup_{d>>0}^{\infty} \mathcal{N}_d,$$

where $d >> 0$ means that d is sufficiently large.

Claim. If $d > \max\{d_1, \ldots, d_p\}$, where d_j is the homogeneous degree of $f_j(X_0, \ldots, X_n, Y_1, \ldots, Y_m)$ in the X_i's, then $\mathcal{N}_d$ is open in $\mathbb{A}^m$. This will finish the proof.

Write $S_d(y)$ for the vector space (over k) of polynomials in $k[y_1, \ldots, y_m][X_0, \ldots, X_n]$ of *exact degree* d. We have a map of vector spaces

$$\psi_d(y)\colon S_{d-d_1}(y) \oplus \cdots \oplus S_{d-d_p}(y) \longrightarrow S_d(y)$$

given by

$$\psi_d(y)(g_1, \ldots, g_p) = \sum_{j=1}^{p} f_j g_j.$$

If we assume that $\psi_d(y)$ is surjective, then all monomials of degree d are in the range of $\psi_d(y)$. Thus, $\mathfrak{A}(y)$ will contain all the generators of $(X_0, \ldots, X_n)^d$, i.e.,

$$(X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y),$$

and this means

$$y \in \mathcal{N}_d.$$

Conversely, if $y \in \mathcal{N}_d$, then $(X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)$, and thus, $\mathfrak{A}(y)$ contains every monomial of degree d. But then, each monomial of degree d is in the range of $\psi_d(y)$, and since these monomials form a basis of $S_d(y)$, the map $\psi_d(y)$ is surjective.

Therefore, $y \in \mathcal{N}_d$ iff $\psi_d(y)$ is surjective.

Pick bases for all of the S_{d-d_j}'s and for S_d. Then, $\psi_d(Y)$ is given by a matrix whose entries are polynomials in the Y_j's. We know that $\psi_d(y)$ is surjective iff $\mathrm{rk}\, \psi_d(y) = n_d$, where $n_d = \dim(S_d)(y)$. Therefore, $\psi_d(y)$ will be surjective iff some $n_d \times n_d$ minor of our matrix is nonsingular. This holds if and only if the determinant of this minor is nonzero. However, these determinants for $\psi_d(y)$ are polynomials $q(Y_1, \ldots, Y_m)$. Therefore, $\psi_d(y)$ will be surjective iff y belongs to the k-open such that some $q(y) \neq 0$. This proves that $\mathcal{N}_d$ is open, and finishes the proof. QED

Remarks:

(1) Homogeneity in the X_i's allowed us to control degrees.

(2) If Y is a separated k-variety, then for any morphism, $\varphi\colon X \to Y$, the graph $\Gamma_\varphi \subseteq X \prod Y$ is closed.

The argument used to establish (2) is a standard categorical argument. Note that

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\Gamma_\varphi"'] & Y \arrow[d, "\Delta"] \\ X \prod Y \arrow[r, "\varphi \prod \mathrm{id}"'] & Y \prod Y \end{tikzcd}$$

is a product diagram, where

$$\Gamma_\varphi(x) = (x, \varphi(x)).$$

To see this, replace X, Y, etc., by the functors they represent and, by choosing any test variety T, we reduce the proof to the case where X, Y, etc. are sets. Observe that

$$Y \prod_{Y \prod Y} (X \prod Y) = \{(y, (x, z)) \mid x \in X;\ y, z \in Y;\ \text{and}\ (y, y) = (\varphi(x), z)\}.$$

Here, $\Delta\colon Y \to Y \prod Y$ and $\varphi \prod \mathrm{id}\colon X \prod Y \to Y \prod Y$. This shows that $y = z$ and $y = \varphi(x)$, and thus, the maps

$$(y, x, z) \mapsto x$$

and

$$x \mapsto (\varphi(x), x, \varphi(x))$$

are inverse isomorphisms.

Then, Γ_φ is the base extension of the diagonal morphism Δ, which is a closed immersion, as Y is separated. Therefore, Γ_φ is a closed immersion, which implies that $\Gamma_\varphi(X)$ is closed in Y, as claimed in (2).

The properness theorem for projective varieties has a number of important corollaries.

Corollary 2.37 *If V is a proper k-variety (e.g., by Theorem 2.36, any projective variety) and W is a separated k-variety, then any morphism $\varphi\colon V \to W$ is a closed map.*

Proof. We factor $\varphi\colon V \to W$ as

$$V \xrightarrow{\Gamma_\varphi} V \prod W \xrightarrow{pr_2} W.$$

By remark (2) above, the map Γ_φ is a closed map, and by Theorem 2.36, the map pr_2 is closed, and the result follows. $\square$

Corollary 2.38 *Let V be a proper k-variety (e.g., by Theorem 2.36, any projective variety). If W is any quasi-affine variety (i.e., an open in an affine) or any affine variety, then for any morphism $\varphi\colon V \to W$, the image,* Im φ, *of φ is a finite set of points. If V is geometrically connected, then φ is constant. In particular, every holomorphic function on V has finitely many values and if V is geometrically connected, φ is constant.*

Proof. Since $\mathbb{A}^n$ is separated, W is separated. We have

$$V \longrightarrow W \hookrightarrow \mathbb{A}^n,$$

and thus, we may assume that $W = \mathbb{A}^n$. Pick j, with $1 \leq j \leq n$, and look at

$$V \longrightarrow \mathbb{A}^n \xrightarrow{pr_j} \mathbb{A}^1.$$

If we knew the result for $\mathbb{A}^1$, by a simple combinatorial argument, we would have the result for $\mathbb{A}^n$. Thus, we are reduced to the case $W = \mathbb{A}^1$. In this case, either Im $\varphi = \mathbb{A}^1$, or a finite set of points, since $\mathbb{A}^1$ is irreducible. Furthermore, in the latter case, if V is geometrically connected, then Im φ consists of a single point. We need to prove that $\varphi\colon V \to \mathbb{A}^1$ is never surjective. Assume it is. Consider the diagram

$$\begin{tikzcd} V \prod \mathbb{A}^1 \arrow[r, "\varphi \prod \mathrm{id}"] \arrow[dr, "\pi_2"'] & \mathbb{A}^1 \prod \mathbb{A}^1 \arrow[d, "\pi_2"] \\ & \mathbb{A}^1 \end{tikzcd}$$

and let

$$D = \{(x, y) \in \mathbb{A}^1 \times \mathbb{A}^1 \ \mid \ xy = 1\}.$$

The map $\varphi \prod \mathrm{id}$ is onto. Therefore, $(\varphi \prod \mathrm{id})^{-1}(D)$ is closed and

$$\varphi \prod \mathrm{id}\colon (\varphi \prod \mathrm{id})^{-1}(D) \to D$$

is surjective. Let $C = (\varphi \prod \mathrm{id})^{-1}(D)$. By the definition of proper, $pr_2(C)$ is closed. However, by surjectivity,

$$pr_2(C) = \pi_2(C),$$

and yet, $\pi_2(C)$ is k-open, a contradiction on the irreducibility of $\mathbb{A}^1$. $\square$

Corollary 2.39 *(Kronecker's main theorem of elimination) Consider p polynomials $f_1(X_0, \ldots, X_n; Y_1, \ldots, Y_m)$, ..., $f_p(X_0, \ldots, X_n; Y_1, \ldots, Y_m)$, with coefficients in k and homogeneous in the X_i's (of varying degrees). Consider further the simultaneous system*

$$f_j(X_0, \ldots, X_n; Y_1, \ldots, Y_m) = 0, \quad \text{for } j = 1, \ldots, p. \tag{$\dagger$}$$

Then, there exist polynomials $g_1(Y_1, \ldots, Y_m), \ldots, g_t(Y_1, \ldots, Y_m)$ with coefficients in k involving only the Y_j's so that ($\dagger$) has a solution in which not all X_i's are 0 iff the system

$$g_j(Y_1, \ldots, Y_n) = 0, \quad \text{for } j = 1, \ldots, t, \tag{$\dagger\dagger$}$$

has a solution. (The X_i's have been eliminated).

Proof. The system ($\dagger$) defines a closed subvariety C of $\mathbb{P}^n \prod \mathbb{A}^m$.

Claim. The set $pr_2(C)$, which, by Theorem 2.36, is closed in $\mathbb{A}^m$, gives us the system ($\dagger\dagger$) by taking the g_j's as a set of polynomials defining $pr_2(C)$. To see this, note that $C = \emptyset$ iff $pr_2(C) = \emptyset$; note further that $(x, y) \in C$ iff ($\dagger$) has a solution with not all X_i's all zero. Consequently, ($\dagger$) has a solution with not all X_i zero iff ($\dagger\dagger$) has a solution in the Y_j's. $\square$

2.6 Linear Projections and Noether Normalization Theorem

Some special morphisms of projective space and associated varieties are extremely important. They, and concatenations of them occur repeatedly throughout the theory. One such morphism is projection from a point.

Let $p \in \mathbb{P}^n$, and let H be a hyperplane such that $p \notin H$. Consider the collection of lines through p, and take any $q \in \mathbb{P}^n$ such that $q \neq p$. Then, p and q define a unique line l_{pq} not contained in H, since otherwise, we would have $p \in H$. The line l_{pq} intersects H in a single point, $\pi_p(q)$. This defines a map

$$\pi_p \colon \mathbb{P}^n - \{p\} \longrightarrow H,$$

called the *projection onto H from p*. We claim that this map is a morphism. For this, let

$$\sum_{j=0}^{n} a_j X_j = 0$$

be an equation defining the hyperplane H; let $p = (p_0 \colon \cdots \colon p_n)$ and $q = (q_0 \colon \cdots \colon q_n)$. The line l_{pq} has the parametric equation

$$(s \colon t) \mapsto (sp_0 + tq_0 \colon \cdots \colon sp_n + tq_n),$$

where $(s \colon t) \in \mathbb{P}^1$. The line l_{pq} intersects H in the point whose coordinates satisfy the equation

$$\sum_{j=0}^{n} a_j(sp_j + tq_j) = 0,$$

and we get

$$s\sum_{j=0}^{n} a_j p_j + t\sum_{j=0}^{n} a_j q_j = 0.$$

However, $\sum_{j=0}^{n} a_j p_j \neq 0$, since $p \notin H$, and thus, we can solve for s in terms of t. We find that $l_{pq} \cap H$ is the point with homogeneous coordinates

$$t\left(-\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_0 + q_0 \colon \cdots \colon -\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_n + q_n\right),$$

and this is,

$$\left(-\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_0 + q_0 \colon \cdots \colon -\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_n + q_n\right),$$

since $t \neq 0$, because $p \notin H$. These coordinates are linear in the q_j's, and thus, the projection map is a morphism.

We may perform a linear change of coordinates so that the equation of the hyperplane H becomes

$$X_n = 0.$$

We get

$$\pi_p(q_0\colon \cdots : q_n) = (l_1(q_0, \ldots, q_n)\colon \cdots : l_n(q_0, \ldots, q_n)\colon 0),$$

where $l_i(q_0, \ldots, q_n)$ is a linear form, for $i = 1, \ldots, n$. Furthermore, these n linear forms do not vanish simultaneously for any $q = (q_0\colon \cdots : q_n)$, unless $q = p$, which implies that they are linearly independent.

Conversely, let us take any n linearly independent linear forms $l_1(X_0, \ldots, X_n)$, $\ldots$, $l_n(X_0, \ldots, X_n)$. These linear forms define some hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^n$ whose intersection is a point $p \in \mathbb{P}^n$. Then, we have the map $\pi_p\colon (\mathbb{P}^n - \{p\}) \to \mathbb{P}^{n-1}$, defined by

$$\pi_p(X_0\colon \cdots : X_n) = (l_1(X_0, \ldots, X_n)\colon \cdots : l_n(X_0, \ldots, X_n)).$$

Geometrically, π_p is the projection from p onto the hyperplane $X_n = 0$. We have the following corollary of Theorem 2.36:

Corollary 2.40 *Let $X \subseteq \mathbb{P}^n$ be a projective variety, and let $p \in \mathbb{P}^n - X$. Then, projection from p, when restricted to X, is a morphism from X to $\mathbb{P}^{n-1}$. Further, we have the following properties:*

(a) If $X' = \pi_p(X)$, then $\pi_p \restriction X\colon X \to X'$ is a morphism.

(b) X' is closed in $\mathbb{P}^{n-1}$.

(c) The fibres of $\pi_p \restriction X$ are finite.

Proof. The map π_p is a morphism outside p, and since $p \notin X$, it is a morphism on X. Since X is closed in $\mathbb{P}^n$, by Theorem 2.36, X' is closed in $\mathbb{P}^{n-1}$. For (c), pick $q \in X'$. Note that $\pi_p^{-1}(q)$ corresponds to the line l_{pq} intersected with X. However, $l_{pq} \not\subseteq X$, since $p \notin X$, and thus, $l_{pq} \cap X \neq l_{pq}$. Then, $l_{pq} \cap X$ is closed in l_{pq}, and since l_{pq} has dimension 1, it follows that $l_{pq} \cap X$ is finite. $\square$

We can iterate Corollary 2.40 to prove *Noether's normalization lemma in the projective case*.

Corollary 2.41 *(Noether's normalization lemma–projective case) Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety, and assume that $\dim(X) = r < n$. Then, there is a morphism $\pi\colon X \to \mathbb{P}^r$ such that the following properties hold:*

(1) The fibres are finite and π is surjective.

(2) The projective coordinate ring, $k[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$, is a finite $k[Y_0, \ldots, Y_r]$-module.

(3) *If k is infinite or we allow ourselves a finite degree field extension, the Y_i's can be taken to be linear functions of the Z_i's.*

Proof. After a finite field extension if k is finite, there exists a $p \in \mathbb{P}^n - X$. Project from p. Corollary 2.40 says that $\pi_p(X) = X_1 \subseteq \mathbb{P}^{n-1}$, and that the fibres are finite. Then, $\dim(X_1) = \dim(X) = r$, by the fibre dimension theorem. If $r \neq n-1$, repeat the process. We get a sequence of projections

$$\pi\colon X \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X' \subseteq \mathbb{P}^r.$$

Since X is irreducible, X' is also irreducible, and $\dim(X') = r = \dim(\mathbb{P}^r)$. Hence, $X' = \mathbb{P}^r$, since $\mathbb{P}^r$ is irreducible. The fibres of π are finite.

In order to prove (2), we only need to consider a single step, since being a finite module is a transitive property, and we can finish by induction. Pick $p \in \mathbb{P}^n - X$. Using a preliminary linear transformation, we may assume that $p = (0\colon \cdots : 0\colon 1)$ and that the linear forms l_j defining p are $l_j(Z_0, \ldots, Z_n) = Z_j$, for $j = 0, \ldots, n-1$. Then,

$$\pi_p(q) = (q_0\colon \cdots : q_{n-1}).$$

Our result is a question about the affine cones $C(X)$ and $C(X')$, whose rings are $k[C(X)] = k[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$ and $k[C(X')] = k[Z_0, \ldots, Z_{n-1}]/\mathfrak{I}(X')$, where the map of affine rings

$$k[C(X')] \longrightarrow k[C(X)]$$

is given by $Z_j \mapsto Z_j$, $j = 0, \ldots, n-1$. There is some $f \in \mathfrak{I}(X)$ such that $f(p) \neq 0$, since $p \notin X$. Let $\deg(f) = \delta$.

Claim. The monomial Z_n^δ appears in f.

If not, the monomials appearing in f are of the form

$$Z_n^\epsilon Z_1^{\alpha_1} \cdots Z_{n-1}^{\alpha_{n-1}}$$

where $\epsilon + \alpha_1 + \cdots + \alpha_{n-1} = \delta$ and $\epsilon < \delta$. But then, some $\alpha_i > 0$, and these monomials all vanish at p, a contradiction. Thus,

$$f(Z_0, \ldots, Z_n) = Z_n^\delta + f_1(Z_0, \ldots, Z_{n-1})Z_n^{\delta-1} + \cdots + f_\delta(Z_0, \ldots, Z_n).$$

We know that the map

$$k[Z_0, \ldots, Z_{n-1}] \longrightarrow k[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$$

factors through $k[C(X')]$. We only need to prove that $k[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$ is a finite $k[Z_0, \ldots, Z_{n-1}]$-module. This will be the case if $k[Z_0, \ldots, Z_n]/(f)$ is a finite $k[Z_0, \ldots, Z_{n-1}]$-module. But $k[Z_0, \ldots, Z_n]/(f)$ is a free $k[Z_0, \ldots, Z_{n-1}]$-module on the basis $1, Z_n, \ldots, Z_n^{\delta-1}$, and this proves (2) and (3). $\square$

From this corollary, we can also derive another proof of *Noether's normalization lemma in the affine case* (DX).

The successive projections of Corollary 2.41 can be viewed as a projection from a linear center. Let L be a linear subspace of $\mathbb{P}^n$, which means that $C(L)$, the cone over L, is a linear subspace of $\mathbb{A}^{n+1}$. Assume that $\dim(L) = \delta$, and let $r = n - \delta - 1$. Then, we can define a morphism $\pi_L\colon (\mathbb{P}^n - L) \to \mathbb{P}^r$. Indeed, if L is cut out by $n - \delta = r + 1$ hyperplanes defined by linear forms $l_0, \ldots l_r$, we let

$$\pi_L(q_0\colon \cdots : q_n) = (l_0(q_0, \ldots, q_n)\colon \cdots : l_r(q_0, \ldots, q_n)).$$

Geometrically, π_L can described as follows: Let H be a linear subspace of $\mathbb{P}^n$ of dimension $n - \delta - 1 = r$, disjoint from L. Consider any linear subspace F of dimension $\delta + 1 = n - r$ through L. Then,

$$\dim(F) + \dim(H) - n = \delta + 1 + r - n = 0.$$

By the projective version of the intersection dimension theorem, $F \cap H$ is nonempty, and $F \cap H$ consists of a single point, $\pi_L(F)$. Thus, we get a map as follows: For every $q \notin L$, if F_q is the span of q and L, then $\dim(F_q) = \delta + 1$, and we let

$$\pi_L(q) = F_q \cap H.$$

In Corollary 2.41, we can take the span of the successive points from which projections are made (call it L), and we get $\pi_L\colon X \to \mathbb{P}^r$ as the resulting morphism.

We can also use Corollary 2.40 to investigate the degree of a curve. Given a curve $C \subseteq \mathbb{P}^n$, we want to find an open set, U, in $\mathbb{P}((\Omega^{n+1})^D) = (\mathbb{P}^n)^D$ (the hyperplane space) so that $H \cap C$ is a constant number of points for all $H \in U$. First, assume that $n > 3$. Then, the secant variety, $\mathrm{Sec}(C)$, of lines touching C in at least two points (including tangent lines) has dimension 3. Therefore, $\mathrm{Sec}(C)$ is strictly contained in $\mathbb{P}^n$, and we can pick some p such that no line in $\mathrm{Sec}(C)$ passes through p. If we project C from p, we get a map

$$\pi_p \restriction C\colon C \longrightarrow C' \subseteq \mathbb{P}^{n-1}.$$

If $q \in C$, then $l_{pq} \cap C$ is the fibre over q of $\pi_p \restriction C$, and since no line in $\mathrm{Sec}(C)$ passe through p, every fibre consists of a single point. Also, the good hyperplanes for C' correspond to the good hyperplanes for C through p. By varying p in $\mathbb{P}^n - \mathrm{Sec}(C)$ (which means that p does not belong to any line in $\mathrm{Sec}(C)$), we see that induction reduces the proof to the case $n = 3$. Actually, if C is nonsingular, then C' and C are isomorphic (this can be shown using the formal implicit function theorem). Thus, we obtain another corollary.

Corollary 2.42 *Every nonsingular projective curve C admits an embedding into $\mathbb{P}^3$. If X is a nonsingular projective variety of dimension d, then X admits an embedding in $\mathbb{P}^{2d+1}$ as a nonsingular variety (the secant variety has dimension $2d + 1$).*

We now have to deal with the case where $C \hookrightarrow \mathbb{P}^3$. Generally, $\mathrm{Sec}(C) = \mathbb{P}^3$. Take any $p \in \mathbb{P}^3$ and consider the projection from p onto $\mathbb{P}^2$. The hyperplanes in $\mathbb{P}^2$ correspond to hyperplanes in $\mathbb{P}^3$ through p. Look at the lines in $\mathbb{P}^3$. They belong to the Grassmannian $\mathbb{G}(2,4)$, which has dimension $2(4-2) = 4$. The lines through p form a subvariety of dimension 3. The family of lines

$$\{l_{pq} \mid l_{pq} \cap C \neq \emptyset\}$$

has dimension 2, and the subfamily of those lines l_{pq} such that $l_{pq} \in \mathrm{Sec}(C)$ has dimension 1. Thus, the fibre of π_p above a point s in C' has cardinality strictly greater than 1 only on a closed subset, Z, of C'. It follows that C' has at most a finite number of extra singularities besides those of C. Using the projection $\pi_p\colon C \to C'$, we get an open set of hyperplanes H in $\mathbb{P}^3$ through p such that $H \cap C$ has a constant number of points (in $\mathbb{P}^2$, we avoid the finite number of singularities).

Finally, we have $C' \subseteq \mathbb{P}^2$. But C' is a hypersurface, and thus, it is given by a single equation

$$f(X_0, X_1, X_2) = 0$$

of degree δ. Then, the lines cutting C' in δ distinct points are those missing a closed algebraically defined set in $\mathbb{P}^2$, which concludes the proof.

A shorter and better proof can be sketched as follows: Consider the product variety $C \prod (\mathbb{P}^n)^D$, and the incidence variety

$$I = \{(\xi, H) \in C \prod (\mathbb{P}^n)^D \mid \xi \in H\}.$$

It is a closed subvariety. Consider the projection

$$pr_2\colon I \longrightarrow (\mathbb{P}^n)^D.$$

By the intersection dimension theorem, pr_2 is surjective. The fibre of H is the set

$$\{(\xi, H) \mid \xi \in C \cap H\}.$$

Assuming that C is nondegenerate (which means that C is not contained in any H), we find that $C \cap H$ is a finite number of points. Since I is a projective variety and pr_2 is surjective, pr_2 is a closed map and has finite fibres. This implies (to be proved later on) that pr_2 is a finite morphism. Now, $\dim(I) = \dim\,(\mathbb{P}^n)^D$, and the proof of Chevalley's theorem says that there is some open $U \subseteq (\mathbb{P}^n)^D$ so that

$$pr_2\colon pr_2^{-1}(U) \cap I \longrightarrow U$$

is an integral morphism. Then, there is a smaller open on which the cardinality of the fibres is constant, as desired.

2.7 Rational Maps

In projective geometry, there are many maps between varieties which are densely defined yet not extendable to the whole space and hence, are not morphisms. These have important geometric content and, in fact, predated the concept of morphism, which has occupied us in the previous work.

Lemma 2.43 *Let X, Y be varieties, with Y separated. For any two morphisms $\varphi\colon X \to Y$ and $\psi\colon X \to Y$ and for any dense subset $U \subseteq X$, if $\varphi = \psi$ on U, then $\varphi = \psi$ on X.*

Proof. Look at the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] & X \prod X \arrow[r, "\varphi \prod \psi"] & Y \prod Y \\ U \arrow[u] \arrow[rr, "\theta"] & & Y. \arrow[u, "\Delta_Y"'] \end{tikzcd}$$

By hypothesis, a morphism $\theta\colon U \to Y$ making the diagram commute exits, as shown. Then,

$$(\varphi \prod \psi) \circ \Delta_X(U) \subseteq \Delta_Y.$$

By continuity,

$$(\varphi \prod \psi) \circ \Delta_X(\overline{U}) \subseteq \overline{(\varphi \prod \psi) \circ \Delta_X(U)} \subseteq \overline{\Delta_Y}.$$

However, Y is separated, so that $\Delta_Y = \overline{\Delta_Y}$, and since U is dense, we get

$$(\varphi \prod \psi) \circ \Delta_X(X) \subseteq \Delta_Y, \qquad \square$$

A counterexample to the above lemma when Y is not separated is the variety constructed as follows: Let $V = \mathbb{A}^1$ and $U = \mathbb{A}^1 - \{0\}$; and glue V to V along U, by the identity. We get a variety W consisting of two half lines and a pair of points at the origin. We can define two morphisms from U to W that agree on U, and yet, take distinct values at 0.

Remark: If we glue the two copies of V but we flip the second copy "upside down," i.e., use the gluing map $x \mapsto 1/x$, then (a) for $k = \mathbb{R}$, we get the circle $\mathbb{RP}^1 = S^1$, and (b) for $k = \mathbb{C}$, we get the sphere $\mathbb{CP}^1 = S^2$. It turns out that for the quaternions $\mathbb{H}$, we get $\mathbb{HP}^1 = S^4$.

Let U be a k-open and k-dense subset of a variety V and let $\varphi_U\colon U \to Z$ be a morphism to a *separated* variety Z. We can define an equivalence relation among pairs (U, φ_U), as follows: Given two pairs (U_1, φ_{U_1}) and (U_2, φ_{U_2}), where $\varphi_{U_1}\colon U_1 \to Z$ and $\varphi_{U_2}\colon U_2 \to Z$ are morphisms, we say that (U_1, φ_{U_1}) and (U_2, φ_{U_2}) are equivalent, denoted by

$$(U_1, \varphi_{U_1}) \sim (U_2, \varphi_{U_2})$$

iff $\varphi_{U_1} = \varphi_{U_2}$ on $U_1 \cap U_2$. We need to check that this is indeed an equivalence relation. The only nontrivial fact is transitivity. If

$$(U_1, \varphi_{U_1}) \sim (U_2, \varphi_{U_2}),$$

then $\varphi_{U_1} = \varphi_{U_2}$ on $U_1 \cap U_2$, and if

$$(U_2, \varphi_{U_2}) \sim (U_3, \varphi_{U_3}),$$

then $\varphi_{U_2} = \varphi_{U_3}$ on $U_2 \cap U_3$. Consider $G = U_1 \cap U_2 \cap U_3$. This set is k-open and k-dense in $U_1 \cap U_3$, and

$$\varphi_{U_1} = \varphi_{U_2} = \varphi_{U_3}$$

on G. By Lemma 2.43, we have $\varphi_{U_1} = \varphi_{U_3}$ on $U_1 \cap U_3$.

Definition 2.12 Given two varieties V and Z where Z is separated, an equivalence class φ of pairs (U, φ_U) as above is a *rational map* of V to Z. It is denoted by $\varphi\colon V \dashrightarrow Z$.

Given a rational map $\varphi\colon V \dashrightarrow Z$, we can define the k-open and k-dense subset

$$\mathcal{U} = \bigcup \{U \mid (U, \varphi_U) \in \varphi\}.$$

Clearly, $\varphi_{\mathcal{U}}$ is defined as a morphism on $\mathcal{U}$, and $\mathcal{U}$ is the largest k-dense, k-open on which $\varphi_{\mathcal{U}} \in \varphi$ is a morphism. We summarize this in the following proposition:

Proposition 2.44 *Every rational map $\varphi\colon V \dashrightarrow Z$ admits a unique maximal k-open, k-dense subvariety where it is defined as a morphism. Thus, a rational map to Z (separated) is a morphism from a k-open, k-dense subset of V admitting no further extension as a morphism.*

We also introduce the following nomenclature and notation: We denote by $\mathrm{Rat}(X, Z)$ the collection of all rational maps $\varphi\colon V \dashrightarrow Z$ (Z is separated). We let $\mathcal{M}\mathrm{er}(X) = \mathrm{Rat}(X, \mathbb{A}^1)$ be the set of rational functions on X, also called *meromorphic functions* on X. A rational map $\varphi\colon V \dashrightarrow Z$ whose image is k-dense in Z is called *dominant*, or *dominating*.

Let X, Z be separated varieties, assume that we have rational maps $\varphi\colon X \dashrightarrow Z$ and $\psi\colon Z \dashrightarrow X$, and that U and V are the maximal domains of definition of φ and ψ. Assume that $\varphi^{-1}(V)$ is dense in X, and $\psi^{-1}(U)$ is dense in Y; further that $U \cap \varphi^{-1}(V)$ is open and dense in U and $V \cap \psi^{-1}(U)$ is open and dense in V. If

$$\varphi \circ \psi = \mathrm{id} \text{ on } V \cap \varphi^{-1}(U)$$

and

$$\psi \circ \varphi = \mathrm{id} \text{ on } U \cap \psi^{-1}(V),$$

then we say that φ and ψ are *birational maps* and birational inverses of each other. In this case, we say that X *is birationally equivalent to* Y. Further, call φ *birational* if ψ, inverting φ, as above, exists.

Two varieties might be birationally equivalent, but not isomorphic. For example, $\mathbb{A}^n$ is birationally equivalent to $\mathbb{P}^n$. Let H be any hyperplane complementary to $\mathbb{A}^n$ in $\mathbb{P}^n$, and let $V = \mathbb{P}^n - H$. Then, the bijection sending $\mathbb{A}^n$ to V is a birational map. However, $\mathbb{P}^n$ is proper but $\mathbb{A}^n$ is not, and $\mathbb{P}^n$ and $\mathbb{A}^n$ are not isomorphic.

Proposition 2.45 *Given an irreducible variety X, the natural inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ yields a bijection*

$$\mathrm{Rat}(X, \mathbb{A}^1) \cong \mathrm{Rat}(X, \mathbb{P}^1).$$

Proof. The map $\mathrm{Rat}(X, \mathbb{A}^1) \longrightarrow \mathrm{Rat}(X, \mathbb{P}^1)$ is clearly an injection. Let $\varphi \in \mathrm{Rat}(X, \mathbb{P}^1)$, and assume that U is the domain of φ. If φ misses "∞" (where ∞ means the complement of $\mathbb{A}^1$ in $\mathbb{P}^1$), then φ maps X to $\mathbb{A}^1$, and we are done. Otherwise, consider the subset $Z = \varphi^{-1}(\infty)$. It is closed in X, and we let $V = X - Z$. Since X is irreducible, V is dense in X, and then, the restriction, ψ, of φ to $U \cap V$ is a morphism $\psi \colon U \cap V \to \mathbb{A}^1$. Hence, φ comes from a morphism from A to $\mathbb{A}^1$. $\square$

Remark: This is false for morphisms. The problem is that a morphism $\varphi \colon X \to \mathbb{P}^1$ gives a rational map $\psi \colon X \dashrightarrow \mathbb{A}^1$, but ψ does not necessarily extend to a morphism.

We define the categories Irred/k and FncFlds/k as follows: The objects of the category Irred/k are all the irreducible separated k-varieties and the maps are dominant rational morphisms. Isomorphism in this category is birational equivalence. The objects of the category FncFlds/k are all the finitely generated fields over k; that is, fields of the form $\mathrm{Frac}(A)$, where A is a finitely generated k-algebra which is a domain. The morphisms in FncFlds/k are the k-monomorphisms (if necessary, we will assume that $\mathrm{Frac}(A)$ is separable over k).

We have the following theorem showing that Irred/k and FncFlds/k are anti-equivalent:

Theorem 2.46 *The functor $X \mapsto \mathcal{M}\mathrm{er}(X)$ establishes an anti-equivalence of the two categories* Irred/k *and* FncFlds/k. *Hence, k-irreducible varieties are birationally equivalent iff their function fields are isomorphic.*

Proof. Let U be open in X. Then, $\varphi \colon U \to \mathbb{A}^1$ gives $\varphi \in \mathcal{M}\mathrm{er}(X) = \mathrm{Rat}(X, \mathbb{A}^1)$. However, $\mathbb{A}^1$ is separated, so that giving φ in $\mathrm{Rat}(X, \mathbb{A}^1)$ is equivalent to giving φ in $\mathrm{Rat}(U, \mathbb{A}^1)$ for any dense open U in X. Hence, the sheaf $U \mapsto \mathcal{M}\mathrm{er}(U)$ is the constant sheaf on X.[2] Pick $U_0 \subseteq X$, open, dense, affine. We know that

$$\mathrm{Rat}(X, \mathbb{A}^1) = \mathrm{Rat}(U_0, \mathbb{A}^1) = \mathrm{Frac}(k[U_0]),$$

and $k[U_0]$ is finitely generated as a k-algebra. This implies that $\mathcal{M}\mathrm{er}(X)$ is indeed a function field. If $\varphi \colon X \dashrightarrow Y$ is a dominant rational map, then pick $\psi \in \mathcal{M}\mathrm{er}(Y)$, that is, a rational

[2] See the Appendix for the definition of a sheaf.

map from Y to $\mathbb{A}^1$. The rational map ψ is defined on an open dense set $V \subseteq Y$, so that $\varphi^{-1}(V)$ is open dense in X, and on $\varphi^{-1}(V)$, we get

$$\psi \circ \varphi \colon \varphi^{-1}(V) \longrightarrow \mathbb{A}^1,$$

a function in $\mathcal{M}\mathrm{er}(X)$. The reader should check that

(1) $\mathcal{M}\mathrm{er}(Y) \longrightarrow \mathcal{M}\mathrm{er}(X)$, as just given, is a k-injection.

(2) $X \mapsto \mathcal{M}\mathrm{er}(X)$ is a cofunctor.

Backwards, first pick K of the form $\mathcal{M}\mathrm{er}(X)$ and L of the form $\mathcal{M}\mathrm{er}(Y)$, and an injection $\theta \colon L \to K$. Take $V \subseteq Y$ open affine, so that $\mathcal{M}\mathrm{er}(V) = \mathcal{M}\mathrm{er}(Y)$ and $\mathcal{M}\mathrm{er}(V) = \mathrm{Frac}(k[V])$. Now, $k[V] = k[Z_1, \ldots, Z_n]/\mathfrak{I}$, where $\sqrt{\mathfrak{I}} = \mathfrak{I}$ and the injection θ is determined by $\theta(Z_1), \ldots, \theta(Z_n) \in \mathcal{M}\mathrm{er}(X)$. There are some open dense subsets U_j such that $\theta(Z_j)$ is holomorphic on U_j; so, let $U = U_1 \cap \cdots \cap U_n$, which is k-open, k-dense, and affine (by choosing the U_j's affine). Then, $\theta(Z_j) \in k[U]$, which implies that we have the commutative diagram

$$\begin{tikzcd} k[V] \arrow[r, "\theta"] \arrow[d] & k[U] \arrow[d] \\ \mathcal{M}\mathrm{er}(Y) \arrow[r, "\theta"'] & \mathcal{M}\mathrm{er}(X) \end{tikzcd}$$

so that $\theta \colon k[V] \to k[U]$ is injective. Hence, we get a morphism $\theta^* \colon U \to V$ with dense image, and so, $\theta^* \in \mathrm{Rat}(X, Y)$. We still have to prove that every function field is of the form $\mathcal{M}\mathrm{er}(X)$. Let K be a finitely generated field extension over k. Then,

$$K = k(z_1, \ldots, z_n) = \mathrm{Frac}(k[z_1, \ldots, z_n]).$$

There exist some indeterminates $T_1, \ldots, T_n$ and a surjective map

$$\varphi \colon k[T_1, \ldots, T_n] \longrightarrow k[z_1, \ldots, z_n]$$

whose kernel is a prime ideal $\mathfrak{p}$. Then,

$$k[V(\mathfrak{p})] = k[T_1, \ldots, T_n]/\mathfrak{p} \cong k[z_1, \ldots, z_n],$$

and $K = \mathcal{M}\mathrm{er}(V(\mathfrak{p}))$. $\square$

Remark: If K is separably generated over k, then V is a variety separably generated over k.

Corollary 2.47 *If X, Y are irreducible varieties, then X and Y are birationally equivalent iff $\mathcal{M}\mathrm{er}(X) \cong \mathcal{M}\mathrm{er}(Y)$.*

Corollary 2.48 *If X is irreducible and separably generated over k, then X is birationally equivalent to a hypersurface in $\mathbb{A}^n$ (or a hypersurface in $\mathbb{P}^n$).*

Proof. Pick a separating transcendence basis for $K = \mathcal{M}\mathrm{er}(X)$ over k, say $Z_1, \ldots, Z_r$ ($\dim(X) = r$). Then, $K/k(Z_1, \ldots, Z_r)$ is separable algebraic and finitely generated, which implies finite and separable. By Kronecker's theorem of the primitive element (Zariski and Samuel [60], Theorem 19, Chapter II, Section 9), there is some θ such that

$$K = k(Z_1, \ldots, Z_r)(\theta)$$

and θ satisfies an equation of the form

$$\theta^n + \alpha_1(Z_1, \ldots, Z_r)\theta^{n-1} + \cdots + \alpha_{n-1}(Z_1, \ldots, Z_r)\theta + \alpha_n(Z_1, \ldots, Z_r) = 0,$$

where $\alpha_j(Z_1, \ldots, Z_r) \in k(Z_1, \ldots, Z_r) = \mathcal{M}\mathrm{er}(\mathbb{A}^r) = \mathcal{M}\mathrm{er}(\mathbb{P}^r)$. Clear denominators, and get an equation of the form

$$\beta_0(Z_1, \ldots, Z_r)\theta^n + \beta_1(Z_1, \ldots, Z_r)\theta^{n-1} + \cdots + \beta_{n-1}(Z_1, \ldots, Z_r)\theta + \beta_n(Z_1, \ldots, Z_r) = 0,$$

where $\beta_j(Z_1, \ldots, Z_r) \in k[Z_1, \ldots, Z_r]$. Therefore, we get the hypersurface V of equation

$$\beta_0(Z_1, \ldots, Z_r)T^n + \beta_1(Z_1, \ldots, Z_r)T^{n-1} + \cdots + \beta_{n-1}(Z_1, \ldots, Z_r)T + \beta_n(Z_1, \ldots, Z_r) = 0$$

in $\mathbb{A}^{r+1}$, or, homogenizing the β_j's, in $\mathbb{P}^{r+1}$. But $\mathcal{M}\mathrm{er}(V) = K$, by construction. By Corollary 2.47, since $\mathcal{M}\mathrm{er}(V) = K = \mathcal{M}\mathrm{er}(X)$, the varieties X and V are birationally equivalent. $\square$

Recall that an irreducible variety X is normal iff for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is integrally closed in $\mathcal{M}\mathrm{er}(X)$. This is equivalent to saying that X is covered by affine patches and the coordinate rings of these patches are integrally closed in $\mathcal{M}\mathrm{er}(X)$.

Proposition 2.49 *If X is normal, then X is nonsingular in codimension 1. That is, the singular locus, $\mathrm{Sing}(X)$, has codimension at least 2 in X.*

Proof. Let x be chosen with $\dim(\overline{\{x\}}) = n - 1$, where $n = \dim(X)$. We must show that $x \notin \mathrm{Sing}(X)$. (As usual, we assume separable generation over k). However, $\mathcal{O}_{X,x}$ is one-dimensional and integrally closed. Thus, $\mathcal{O}_{X,x}$ is Noetherian, local, one-dimensional, and an integrally closed domain, which implies that $\mathcal{O}_{X,x}$ is a local Dedekind ring. So, $\mathcal{O}_{X,x}$ is a DVR (a discrete valuation ring, see Zariski and Samuel [60], Theorem 15, Chapter V, Section 6). This implies that $\mathcal{O}_{X,x}$ is a regular local ring. Therefore, x is nonsingular. $\square$

Sometimes, a rational map to a suitable variety can be extended to a really big domain of definition. The theorem below makes this precise and is used very often in the succeeding theory of projective varieties and their generalizations.

Theorem 2.50 *Let X be an irreducible variety which is nonsingular in codimension 1 (e.g., normal), and let Y be a quasi-projective variety (open subvariety in some projective variety). Then, every rational map $\varphi\colon X \dashrightarrow Y$ admits an extension to a rational map $\Phi\colon X \dashrightarrow \overline{Y}$ (where $\overline{Y}$ is the closure of Y in $\mathbb{P}^n$), and the locus where Φ is not defined has codimension at least 2 in X.*[3]

[3]For example, when X is a surface, nonsingular in codimension 1, then the locus where Φ is possibly not defined is just a finite set of points.

Proof. We may assume that X is affine, since $\mathrm{dom}(\varphi)$ and $\mathrm{dom}(\Phi)$ are open and dense for any extension, Φ, of φ. Let ξ be generic for X, by hypothesis, $Y \hookrightarrow \mathbb{P}^n$ for some n, and by composition, we get a rational map $\widetilde{\varphi}\colon X \dashrightarrow \mathbb{P}^n$. If we show that every point $x \notin \mathrm{Sing}(X)$ is in $\mathrm{dom}(\widetilde{\varphi})$, then, as x is a specialization of ξ, we see that $\widetilde{\varphi}(x)$ is a specialization of $\varphi(\xi)$, and thus, $\widetilde{\varphi}(x) \in \overline{Y}$. Thus, we may assume that Y is closed and more, that $Y = \mathbb{P}^n$. We are down to the essential case where X is affine, irreducible, and $Y = \mathbb{P}^n$.

We have $\varphi(\xi) \in \mathbb{P}^n$, where ξ is generic for X. Thus, $\varphi(\xi)$ is in one of the standard opens, say $\varphi(\xi) \in U_j = \{(\alpha) \in \mathbb{P}^n \mid \alpha_j \neq 0.\}$. However, U_j is affine, so that there are holomorphic functions θ_i (near ξ) such that

$$\varphi(\xi) = (\theta_0(\xi), \ldots, \widehat{\theta_j(\xi)}, \ldots, \theta_n(\xi)),$$

where, as usual, the hat over the argument means that it should be omitted. Each θ_j has a denominator, and by multiplying through by all these denominators, we get

$$\varphi(\xi) = (\Lambda_0(\xi)\colon \cdots : \Lambda_j(\xi)\colon \cdots : \Lambda_n(\xi)),$$

where Λ_i is holomorphic in X. Look at Λ_l in $\mathcal{O}_{X,x}$. Now, $\mathcal{O}_{X,x}$ is a DVR, so let π be a local uniformizer (i.e., $\mathfrak{m}_x = (\pi) = \pi\mathcal{O}_{X,x}$). Write $\mathrm{ord}_x(\Lambda_j) = \alpha_j$ (α_j is the largest integer d such that $\Lambda_j \in \mathfrak{m}_x^d$). If $\varphi(x)$ is to exists, it is the point

$$(\Lambda_0(x) \bmod \mathfrak{m}_x\colon \cdots : \Lambda_n(x) \bmod \mathfrak{m}_x).$$

Each $\Lambda_l(\xi)$ has the form

$$\Lambda_l(\xi) = \pi^{\alpha_l} M_l(\xi)$$

where M_l is a unit in $\mathcal{O}_{X,x}$. One of the orders α_l is minimal, say α_r. Then, we have

$$\Lambda_l(\xi) = \pi^{\alpha_r}\pi^{\alpha_l - \alpha_r} M_l(\xi),$$

and so,

$$\begin{aligned}\varphi(\xi) &= (\pi^{\alpha_r}\pi^{\alpha_0-\alpha_r}M_0(\xi)\colon \cdots : \pi^{\alpha_r}\pi^{\alpha_n-\alpha_r}M_n(\xi))\\ &= (\pi^{\alpha_0-\alpha_r}M_0(\xi)\colon \cdots : M_r(\xi)\colon \cdots : \pi^{\alpha_n-\alpha_r}M_n(\xi)).\end{aligned}$$

We obtain $\varphi(x)$ by replacing ξ by x, and reducing mod $\mathfrak{m}_x$. But $M_r(\xi) \neq 0 \pmod{\mathfrak{m}_x}$, since M_r is a unit in $\mathcal{O}_{X,x}$. Thus, $\varphi(x)$ exists in $\mathbb{P}^n$, as desired. $\square$

Corollary 2.51 *Let X be an irreducible curve and Y a quasi-projective variety. If X is normal, then every rational map $\varphi\colon X \dashrightarrow Y$ extends to a morphism $\widehat{\varphi}\colon X \to \overline{Y}$. In particular, if Y is projective, then φ is already a morphism. Consequently if X and Y are projective nonsingular curves, birational equivalence of X and Y is the same as isomorphism. To classify nonsingular (projective) curves is the same as classifying function fields in one variable (i.e., of dimension 1).*

2.8 Blow-Ups

In the last corollary of the previous section dealing with rational morphisms of curves, we observed that for nonsingular curves, birational equivalence of nonsingular curves is the same as isomorphism. This is far from being the case in every dimension bigger than one. In fact, one tries to find in each birational equivalence class of higher dimensional varieties a "simplest" examplar. The exact meaning of simplest will be discussed later on. For surfaces, it turns out that in most cases there is a unique nonsingular surface in each birational class; so, the reader will see that many varieties can be birationally equivalent to a nonsingular variety, yet not isomorphic to it. To make a construction yielding nonisomorphic yet birational varieties is the aim of this section. Here is the construction.

Consider $\mathbb{A}^n$, $\mathbb{P}^{n-1}$, pick some point $p \in \mathbb{A}^n$, and choose coordinates $(x_1, \ldots, x_n)$ in $\mathbb{A}^n$ so that $p = (0, \ldots, 0)$, and let $(y_1\colon \cdots : y_n)$ be homogeneous coordinates in $\mathbb{P}^{n-1}$.

Definition 2.13 The subvariety, $B_p(\mathbb{A}^n)$, of $\mathbb{A}^n \prod \mathbb{P}^{n-1}$, is the variety defined by the equations

$$x_i y_j = x_j y_i \quad 1 \leq i, j \leq n.$$

It is called the *blow-up of* $\mathbb{A}^n$ *at* p.

The restriction of the projection maps

$$\begin{tikzcd} & \mathbb{A}^n \prod \mathbb{P}^{n-1} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{A}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

to $B_p(\mathbb{A}^n)$ yields the maps:

$$\begin{tikzcd} & B_p(\mathbb{A}^n) \arrow[dl, "\pi_p"'] \arrow[dr, "\sigma_p"] & \\ \mathbb{A}^n & & \mathbb{P}^{n-1}. \end{tikzcd}$$

Look at $\pi_p\colon B_p(\mathbb{A}^n) \to \mathbb{A}^n$. It is a morphism, and in fact, a birational map. Let $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi \neq p$; what is $\pi_p^{-1}(\xi)$?

Since $\xi \neq p$, we have $\xi_j \neq 0$ for some j; then

$$\pi_p^{-1}(\xi) = \{(\xi, y) \mid \xi_i y_j = \xi_j y_i,\ 1 \leq i, j \leq n\}.$$

But $\xi_j \neq 0$, and thus,

$$y_i = \frac{\xi_i}{\xi_j} y_j,$$

so that

$$(y_1\colon \cdots : y_n) = \left(\frac{\xi_1}{\xi_j}\colon \cdots : \frac{\xi_{j-1}}{\xi_j}\colon 1\colon \frac{\xi_{j+1}}{\xi_j}\colon \cdots : \frac{\xi_n}{\xi_j}\right) = (\xi_1\colon \cdots : \xi_n).$$

Therefore, $\pi_p^{-1}(\xi)$ consists of the single point

$$\pi_p^{-1}(\xi) = \{((\xi_1, \ldots, \xi_n); (\xi_1\colon \cdots : \xi_n))\}.$$

As a consequence, the restriction of π_p is an isomorphism

$$B_p(\mathbb{A}^n) - \pi^{-1}(p) \cong \mathbb{A}^n - \{p\}.$$

Now, for p, we have

$$\pi_p^{-1}(p) = \{(0, y) \mid 0y_j = 0y_i,\ 1 \leq i, j \leq n\} = \{0\} \times \mathbb{P}^{n-1}.$$

To simplify the notation, from now on, we will drop the subscript p in π_p and σ_p, unless confusion arises. The above yields the following proposition:

Proposition 2.52 *The map $\pi\colon B_p(\mathbb{A}^n) \to \mathbb{A}^n$ is a surjective morphism that induces an isomorphism between $B_p(\mathbb{A}^n) - \pi^{-1}(p)$ and $\mathbb{A}^n - \{p\}$. Each fibre over $\xi \neq p$ is a single point, and $\pi^{-1}(p) \cong \mathbb{P}^{n-1}$.*

Let us now look at all the lines through p. Parametrically, such a line $L_{(\alpha)}$ is given by $(\alpha_1 t, \ldots, \alpha_n t)$, where $\alpha_j \neq 0$ for some j, and t is the parameter. If $t \neq 0$, we have a point $(\alpha_1 t, \ldots, \alpha_n t) \in \mathbb{A}^n$ distinct from 0, and

$$\pi^{-1}(\alpha_1 t, \ldots, \alpha_n t) = \{((\alpha_1 t, \ldots, \alpha_n t); (\alpha_1\colon \cdots : \alpha_n))\}.$$

So,

$$\pi^{-1}(L_{(\alpha)} - \{p\}) = \{((\alpha_1 t, \ldots, \alpha_n t); (\alpha_1\colon \cdots : \alpha_n)) \mid t \neq 0\} \cong L_{(\alpha)} - \{p\}.$$

Then,

$$\overline{\pi^{-1}(L_{(\alpha)} - \{p\})} \cap \pi^{-1}(p) = \{((0, \ldots, 0); (\alpha_1\colon \cdots : \alpha_n))\}.$$

Hence, we get the following proposition:

Proposition 2.53 *Given a line $L_{(\alpha)}$ through p in $\mathbb{A}^n$, the line $L_{(\alpha)}$ is defined parametrically by $(\alpha_1 t, \ldots, \alpha_n t)$, where $\alpha_j \neq 0$ for some j, and the fibre $\pi^{-1}(p)$ intersects the closure of $\pi^{-1}(L_{(\alpha)} - \{p\})$ in exactly one point $((0), (\alpha_1\colon \cdots : \alpha_n))$. Hence, the correspondence*

$$L_{(\alpha)} \longleftrightarrow \overline{\pi^{-1}(L_{(\alpha)} - \{p\})} \cap \pi^{-1}(p)$$

is a bijection between lines through the point p in $\mathbb{A}^n$ and points of $\pi^{-1}(p)$. Consequently, the fibre above $(\alpha_1\colon \cdots : \alpha_n)$ of the other projection, $pr_2\colon B_p(\mathbb{A}^n) \to \mathbb{P}^{n-1}$, is (isomorphic to) the line $L_{(\alpha)}$. Thus, $B_p(\mathbb{A}^n) - \pi^{-1}(p)$ is dense in $B_p(\mathbb{A}^n)$, and $B_p(\mathbb{A}^n)$ is irreducible.

Proof. The only statements we haven't proved are those following the word "Consequently." We have

$$pr_2^{-1}(\alpha_1\colon \cdots \colon \alpha_n) = \{((z_1,\ldots,z_n);(\alpha_1\colon \cdots \colon \alpha_n)) \mid z_i\alpha_j = z_j\alpha_i,\ 1 \leq i,j \leq n\}.$$

Since $\alpha_j \neq 0$ for some j and the α_j are fixed, the equations $z_i\alpha_j = z_j\alpha_i$ define the line $L_{(\alpha)}$. Moreover, we saw that all points of $B_p(\mathbb{A}^n)$ lie in $B_p(\mathbb{A}^n) - \pi^{-1}(p)$ or on the closure of some line $L_{(\alpha)}$. Hence, as

$$\pi^{-1}(L_{(\alpha)} - \{p\}) \subseteq B_p(\mathbb{A}^n) - \pi^{-1}(p),$$

the points of $B_p(\mathbb{A}^n)$ lie in the closure of $B_p(\mathbb{A}^n) - \pi^{-1}(p)$. But,

$$B_p(\mathbb{A}^n) - \pi^{-1}(p) \cong \mathbb{A}^n - \{p\},$$

the latter being irreducible. Thus, $B_p(\mathbb{A}^n)$ is irreducible (as the closure of an irreducible). □

We also denote $\pi^{-1}(p)$ by E, and call it the *exceptional locus*, or *exceptional divisor*. It is a Weil divisor (see Section 5.1).

Example 2.11 Consider the rational map $\varphi\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ given by

$$(x,y) \mapsto \left(\frac{x}{y}, \frac{y}{x}\right).$$

Its open set of definition is the complement of the axes $x = 0$ and $y = 0$. Use coordinates (z_1, z_2) in the image $\mathbb{A}^2$. Then,

$$z_1 = \frac{x}{y}, \quad z_2 = \frac{y}{x}. \tag{$*$}$$

The image of φ is the curve $z_1z_2 = 1$. Embed $\mathbb{A}^2$ into $\mathbb{P}^2$ via the map

$$(z_1, z_2) \mapsto (1\colon z_1\colon z_2).$$

Using homogeneous coordinates T_0, T_1, T_2 in $\mathbb{P}^2$, we have

$$z_j = \frac{T_j}{T_0}, \quad \text{for } j = 1, 2.$$

The map φ is now given by

$$\frac{x}{y} = \frac{T_1}{T_0}, \quad \frac{y}{x} = \frac{T_2}{T_0}. \tag{$*'$}$$

From $(*')$, we get

$$T_0x = T_1y \quad \text{and} \quad T_0y = T_2x. \tag{$**$}$$

Note that $(**)$ gives a rational map $\varphi_1\colon \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ extending φ to all of $\mathbb{A}^2$, except the origin. Indeed, if $x = 0$ and $y \neq 0$, by $(**)$,

$$T_0 = T_1 = 0,$$

and we let

$$\varphi_1(0, y) = (0\colon 0\colon 1).$$

On the other hand, if $x \neq 0$ and $y = 0$, by $(**)$,

$$T_0 = T_2 = 0,$$

and we let

$$\varphi_1(x, 0) = (0\colon 1\colon 0).$$

Now, $\varphi_1\colon \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ still has the origin as a point of *indeterminacy*. Let us blow up $\mathbb{A}^2$ at the origin. We get $B_0(\mathbb{A}^2) \subseteq \mathbb{A}^2 \prod \mathbb{P}^1$, given by the equation

$$xv = yu,$$

where we use $(u\colon v)$ as homogeneous coordinates in $\mathbb{P}^1$. Then, we can extend φ_1 to $\Phi\colon B_0(\mathbb{A}^2) \dashrightarrow \mathbb{P}^2$, and we claim that Φ is defined everywhere, and is a morphism:

On points where $x \neq 0$ or $y \neq 0$, we have $\Phi = \varphi_1$. Assume $x = y = 0$. We know that either $u \neq 0$ or $v \neq 0$.

(1) If $u \neq 0$ and $v = 0$, then
$$((0, 0); (u, 0)) \mapsto (0\colon 1\colon 0).$$

(2) If $u = 0$ and $v \neq 0$, then
$$((0, 0); (0, v)) \mapsto (0\colon 0\colon 1).$$

(3) If $u \neq 0$ and $v \neq 0$, then
$$((0, 0); (u, v)) \mapsto \left(1\colon \frac{u}{v}\colon \frac{v}{u}\right).$$

The reader should check that (1)–(3) actually defined Φ everywhere on $B_0(\mathbb{A}^2)$ as a morphism.

The procedure above is a general fact in dimension 2. Namely, if X is a normal surface and $\varphi\colon X \dashrightarrow Y$ is a rational map, where Y is a projective variety, then for the points of indeterminacy, $p_1, \ldots, p_t$, of φ, repeated blow-ups at $p_1, \ldots, p_t$ give a surface Z and a morphism $\pi\colon Z \to X$ which is birational, and further there exists a *morphism* $\psi\colon Z \to Y$ so that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, "\varphi"] & & Y. \end{tikzcd}$$

This fact is a consequence of a theorem of Zariski, and the simplest example is the one we have given using (1)–(3).

A number of comments and remarks about blow-ups are in order:

Remarks:

(1) The blow-up $B_p(\mathbb{A}^n)$ depends only on a local neighborhood of p. Therefore, if X is covered by open subsets isomorphic to $\mathbb{A}^n$ (e.g., $\mathbb{P}^n$ and the Grassmannians), then $B_p(X)$ makes sense (just remove the open neighborhood of p, do $B_p(\mathbb{A}^n)$, reglue).

(2) If $p \neq q$ in $\mathbb{A}^n$, then
$$B_{q,p}(\mathbb{A}^n) = B_{p,q}(\mathbb{A}^n).$$

We now define the blow-up of any affine variety $X \subseteq \mathbb{A}^n$ at a point $p \in X$. First, make $B_p(\mathbb{A}^n)$. We know that
$$B_p(\mathbb{A}^n) - \pi^{-1}(p) \cong \mathbb{A}^n - \{p\},$$
and we have an inclusion $X - \{p\} \hookrightarrow \mathbb{A}^n - \{p\}$, so
$$\pi^{-1}(X - \{p\}) \cong X - \{p\}.$$

Definition 2.14 The *blow-up, $B_p(X)$, of an affine variety $X \subseteq \mathbb{A}^n$ at a point $p \in X$* is the closure of $\pi^{-1}(X - \{p\})$ in $B_p(\mathbb{A}^n)$.

What is $\pi^{-1}(p) \cap B_p(X)$? We can define X near p by some equations
$$f_1(Z_1, \ldots, Z_n) = \cdots = f_t(Z_1, \ldots, Z_n) = 0.$$
Arrange coordinates so that p is the origin, and write
$$f_i = f^{(i)}_{d_i} + f^{(i)}_{d_i+1} + \cdots,$$
where $f^{(i)}_{d_i}$ is the lowest degree homogeneous form appearing in f_i, with degree d_i, which is at least one, since $f_i(0) = 0$ (remember, $p = 0$). Look at the total inverse image in $B_p(\mathbb{A}^n)$ of X near p. The equations are
$$f_l(Z_1, \ldots, Z_n) = 0, \quad Z_iY_j - Z_jY_i = 0, \quad \text{where} \quad 1 \leq l \leq t, \text{ and } 1 \leq i, j \leq n.$$
Here, we use homogeneous coordinates $(Y_1 \colon \cdots \colon Y_n)$ in $\mathbb{P}^{n-1}$. Look at the patch, U_l, in $\mathbb{P}^{n-1}$, so that $Y_l \neq 0$. We get
$$Z_i = \frac{Y_i}{Y_l} Z_l;$$
thus,
$$\begin{aligned}
0 = f_j(Z_1, \ldots, Z_n) &= f_j\left(\frac{Y_1}{Y_l}Z_l, \ldots, \frac{Y_n}{Y_l}Z_l\right) \\
&= f^{(j)}_{d_j}\left(\frac{Y_1}{Y_l}Z_l, \ldots, \frac{Y_n}{Y_l}Z_l\right) + f^{(j)}_{d_j+1}\left(\frac{Y_1}{Y_l}Z_l, \ldots, \frac{Y_n}{Y_l}Z_l\right) + \cdots \\
&= Z_l^{d_j} f^{(j)}_{d_j}\left(\frac{Y_1}{Y_l}, \ldots, \frac{Y_n}{Y_l}\right) + Z_l^{d_j+1} f^{(j)}_{d_j+1}\left(\frac{Y_1}{Y_l}, \ldots, \frac{Y_n}{Y_l}\right) + \cdots
\end{aligned}$$
The total inverse image has two components:

(a) A copy of $\mathbb{P}^{n-1}$ when $Z_l = 0$.

(b) When $Z_l \neq 0$, the locus cut out by the equations

$$f^{(j)}_{d_j}\left(\frac{Y_1}{Y_l}, \ldots, \frac{Y_n}{Y_l}\right) + Z_l f^{(j)}_{d_j+1}\left(\frac{Y_1}{Y_l}, \ldots, \frac{Y_n}{Y_l}\right) + \cdots = 0, \quad \text{for} \quad j = 1, \ldots, t.$$

The latter are the equations of $B_p(X)$ on the local piece of $\mathbb{A}^n \prod \mathbb{P}^{n-1}$ corresponding to $Y_l \neq 0$. By homogenizing, we get

$$f^{(j)}_{d_j}(Y_1, \ldots, Y_n) + \frac{Z_l}{Y_l} f^{(j)}_{d_j+1}(Y_1, \ldots, Y_n) + \cdots = 0. \tag{$*$}$$

Equations $(*)$ describe $B_p(X)$ on the patch $Y_l \neq 0$. We get the points on $B_p(X) \cap \pi^{-1}(p)$ when we take $Z_1 = \cdots = Z_n = 0$ in equation $(*)$. Thus, for $B_p(X) \cap \pi^{-1}(p)$, we get the projective variety whose equations are

$$f^{(j)}_{d_j}(Y_1, \ldots, Y_n) = 0, \quad \text{for} \quad j = 1, \ldots, t.$$

Now, the *tangent cone to X at p* is defined to be the variety

$$\operatorname{Spec} k[T_1, \ldots, T_n]/(f^{(1)}_{d_1}, \ldots, f^{(t)}_{d_t}),$$

it is indeed a cone in $\mathbb{A}^n$. Our discussion shows that

$$\pi^{-1}(p) \cap B_p(X) = \operatorname{Proj}(k[T_1, \ldots, T_n]/(f^{(1)}_{d_1}, \ldots, f^{(t)}_{d_t}),$$

the *projectivized tangent cone to X at p*. We see that $B_p(X)$ is X with the projectivized tangent cone sewn in at p in place of p.

Let us give a few examples.

Example 2.12 Consider the cuspidal cubic, X, given by

$$Z_2^2 = Z_1^3,$$

and blow up the singular point $p = (0, 0)$. The lowest degree term is Z_2^2, and we expect the projectivized tangent cone to be $\operatorname{Proj}(k[Z_1, Z_2]/(Z_2^2))$. As a variety (not a scheme, see Chapter 3), this is $\operatorname{Proj}(k[Z_1]) = \mathbb{P}^0$, namely, a single point. We know that $\pi^{-1}(X)$ is the variety given by

$$Z_2^2 = Z_1^3 \quad \text{and} \quad Z_2 Y_1 = Z_1 Y_2.$$

Consider the two affine patches $Y_1 \neq 0$ and $Y_2 \neq 0$.

Case 1: $Y_1 \neq 0$. From the equations, we get

$$Z_2 = \frac{Y_2}{Y_1} Z_1 = Z_1 U,$$

where we let

$$U = \frac{Y_2}{Y_1}.$$

Note that introducing this new coordinate, U, has the effect that $U = \frac{Z_2}{Z_1}$; so, $U^2 = Z_1$, which means that U is integral over $\mathcal{O}_{X,p}$. We deduce that

$$Z_1^2 U^2 = Z_1^3,$$

and this defines two components. The first component corresponds to $Z_1 = 0$, in which case, we get the affine part of the fibre $\pi^{-1}(p)$. The second component corresponds to

$$U^2 = Z_1.$$

This is a nonsingular curve; and above p (i.e., when $Z_1 = 0$), we get the point $U = 0$ (actually, a double point, because the equation is $U^2 = 0$).

Case 2: $Y_2 \neq 0$. From the equations, we get

$$Z_1 = \frac{Y_1}{Y_2} Z_2 = Z_2 V,$$

where we let

$$V = \frac{Y_1}{Y_2} = \frac{1}{U}.$$

We deduce that

$$Z_2^3 V^3 = Z_2^2,$$

and this defines two components. The first component corresponds to $Z_2 = 0$, in which case, we get the other affine part of the fibre $\pi^{-1}(p)$. The second component corresponds to

$$Z_2 V^3 = 1.$$

This is a nonsingular curve. This time, we note that nothing lies over p, since $Z_2 = 0$ does not satisfy the equation $Z_2 V^3 = 1$. There is only one point over p in $B_p(X)$, and it lies on the part of the locus when $Y_1 \neq 0$.

Example 2.13 Consider the nodal cubic, X, given by

$$Z_2^2 = Z_1^2(Z_1 + 1).$$

and blow up the singular point $p = (0, 0)$. The lowest degree term is $Z_2^2 - Z_1^2$, and we expect the projectivized tangent cone to be $\mathrm{Proj}(k[Z_1, Z_2]/(Z_2^2 - Z_1^2))$. It consists of the two points $(1\colon 1)$ and $(1\colon -1)$. We know that $\pi^{-1}(X)$ is the variety given by

$$Z_2^2 = Z_1^2(Z_1 + 1) \quad \text{and} \quad Z_2 Y_1 = Z_1 Y_2.$$

Consider the two affine patches $Y_1 \neq 0$ and $Y_2 \neq 0$.

Case 1: $Y_1 \neq 0$. From the equations, we get

$$Z_2 = \frac{Y_2}{Y_1} Z_1 = Z_1 U,$$

where we let

$$U = \frac{Y_2}{Y_1}.$$

Note that introducing this new coordinate, U, has the effect that $U = \frac{Z_2}{Z_1}$, and so, $U^2 = Z_1 + 1$, which means that U is integral over $\mathcal{O}_{X,p}$. We deduce that

$$Z_1^2 U^2 = Z_1^2 (Z_1 + 1),$$

and this defines two components. The first component corresponds to $Z_1 = 0$, in which case, we get the affine part of the fibre $\pi^{-1}(p)$. The second component corresponds to

$$U^2 = Z_1 + 1.$$

This is a nonsingular curve. Above p (i.e., when $Z_1 = 0$), we get the two points $U = \pm 1$.

Case 2: $Y_2 \neq 0$. From the equations, we get

$$Z_1 = \frac{Y_1}{Y_2} Z_2 = Z_2 V,$$

where we let

$$V = \frac{Y_1}{Y_2} = \frac{1}{U}.$$

We deduce that

$$Z_2^2 = Z_2^2 V^2 (Z_2 V + 1),$$

and this defines two components. The first component corresponds to $Z_2 = 0$, in which case, we get the other affine part of the fibre $\pi^{-1}(p)$. The second component corresponds to

$$1 = V^2 (Z_2 V + 1).$$

This is a nonsingular curve. Above p (i.e., when $Z_2 = 0$), we get the two points $V = \pm 1$.

In the general case, how about the second projection $\sigma \colon B_p(\mathbb{A}^n) \to \mathbb{P}^{n-1}$? What are the fibres?

Pick $(y_1 \colon \cdots \colon y_n) \in \mathbb{P}^{n-1}$. We have

$$\sigma^{-1}(y_1 \colon \cdots \colon y_n) = \{((Z_1, \ldots, Z_n); (y_1 \colon \cdots \colon y_n)) \mid Z_i y_j = Z_j y_i,\ 1 \leq i, j \leq n\},$$

with the y_js fixed. This is a line through the origin, in fact, the line describing $(y_1 \colon \cdots \colon y_n)$ as a point of the dual space, $(\mathbb{P}^{n-1})^D$, corresponding to the lines through (0) in $\mathbb{A}^n$. This line is the *tautological line* of the point $(y_1\colon \cdots \colon y_n)$. Thus, $B_p(\mathbb{A}^n)$ is a *line family* over $\mathbb{P}^{n-1}$ (in fact, a line bundle see Section 5.1).

What about sections of $B_p(\mathbb{A}^n)$?

We say that a function $s\colon \mathbb{P}^{n-1} \to B_p(\mathbb{A}^n)$ is a *holomorphic section* if s is a morphism and $\sigma \circ s = \mathrm{id}$. This means that $s(q) \in \sigma^{-1}(q)$, the line over q.

Proposition 2.54 *Every section, s, of $B_p(\mathbb{A}^n)$ over $\mathbb{P}^{n-1}$ is the trivial section, that is, we have $s(q) = 0$ in the line $\sigma^{-1}(q)$.*

Proof. Consider the composed morphism

$$\mathbb{P}^{n-1} \xrightarrow{s} B_p(\mathbb{A}^n) \xrightarrow{\pi} \mathbb{A}^n.$$

Then, the image is reduced to a single point of $\mathbb{A}^n$ (see Corollary 2.38). We have $s(y) \in l_y = \sigma^{-1}(y)$, and we know that $\pi \circ s(y) = \pi \circ s(z)$ for all z, since the image of the composed map is a single point. The points $s(y)$ each being in the line l_y lying over y in $\mathbb{P}^{n-1}$ go to a single point in $\mathbb{A}^n$, which lies on the image of every line l_y considered as a line of $\mathbb{A}^n$. Yet,

$$\bigcap_{y \in \mathbb{P}^{n-1}} l_y = (0),$$

which implies that $s(y) = 0$ in each line l_y. $\square$

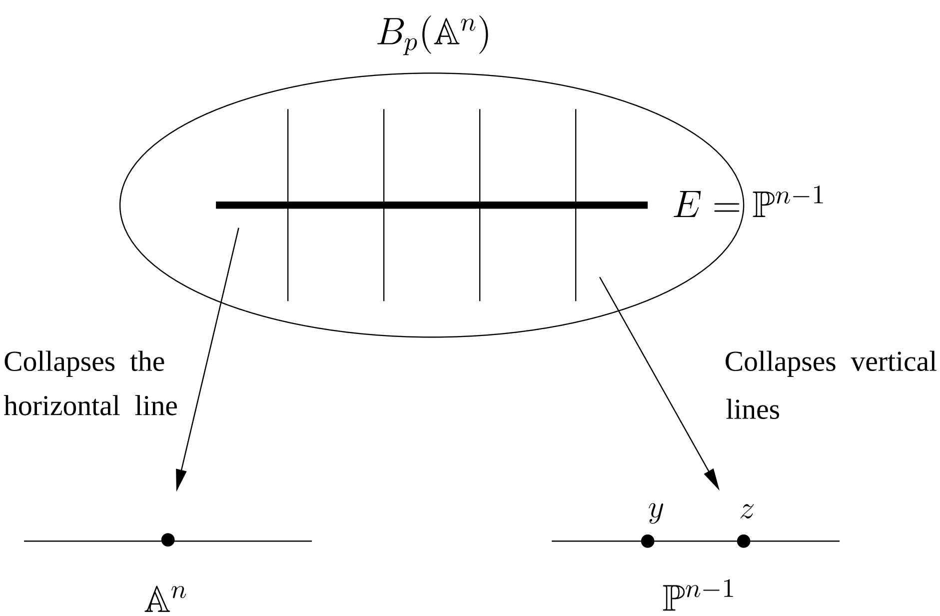

Figure 2.7: The line bundle $B_p(\mathbb{A}^n)$

Proposition 2.54, in contradiction to the simplified picture shown in Figure 2.7, shows that $B_p(\mathbb{A}^n)$ is a twisted bundle (as we will see later, it is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$).

Let's now look at the exceptional divisor $E = \pi^{-1}(p)$ in $B_p(\mathbb{A}^n)$. We know that E has codimension 1 (in fact, it is isomorphic to $\mathbb{P}^{n-1}$). If π_X is the projection from $B_p(X)$ to X, then $\pi_X^{-1}(p) = E \cap B_p(X)$. Also, $E \not\subseteq B_p(X)$, unless X is an open and $p \in X$.

Consider E as a subvariety of $B_p(\mathbb{A}^n)$. Look at the point $((0, \ldots, 0); (y_1 \colon \cdots : y_n)) \in E \subseteq B_p(\mathbb{A}^n)$, where we view E as $\mathbb{P}^{n-1}$. We have $y_l \neq 0$ for some l, with $1 \leq l \leq n$, and near E on $B_p(\mathbb{A}^n)$ we have the equations

$$Z_i y_l = Z_l y_i, \quad \text{for all} \quad i = 1, \ldots, n.$$

Locally on E, the equation $y_l = 1$ gives affine coordinates, and locally on $B_p(\mathbb{A}^n)$, over the open where $y_l \neq 0$, we have

$$Z_i = Z_l y_i.$$

The equation $Z_l = 0$ gives E locally on $B_p(\mathbb{A}^n)$. Therefore, E indeed has codimension 1 and is given locally by one equation. Consequently, E is what is called a Cartier divisor on $B_p(\mathbb{A}^n)$ (see Section 5.1). Hence, the exceptional divisor $B_p(X) \cap E$ of $B_p(X)$ is also a Cartier divisor on $B_p(X)$ (see Section 5.1).

Obviously one wants to blow-up more than one point in a given variety. With our current set-up, this is somewhat tricky to define rigorously. However, we can use geometric intuition to at least describe what goes on. So, consider a subvariety Y in X, where Y is nonsingular in X. Let T_Y be the union of all tangent spaces to Y, called the *tangent bundle of* Y, and similarly for X. We have the exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_X \restriction Y \longrightarrow \mathcal{N}_{Y \hookrightarrow X} \longrightarrow 0,$$

where $\mathcal{N}_{Y \hookrightarrow X}$ is a vector space family, whose fibres have dimension $\mathrm{codim}(Y \hookrightarrow X)$ (for all of this, see Section 5.1). The vector space family $\mathcal{N}_{Y \hookrightarrow X}$ is actually a vector bundle called the *normal bundle of* Y *in* X. We want to blow-up Y as a subvariety of X. Since Y is nonsingular in X, we know that locally, Y is given by d equations in the variables of X, where d is the codimension of Y in X. These equations give us a coordinate system locally on X in which we can define a sort of complement, Z, to Y at X (in the C^∞-case, this is just the implicit function theorem), and Y intersects this complement Z in a point, p. Blow up $p = Y \cap Z$ on Y. At p, we sew in $\mathbb{P}((\mathcal{N}_{Y \hookrightarrow X})_p)$, the projectivized fibre of the normal bundle at p. If we do this for all p, we get the *projectivized normal bundle of* Y *in* X as exceptional divisor. All this has been very intuitive, relying on the reader's intuition in the C^∞-case with the norm topology. Later on, we shall make everything precise in a manner which is correct and agrees with our intuition.

Question: For an affine variety Y, does the blow up of p on Y depend on the embedding of Y in $\mathbb{A}^n$? The answer is no.

In the general case, where Y is a subvariety of X, let $\mathfrak{A}$ be the ideal sheaf defining Y in X. Make the graded ring

$$\mathrm{Pow}(\mathfrak{A}) = \coprod_{n\geq 0} \mathfrak{A}^n t^n.$$

Then, make $\mathrm{Proj}(\mathrm{Pow}(\mathfrak{A}))$. By definition, this is the blow-up, $B_{\mathfrak{A}}(X)$, of X, along $\mathfrak{A}$ (which we have called the blow-up of Y in X as above). We offer no proof at this stage that the fancy definition, $\mathrm{Proj}(\mathrm{Pow}(\mathfrak{A}))$, agrees with previous notions of the blow-up or with our C^∞-intuition.

2.9 Proof of The Comparison Theorem

Recall that in Section 2.3, we gave a theorem comparing the norm topology and the Zariski topology. There, we reduced the general case to the case of a projective variety; now, we must prove this theorem in the projective case. In order to prove the projective comparison theorem, we will need a refined version of Noether's normalization.

Theorem 2.55 *Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety of dimension r, let L be a linear subspace of dimension $n-r-1$ so that $L\cap X = \emptyset$, and let p_L be the projection with center L. For any $\xi \in X$, there is some linear subspace M of L of dimension $n-r-2$, so that the following properties hold:*

(1) If $\pi = p_M \restriction X$, then

$$(\pi)^{-1}(\pi(\xi)) = \{\xi\}.$$

(2) p_L factors as

$$p_L = p_x \circ \pi$$

according to the following commutative diagram, for any $x \notin p_M(X)$:

$$\begin{tikzcd} & \mathbb{P}^{r+1} - \{x\} \arrow[r, "p_x"] & \mathbb{P}^r \\ X \arrow[r, "\pi"'] & p_M(X) \arrow[u] \arrow[ur, "p_x"'] & \end{tikzcd}$$

Proof. We have

$$p_L(\xi) = L(\xi) \cap \mathbb{P}^r,$$

where $L(\xi)$ is the join of L and ξ. Given y, we have

$$p_L(y) = p_L(\xi) \quad \text{iff} \quad y \in L(\xi).$$

Thus, $y \in p_L^{-1}(p_L(\xi))$ iff $y \in L(\xi)$. By the standard version of Noether's normalization, $L(\xi)\cap X$ is a finite set containing ξ, i.e.,

$$L(\xi) \cap X = \{\xi, \eta_1, \ldots, \eta_t\}.$$

Let $L^0(\xi)$ be a hyperplane in $L(\xi)$ so that $\xi \in L^0(\xi)$ but $\eta_j \notin L^0(\xi)$ for $j = 1, \ldots, t$. Write $M = L^0(\xi) \cap L$. Then, M is a hyperplane in L, since $\xi \notin L$ (recall that $L \cap X = \emptyset$ and $\xi \in X$). Observe that

$$M(\xi) = L^0(\xi).$$

For any $y \in X$, we have

$$\pi(y) = \pi(\xi) \quad \text{iff} \quad y \in M(\xi) \cap X \quad \text{iff} \quad y \in L^0(\xi) \cap X.$$

But $L^0(\xi) \cap X = \{\xi\}$, by construction of $L^0(\xi)$. Thus, $y \in (\pi)^{-1}(\pi(\xi))$ iff $y = \xi$, proving (1).

To prove (2) is now very easy. Take x so that $x \notin p_M(X)$ and $M(x) = L$. The rest is clear. $\square$

Theorem 2.56 *(Projective comparison theorem) If X is a projective variety over $\mathbb{C}$ and X_0 is Zariski-open and Zariski-dense in X, then X_0 is $\mathbb{C}$-open and $\mathbb{C}$-dense in X.*

Proof. (Mumford and Stolzenberg) We may assume (using the usual type of argument) that X is irreducible. If so, X_0 is automatically Z-dense. Pick $\xi \in X - X_0$. We'll show that ξ is the limit in the norm topology of a sequence of points in X_0. Now, $\dim(X) = r$, and we can pick M and L as in the refined version of Noether's normalization theorem with respect to ξ (Theorem 2.55). We also choose $x \notin p_M(X)$. We may choose coordinates so that

(1) M is cut out by $X_0 = \cdots = X_{r+1} = 0$.

(2) $\xi = (1\colon 0\colon \cdots\colon 0)$.

(3) L is cut out by $X_0 = \cdots = X_r$, and

$$x = (\underbrace{0\colon \cdots\colon 0}_{r+1}\colon 1).$$

Look at $p_L(X - X_0) \subseteq \mathbb{P}^r$. The image is closed, and thus, contained in some hypersurface $f = 0$, for some homogeneous polynomial, $f(X_0, \ldots, X_r)$. Therefore,

$$\{x \in X \mid f(p_L(x)) \neq 0\} \subseteq X_0,$$

and we may replace X_0 by the above open set. By (2) of Theorem 2.55, $p_M(X)$ has dimension r, and $p_M(X) \subseteq \mathbb{P}^{r+1}$, which implies that $p_M(X)$ is a hypersurface. Thus,

$$p_M(X) = \{y = (y_0\colon \cdots\colon y_{r+1}) \mid F(y) = 0\},$$

for some homogeneous form, $F(Y_0, \ldots, Y_{r+1})$ (of degree d). The rest of the argument has three stages:

Stage 1: Approximating in $\mathbb{P}^r$. Since $f \neq 0$, there is some nontrivial $(\alpha_0, \ldots, \alpha_r) \in \mathbb{C}^{r+1}$ such that $f(\alpha_0, \ldots, \alpha_r) = 0$ (because $\mathbb{C}$ is algebraically closed). Let

$$\xi_0 = p_L(\xi) \in \mathbb{P}^r.$$

By choice, $\xi_0 = (1\colon 0\colon \cdots : 0) \in \mathbb{P}^r$. Look at points

$$\xi_0 + t\alpha = (1 + t\alpha_0, t\alpha_1, \ldots, t\alpha_r).$$

Then, $f(\xi_0 + t\alpha) = f(1 + t\alpha_0, t\alpha_1, \ldots, t\alpha_r)$ is a polynomial in t. However, a polynomial in one variable has finitely many zeros. Thus, there exists a sequence $(t_i)_{i=1}^{\infty}$ so that

(1) $f(\xi_0 + t_i\alpha) \neq 0$.

(2) $t_i \to 0$ as $i \to \infty$.

(3) $\xi_0 + t_i\alpha \to \xi_0$ as $i \to \infty$.

Stage 2: Approximating in $\mathbb{P}^{r+1}$. We know that $p_M(X)$ is the hypersurface given by $F(X_0, \ldots, X_{r+1}) = 0$, and $x = (0\colon \cdots : 0\colon 1)$. Write F as

$$F(X_0, \ldots, X_{r+1}) = \gamma X_{r+1}^d + a_1(X_0, \ldots, X_r)X_{r+1}^{d-1} + \cdots + a_d(X_0, \ldots, X_r). \tag{$*$}$$

Claim. There exists a sequence (b_i) so that

(1) $b_i \in p_M(X)$.

(2) $b_i \to \xi_0 + t_i\alpha$ (under p_x).

(3) $\lim_{i\to\infty} b_i = (1\colon 0\colon \cdots : 0) = p_M(\xi)$.

In order to satisfy (2), the b_i must be of the form

$$b_i = (1 + t_i\alpha_0\colon t_i\alpha_1\colon \cdots : t_i\alpha_r\colon \beta^{(i)}),$$

for some $\beta^{(i)}$ yet to be determined. We also need to satisfy (1); that is, we must have

$$F(1 + t_i\alpha_0\colon t_i\alpha_1\colon \cdots : t_i\alpha_r\colon \beta^{(i)}) = 0.$$

We know that $x \notin p_M(X)$, which implies that $F(x) \neq 0$, and since $x = (0\colon \cdots : 0\colon 1)$, by $(*)$, we must have $\gamma \neq 0$. The fact that $p_M(\xi) \in p_M(X)$ implies that $F(p_M(\xi)) = 0$. Since $p_M(\xi) = (1\colon 0\colon \cdots : 0)$, from $(*)$, we get $a_d(\xi_0) = F(p_M(\xi)) = 0$. Also, by $(*)$, $\beta^{(i)}$ must be a root of

$$\gamma Y^d + a_1(\xi_0 + t_i\alpha)Y^{d-1} + \cdots + a_d(\xi_0 + t_i\alpha) = 0. \tag{$**$}$$

Thus, we get (2). To get (3), we need $\beta^{(i)} \to 0$ when $i \to \infty$. Now, as $i \to \infty$, $t_i \to 0$; but the product of the roots in $(**)$ is

$$\pm\frac{a_d(\xi_0 + t_i\alpha)}{\gamma},$$

and this term tends to 0 as i tends to infinity. Then, some root must tend to 0, and we can pick $\beta^{(i)}$ in such a manner, so that $\lim_{i\to\infty} \beta^{(i)} = 0$. Thus, we get our claim.

Stage 3: Lifting back to $\mathbb{P}^n$. Lift each b_i in any arbitary manner to some $\eta_i \in X \subseteq \mathbb{P}^n$. We know that $\mathbb{P}^n$ is compact, since $\mathbb{C}$ is locally compact. Thus, the sequence (η_i) has a convergent subsequence. By restriction to this subsequence, we may assume that (η_i) converges, and we let η be the limit. Now, $\eta_i \in X$ and X is closed, so that $\eta \in X$. We have

$$p_M(\eta) = \lim_{i\to\infty} p_M(\eta_i) = \lim_{i\to\infty} b_i = p_M(\xi),$$

since p_M is continuous. Therefore,

$$\eta \in p_M^{-1}(p_M(\xi)) = \{\xi\},$$

and thus, $\eta = \xi$. Now,

$$f(p_L(\eta_i)) = f(p_x(p_M(\eta_i))) = f(p_x(b_i)) = f(\xi_0 + t_i\alpha) \neq 0,$$

and thus, $\eta_i \in X_0$. This proves that X_0 is norm-dense. $\square$

2.10 Further Readings

Other presentations of the material of this chapter (some more complete, some less) can be found in the references listed below: Shafarevich [53], Chapter I and II, and Dieudonné [13], Chapter 1–6, are the closest in spirit; Hartshorne [33], Chapter 1; Mumford [43], Chapter 1 and 2, Mumford [42], Chapter 1, Fulton [17], Chapter 1, 2, 4, 6; Perrin [45], Chapter 1, 2, 4, 5; Kempf [36], Chapter 1, 2, 3, 6; Harris [31]. An excellent tutorial on algebraic geometry can also be found in Danilov's article in [11], and Volume I of Ueno [56] is worth consulting. Although it is devoted to algebraic geometry over the complex field, Griffiths and Harris [20] must be cited as a major reference in algebraic geometry. For a treatment of algebraic curves, one may consult Griffith [19], Kendig [37], Miranda [41], Narasimham [44], Clemens [9], and Walker [58]. As to general background in commutative algebra, we primarily recommend Zariski and Samuel [60, 61], Atiyah and Macdonald [2], Kunz [38], and Peskine [46]. Other useful sources include Eisenbud [14], Bourbaki [7] (Commutative algebra), and Matsumura [40].

Chapter 3

Affine Schemes and Schemes in General

In this chapter and succeeding chapters, we shall make heavy use of the material on sheaves and cohomology which is placed in the appendices for the convenience of the reader. Occasionally, we shall make a direct reference to material in the appendices.

In the development of algebraic geometry, from a historical perspective, we can see several distinct periods. Of course, if one begins with "antiquity," there is all the material in analytic geometry in the sense of Descartes and his followers. But, the period which began essentially with Riemann and ended roughly at the beginning of the twientieth century, was the first where algebraic geometry *per se* was studied, albeit purely from a complex analytic viewpoint and with function-theoretic tools. However, we should mention the algebraic work of the german school of Halphen and Noether (together with the contributions of Hilbert) in the last twenty years of the nineteenth century. Though this period ended around 1900 its spiritual heirs are very active in the wonderful development of complex geometry and complex analysis up to the present time.

The next period was dominated by the use of direct geometric intuitions, geometric language and the introduction of topological ideas into algebraic geometry–principally at the hand of the three great Italian geometers: Castelnuovo, Enriques and Severi (and the early Zariski), together with Lefschetz and some others on the topological side. These methods, though directly geometrically appealing, sometimes led to overlooking of certain important (though degenerate) phenomena and consequently were prone to error in the hands of less gifted practitionners than those mentionned above. Also, they were totally inadequate to reveal the growing number-theoteric connections of algebraic geometry.

A new period was initiated principally by Zariski and Weil with important contributions by Van der Warden. One may take the 1930's as the beginning of this, newest, period. It was characterized by heavy use and development of commutative algebraic machinery, it revealed the connections with number theory, but unfortunately in its language of not everywhere defined maps, it seemed to be a place apart from the rest of burgeonning geometric mathematics. The first two chapters of these notes have given an exposition of some of the material of this period, smoothed out by the introduction of everywhere defined maps and guided by the material yet to come.

The rise of topology and in particular algebraic topology and the renaissance in complex analysis, came together in the 1950's in the work of Serre with the use of topology and the sheaf-theoretic language for algebraic geometry. This work was taken up almost immediately by Grothendieck and his followers who revamped, deepened and even revolutionized algebraic geometry. It is this last period–the modern period–to which we turn in the rest of these notes.

From now on, all rings are assumed to be commutative with unit element (1), and ring homomorphisms preserve unit elements.

3.1 Definition of Affine Schemes: First Properties

We have the category, LRS, of *local ringed spaces* $(X, \mathcal{O}_X)$, where

1. X is a topological space.

2. $\mathcal{O}_X$ is a sheaf of rings.

3. $\mathcal{O}_{X,x}$ (the stalk of the sheaf $\mathcal{O}_X$ at $x \in X$) is a local ring for every $x \in X$.

The morphisms of local ringed spaces are pairs $(\varphi, \varphi^{\mathrm{alg}})$, where $\varphi\colon X \to Y$ is a continuous map and $\varphi^{\mathrm{alg}}\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is a map of sheaves of rings over Y.

Recall that given a sheaf $\mathcal{F}$ on X, the sheaf $\varphi_*\mathcal{F}$, called the *direct image of* $\mathcal{F}$ *by* φ, is the sheaf on Y defined by

$$\varphi_*\mathcal{F}(V) = \mathcal{F}(\varphi^{-1}(V))$$

for every open subset V of Y. Also, given a sheaf $\mathcal{G}$ on Y, we define the presheaf $\varphi_P\mathcal{G}$ on X by

$$\varphi_P\mathcal{G}(U) = \varinjlim_{V \supseteq \varphi(U)} \mathcal{G}(V),$$

where V ranges over open subsets of Y. In general, this is not a sheaf, and we define the sheaf $\varphi^*\mathcal{G}$ on X, called the *inverse image of* $\mathcal{G}$ *by* φ, as the sheaf, $(\varphi_P\mathcal{G})^\sharp$, associated with $\varphi_P\mathcal{G}$ (in the terminology of Hartshorne [33], the *sheafification* of $\varphi_P\mathcal{G}$).

Beware that Hartshorne uses the notation $\varphi^*\mathcal{G}$ for something *different* from what has just been defined here! His notation is $\varphi^{-1}\mathcal{G}$ for the above, and his $\varphi^*\mathcal{G}$ will be considered shortly.

From Appendix A, we know that the functors φ_* and φ^* are adjoint, which means that there is a natural (canonical) isomorphism

$$\theta(\mathcal{F}, \mathcal{G})\colon \mathrm{Hom}_{\mathcal{S}(X)}(\varphi^*\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{S}(Y)}(\mathcal{G}, \varphi_*\mathcal{F}),$$

for all $\mathcal{F} \in \mathcal{S}(X)$ and all $\mathcal{G} \in \mathcal{S}(Y)$, where $\mathcal{S}(X)$ denotes the category of sheaves on X (taking values in some given category).

Thus, having a map $\varphi^{\mathrm{alg}} \colon \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ of sheaves of rings over Y is equivalent to having a map $\varphi^{\mathrm{alg}} \colon \varphi^* \mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings over X.

Because we are considering sheaves of rings whose stalks are local rings, in order for some of our results to hold, we must demand that our morphisms be local morphisms of local rings. This means the following: For every $x \in X$, the map φ^{alg} induces a ring morphism on stalks

$$\varphi_x^{\mathrm{alg}} \colon \mathcal{O}_{Y,\varphi(x)} \longrightarrow \mathcal{O}_{X,x},$$

and we demand that

$$\varphi_x^{\mathrm{alg}} \colon \mathfrak{m}_{Y,\varphi(x)} \longrightarrow \mathfrak{m}_{X,x},$$

where, as usual, $\mathfrak{m}$ denotes the maximal ideal of the local ring in question.

We can define the *values* of a section (as some kind of numerical valued function) as follows: For every $\sigma \in \Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$, the "value" of σ at x is

$$\overline{\sigma(x)} \in \kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}.$$

Then, we have a better idea of what the notion of local homomorphism means. If U is an open subset around $\varphi(x)$ and a section σ has zero *values* on U, we want the section $\varphi^{\mathrm{alg}}(\sigma)$ on $\varphi^{-1}(U)$ to have zero values, too.

Now, given a (commutative) ring A, we would like to make a local ringed space $\widetilde{A}$, from A. We proceed as follows:

The topological space X associated with the ring A is the set

$$X = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } A\},$$

with the *Zariski topology* (also called the *spectral topology*), in which a closed set in X is a set of prime ideals of the form

$$V(\mathfrak{A}) = \{\mathfrak{p} \in X \mid \mathfrak{A} \subseteq \mathfrak{p}\},$$

where $\mathfrak{A}$ is any ideal of A.

The topological space X associated with the ring A is not Hausdorff, but it satisfies a weaker separation property, the T_0-separation property. A topological space X has the T_0-*separation property* (or is a T_0-*space*, or a *Kolmogoroff space*), if for any two distinct points $x, y \in X$, there is some open subset $U \subseteq X$ so that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$. The following proposition will be needed later in Serre's characterization of affine schemes.

Proposition 3.1 *If X is a quasi-compact T_0-space, then every nonempty closed subset, F, of X contains some closed point (i.e., a point $x \in X$ so that $\{x\} = \overline{\{x\}}$).*

Proof. First, we claim that since X is quasi-compact, it has some minimal nonempty closed subset. Dually, consider the family of proper open subsets U of X (i.e., those open subsets U so that $U \neq X$), we claim that it is inductive. Indeed, if (U_α) is any chain of proper open sets under inclusion, its union $\bigcup_\alpha U_\alpha$ is also open. Were

$$\bigcup_\alpha U_\alpha = X,$$

then, by quasi-compactness, there would be a finite subfamily $(U_{\alpha_0}, \ldots, U_{\alpha_r})$ so that

$$X = U_{\alpha_0} \cup \cdots \cup U_{\alpha_r}.$$

However, (U_α) is a chain, so there is some α_j with

$$X = U_{\alpha_0} \cup \cdots \cup U_{\alpha_r} = U_{\alpha_j},$$

which is a clear contradiction. Hence, by Zorn's lemma, there is a maximal proper open subset U of X, and its complement, U^c, is a minimal, closed, nonempty subset of X.

Apply this property to the closed subset, F, of X. We find the required nonempty minimal closed subset, F_0, in F and we prove that F_0 is reduced to a point. If not, there are at least two distinct points $x, y \in F_0$, and by the T_0-separation axiom, there is some open subset, V, so that one of x or y is in V and the other is excluded from V. Then, $F_1 = V^c \cap F$ is a smaller closed subset of F which is nonempty, contradicting the minimality of F_0. $\square$

The reader should check that:

$$\begin{aligned}
V\left(\sum_i \mathfrak{A}_i\right) &= \bigcap_i V(\mathfrak{A}_i) \\
V(\mathfrak{A}\mathfrak{B}) &= V(\mathfrak{A} \cap \mathfrak{B}) = V(\mathfrak{A}) \cup V(\mathfrak{B}) \\
V(\mathfrak{A}) \subseteq V(\mathfrak{B}) &\text{ iff } \sqrt{\mathfrak{B}} \subseteq \sqrt{\mathfrak{A}},
\end{aligned}$$

where

$$\sqrt{\mathfrak{A}} = \bigcap \{\mathfrak{p} \in X \mid \mathfrak{A} \subseteq \mathfrak{p}\}.$$

An open base for the Zariski topology is the family of open sets

$$X_f = (V((f))^c = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

We need a sheaf, $\mathcal{O}_X$, and it is defined as follows: For every open subset U in X,

$$\Gamma(U, \mathcal{O}_X) = \left\{ \sigma \colon U \longrightarrow \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{l} (1)\ \sigma(\mathfrak{p}) \in A_{\mathfrak{p}} \\ (2)\ (\forall \mathfrak{p} \in U)(\exists f, g \in A)(g \notin \mathfrak{p},\ \text{i.e., } \mathfrak{p} \in X_g) \\ (3)\ (\forall \mathfrak{q} \in X_g \cap U)\left(\sigma(\mathfrak{q}) = \text{image}\left(\frac{f}{g}\right) \text{ in } A_{\mathfrak{q}}\right). \end{array} \right\}$$

One can check that $\mathcal{O}_X$ is indeed a sheaf. Sometimes, $\mathcal{O}_X$ is denoted $\widetilde{A}$ to render clear its provenance from A. We can even do the same with an A-module, M; that is, we make an $\mathcal{O}_X$-module, $\widetilde{M}$, as follows: For every open subset U in X,

$$\Gamma(U, \widetilde{M}) = \left\{ \sigma\colon U \longrightarrow \bigcup_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} (1)\ \sigma(\mathfrak{p}) \in M_{\mathfrak{p}} \\ (2)\ (\forall \mathfrak{p} \in U)(\exists m \in M, \exists g \in A)(g \notin \mathfrak{p}, \text{ i.e., } \mathfrak{p} \in X_g) \\ (3)\ (\forall \mathfrak{q} \in X_g \cap U) \left(\sigma(\mathfrak{q}) = \text{image}\left(\frac{m}{g}\right) \text{ in } M_{\mathfrak{q}}\right). \end{array} \right\}$$

Again, one can check that $\widetilde{M}$ is indeed a sheaf. Also, $\Gamma(U, \widetilde{M})$ is a $\Gamma(U, \widetilde{A})$-module for every open U, and $\widetilde{M}$ is an $\mathcal{O}_X$-module.

The ringed space $(X, \mathcal{O}_X) = (X, \widetilde{A})$ is denoted $\operatorname{Spec} A$. We also denote the underlying space, X, by $|\operatorname{Spec} A|$.

Theorem 3.2 *The ringed space* $\operatorname{Spec} A$ *is an LRS. In fact, there is a canonical isomorphism*

$$\theta\colon (\widetilde{A})_{\mathfrak{p}} \longleftrightarrow A_{\mathfrak{p}},$$

where $\mathfrak{p}$ *is a prime ideal. The map*

$$A \to \operatorname{Spec} A$$

is a cofunctor and establishes a full anti-embedding of commutative rings into the category LRS. Moreover, for every $f \in A$*, there is a canonical isomorphism*

$$M_f \cong \Gamma(X_f, \widetilde{M}), \quad \textit{for every } A\textit{-module } M,$$

where $(X, \mathcal{O}_X) = \operatorname{Spec} A$*. In particular, when* $f = 1$*, we get the isomorphisms*

$$M \cong \Gamma(X, \widetilde{M}) \quad \textit{and} \quad A \cong \Gamma(X, \mathcal{O}_X).$$

Proof. Let $\mathfrak{p} \in |\operatorname{Spec} A|$ (a prime ideal of A), and let M be an A-module. Any $\xi \in (\widetilde{M})_{\mathfrak{p}}$ (where $(\widetilde{M})_{\mathfrak{p}}$ is the stalk of $\widetilde{M}$ at $\mathfrak{p}$) is represented by some pair (σ, U), where σ is some local section $\sigma \in \Gamma(U, \widetilde{M})$. Define

$$\theta\colon (\widetilde{M})_{\mathfrak{p}} \to M_{\mathfrak{p}}$$

by

$$\theta(\xi) = \sigma(\mathfrak{p}) \in M_{\mathfrak{p}}.$$

If (τ, V) and (σ, U) are equivalent sections representing ξ, then as $\mathfrak{p}$ belongs to $U \cap V$, we can take any smaller open subset of $U \cap V$ containing $\mathfrak{p}$ and we find that

$$\sigma(\mathfrak{p}) = \tau(\mathfrak{p}),$$

so that θ is well–defined.

Given $m/g \in M_{\mathfrak{p}}$ (where $g \notin \mathfrak{p}$), look at m/g in M_g. This defines a local section, σ, on X_g near $\mathfrak{p}$. Thus, the map is onto.

That θ is injective is seen as follows: Assume that $\sigma(\mathfrak{p}) = 0$. In a smaller open set containing $\mathfrak{p}$, the section σ looks like m/f, where $m \in M$ and $f \notin \mathfrak{p}$. Since we are assuming that $\sigma(\mathfrak{p}) = 0$, we have $m/f = 0$ in $M_{\mathfrak{p}}$. Thus, there is some $h \notin \mathfrak{p}$ so that $hm = 0$. We may assume that $X_h \subseteq X_f$, and thus, on X_h, h is invertible, and thus $\sigma \restriction X_h = 0$. Therefore, the element ξ represented by σ is zero.

Observe that we actually proved the following fact: If $\mathcal{F}$ is a sheaf of the form $\widetilde{M}$ for some A-module M and σ is any local section of $\mathcal{F}$ (i.e., $\sigma \in \Gamma(U, \mathcal{F})$ for some open U), if $\sigma(\mathfrak{p}) = 0$, then there is an open subset $V \subseteq U$ with $\mathfrak{p} \in V$ such that $\sigma = 0$ on V.

Next, we prove that $\Gamma(X_f, \widetilde{M}) \cong M_f$. Define a map from M_f to $\Gamma(X_f, \widetilde{M})$ as follows:

$$\frac{m}{f^r} \mapsto \left(\mathfrak{p} \mapsto \iota_{\mathfrak{p}}\left(\frac{m}{f^r}\right)\right), \quad \text{for all } \mathfrak{p} \in X_f.$$

(a) This map is injective. Assume that the section

$$\mathfrak{p} \mapsto \iota_{\mathfrak{p}}\left(\frac{m}{f^r}\right)$$

is zero. Thus,

$$\frac{m}{f^r} = 0 \quad \text{in } M_{\mathfrak{p}}, \text{ for all } \mathfrak{p} \in X_f.$$

Let

$$\mathfrak{A} = \mathrm{Ann}(m) = \{a \in A \mid am = 0\}$$

be the annihilator of m. Then, for $\mathfrak{p} \in X_f$, there is some $h \in A$ such that $h \notin \mathfrak{p}$ and $hm = 0$; thus, $h \in \mathfrak{A}$, and yet, $h \notin \mathfrak{p}$. This implies

$$\mathfrak{A} \not\subseteq \mathfrak{p},$$

and thus, $\mathfrak{p} \notin V(\mathfrak{A})$. So, for every $\mathfrak{p} \in X_f$, we have $\mathfrak{A} \not\subseteq \mathfrak{p}$. Consequently, we find that

$$X_f \cap V(\mathfrak{A}) = \emptyset,$$

which means

$$V(\mathfrak{A}) \subseteq V(f) = (X_f)^c.$$

But then, $f \in \sqrt{\mathfrak{A}}$; so, as $\mathfrak{A} = \mathrm{Ann}(m)$, we get

$$f^n m = 0, \quad \text{hence} \quad \frac{m}{f^r} = 0 \quad \text{in } M_f.$$

The map is injective.

(b) Surjectivity is a bit harder. Let $\sigma \in \Gamma(X_f, \widetilde{M})$. We can cover X_f by X_{g_i}'s so that the restriction, $\sigma \restriction X_{g_i}$, of σ to X_{g_i} is of the form m_i/g_i (in M_{g_i}).

Claim. Only finitely many g_i's are needed (this argument shows that X_f is quasi-compact). Observe that

$$X_f \subseteq \bigcup_i X_{g_i}.$$

Thus,

$$V(f) \supseteq V\left(\sum_i (g_i)\right),$$

and thus,

$$f \in \sqrt{\left(\sum_i (g_i)\right)}.$$

This means that $f^n \in \sum_i (g_i)$ for some $n \geq 1$, and thus, there exist some $g_{i_1}, \ldots, g_{i_t}$ so that

$$f^n = \alpha_1 g_{i_1} + \cdots + \alpha_t g_{i_t}, \quad \text{where } \alpha_j \in A.$$

Now, $X_{f^n} = X_f$, because $\mathfrak{p}$ is prime. However, $f^n \notin \mathfrak{p}$ implies that $g_{i_j} \notin \mathfrak{p}$ for some j $(1 \leq j \leq t)$ and thus, we must have $\mathfrak{p} \in X_{g_{i_j}}$ for some j as above, and this shows

$$X_f \subseteq X_{g_{i_1}} \cup \cdots \cup X_{g_{i_t}},$$

which proves our claim.

We may assume after renumbering that

$$X_f \subseteq X_{g_1} \cup \cdots \cup X_{g_t}.$$

Now, $\sigma = m_i/g_i$ on X_{g_i} and $\sigma = m_j/g_j$ on X_{g_j}, and thus

$$\left.\frac{m_i}{g_i}\right| X_{g_i g_j} = \left.\frac{m_j}{g_j}\right| X_{g_i g_j}.$$

By injectivity (part (a)), we must have

$$\frac{m_i}{g_i} = \frac{m_j}{g_j} \quad \text{in } M_{g_i g_j},$$

so, there is some $n_{ij} \geq 0$ with

$$(g_i g_j)^{n_{ij}} (g_j m_i - g_i m_j) = 0 \quad \text{in } M.$$

Since there exist finitely many X_{g_i}'s covering X_f, let $N = \max\{n_{ij}\}$; it follows that

$$(g_i g_j)^N (g_j m_i - g_i m_j) = 0, \quad \text{for all } i, j.$$

This can be written as

$$g_i^N g_j^{N+1} m_i = g_i^{N+1} g_j^N m_j, \quad \text{for all } i, j. \qquad (*)$$

However, we know that $X_{g_i} = X_{g_i^{N+1}}$, and these sets cover X_f. By the previous argument (proof of the claim), there is some $n \geq 1$ so that

$$f^n = \sum_{i=1}^{t} \beta_i g_i^{N+1}.$$

Let

$$m = \sum_{i=1}^{t} \beta_i g_i^N m_i.$$

By $(*)$, we get

$$g_j^{N+1} m = \sum_{i=1}^{t} \beta_i g_j^{N+1} g_i^N m_i = \left(\sum_{i=1}^{t} \beta_i g_i^{N+1}\right) g_j^N m_j = f^n g_j^N m_j.$$

If we restrict to $X_{g_i} \subseteq X_f$, we see, since g_i and f are invertible on X_{g_i}, that

$$\frac{m_j}{g_j} = \frac{m}{f^n} = \sigma \restriction X_{g_j}.$$

Thus, there is some $m/f^n \in M_f$ having σ as image, and this proves surjectivity.

If X is an LRS, we let $|X|$ denote the underlying topological space, and $\mathcal{O}_X$ denote the sheaf of rings. Assume that we have a map of rings $\theta\colon A \to B$.

(1) Define the topological map $|\theta|\colon |\mathrm{Spec}\, B| \to |\mathrm{Spec}\, A|$ by

$$|\theta|(\mathfrak{q}) = \theta^{-1}(\mathfrak{q}), \quad \text{for every } \mathfrak{q} \in |\mathrm{Spec}\, B|.$$

Let $V(\mathfrak{A}) \subseteq |\mathrm{Spec}\, A|$, then, $\mathfrak{q} \in |\theta|^{-1}(V(\mathfrak{A}))$ iff $|\theta|(\mathfrak{q}) \in V(\mathfrak{A})$ iff $\theta^{-1}(\mathfrak{q}) \supseteq \mathfrak{A}$ iff $\mathfrak{q} \supseteq \theta(\mathfrak{A})$ iff $\mathfrak{q} \supseteq B \cdot \theta(\mathfrak{A})$. Thus,

$$|\theta|^{-1}(V(\mathfrak{A})) = V(B \cdot \theta(\mathfrak{A})),$$

a closed set, and $|\theta|$ is continuous. (The reader should check It can be shown (DX) that

$$|\theta|^{-1}(|\mathrm{Spec}\, A|_f) = |\mathrm{Spec}\, B|_{\theta(f)}.)$$

(2) Let $Y = \mathrm{Spec} B$ and $X = \mathrm{Spec} A$. We need a map from $\mathcal{O}_X$ to $|\theta|_* \mathcal{O}_Y$, or, equivalently, from $|\theta|^* \mathcal{O}_X$ to $\mathcal{O}_Y$. Thus, for every open $U \subseteq |X|$, we need a map from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(U, |\theta|_* \mathcal{O}_Y)$. We may assume that $U = |X|_f$, where $f \in A$. Then, by definition,

$$\Gamma(U, |\theta|_* \mathcal{O}_Y) = \Gamma(|\theta|^{-1}(U), \mathcal{O}_Y),$$

but, we showed that

$$\Gamma(|X|_f, \mathcal{O}_X) = A_f,$$

and our remark above shows

$$\Gamma(|\theta|^{-1}(|X|_f), \mathcal{O}_Y) = \Gamma(|Y|_{\theta(f)}, \mathcal{O}_Y) = B_{\theta(f)}.$$

The ring map $\theta\colon A \to B$ clearly induces a map from A_f to $B_{\theta(f)}$.

[If one wishes to ue the inverse image of $\mathcal{O}_X$ by $|\theta|$, one sees that

$$(|\theta|^*\mathcal{O}_X)_{\mathfrak{q}} = (\mathcal{O}_X)_{|\theta|(\mathfrak{q})} = A_{|\theta|(\mathfrak{q})} = A_{\theta^{-1}(\mathfrak{q})}.$$

However, $\mathcal{O}_{Y,\mathfrak{q}} = B_{\mathfrak{q}}$, and $\theta\colon A \to B$ induces a map from $A_{\theta^{-1}(\mathfrak{q})}$ to $B_{\mathfrak{q}}$. Observe that this is a local homomorphism.]

The morphism just defined, namely, $(|\theta|, \theta^{\mathrm{a}})$, is clearly functorial; and so, $A \mapsto \mathrm{Spec}\, A$ is indeed a cofunctor.

Let $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$, and consider the ring of global sections, $\Gamma(|X|, \mathcal{O}_X)$, of $\mathcal{O}_X$. By the foregoing argument, $\Gamma(|X|, \mathcal{O}_X) = A$. If $\varphi\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a morphism, we have a sheaf morphism $\varphi^a\colon \mathcal{O}_X \to |\varphi|_*\mathcal{O}_Y$; so, given a section $\sigma \in \Gamma(|X|, \mathcal{O}_X) = A$, we get the composition $\varphi^a \circ \sigma$. This yields a map

$$\sigma \mapsto \varphi^a \circ \sigma$$

from $\Gamma(|X|, \mathcal{O}_X) = A$ to $\Gamma(|X|, |\varphi|_*\mathcal{O}_Y) = \Gamma(|Y|, \mathcal{O}_Y) = B$, and we call this map Φ. The map Φ commutes with taking stalks and restriction to opens, and so we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d] & B \arrow[d] \\ A_{|\varphi|(\mathfrak{q})} \arrow[r] & B_{\mathfrak{q}}. \end{tikzcd}$$

However, the morphism of local ringed spaces, φ, is a *local* local morphism and the diagram implies that $|\varphi|(\mathfrak{q}) = \Phi^{-1}(\mathfrak{q})$, showing our result. $\square$

Corollary 3.3 *If $\mathcal{F}$ is a sheaf on $X = \mathrm{Spec}\, A$ of the form $\widetilde{M}$ (where M is a module over A) and if $\sigma \in \Gamma(U, \mathcal{F})$ is a section of $\mathcal{F}$ over U and $u \in U$, then $\sigma(u) = 0$ iff there is a small open set $V = |X|_h$ such that $u \in V$ and $\sigma \restriction_V = 0$. In other words, the vanishing of a section (qua-section) is an open condition).*

Corollary 3.4 *The functor $M \mapsto \widetilde{M}$ is an exact and full embedding of the category of A-modules to the full subcategory of $\mathcal{O}_X$-modules ($X = \mathrm{Spec}\, A$) of the form $\widetilde{M}$. In particular,*

$$0 \longrightarrow \widetilde{M_1} \longrightarrow \widetilde{M_2} \longrightarrow \widetilde{M_3} \longrightarrow 0$$

is exact iff

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact.

Corollary 3.5 *For $X = \mathrm{Spec}\, A$ and A-modules, M and N, we have*

$$\widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

If $M = \varinjlim M_\lambda$, then $\widetilde{M} = \varinjlim \widetilde{M_\lambda}$. If M' and M'' are A-modules, then

$$\widetilde{M' \coprod M''} = \widetilde{M'} \coprod \widetilde{M''},$$

and if $M', M'' \subseteq M$, then

$$\widetilde{M' \cap M''} = \widetilde{M'} \cap \widetilde{M''}.$$

Proof. (DX)

Corollary 3.6 *Given two A-modules, M and N, we have*

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

If M is finitely presented, then

$$\widetilde{\mathrm{Hom}_A(M, N)} = \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}),$$

where $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ is the sheaf of modules defined by

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})) = \mathrm{Hom}_{\mathcal{O}_X \restriction U}(\widetilde{M} \restriction U, \widetilde{N} \restriction U).$$

We leave this to the reader but only remark that one applies the functor $\mathrm{Hom}_A(-, N)$ to the right-exact finite presentation sequence for M, and uses the five–lemma.

3.2 Quasi-Coherent Sheaves on Affine Schemes

Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. Given an $\mathcal{O}_X$-module, $\mathcal{F}$, assume that there is a sheaf morphism $v\colon \mathcal{O}_X^{(I)} \to \mathcal{F}$, where $\mathcal{O}_X^{(I)}$ is the coproduct sheaf defined as the sheaf associated with the presheaf

$$U \mapsto \Gamma(U, \mathcal{O}_X)^{(I)}, \quad \text{where } I \text{ is any index set.}$$

Note that to give a sheaf map $\rho\colon \mathcal{O}_X \to \mathcal{F}$ is equivalent to giving a global section $\sigma \in \Gamma(|X|, \mathcal{F})$. Indeed, assume that $\sigma \in \Gamma(|X|, \mathcal{F})$. For every open U in X, we define a map ρ_U from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(U, \mathcal{F})$ as follows: Given $a \in \Gamma(U, \mathcal{O}_X)$, let

$$\rho_U(a) = a \cdot (\sigma \restriction U).$$

Conversely, $v\colon \mathcal{O}_X \to \mathcal{F}$ determines the global section

$$\sigma = \rho_X(1) \in \Gamma(|X|, \mathcal{F}),$$

where 1 is the unit element of the ring $\Gamma(|X|, \mathcal{O}_X)$. Thus, we have a bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(|X|, \mathcal{F}).$$

More generally, sheaf maps $\rho\colon \mathcal{O}_X^{(I)} \to \mathcal{F}$ and families $(\sigma_i)_{i\in I}$ of global sections, $\sigma_i \in \Gamma(|X|, \mathcal{F})$, are in one–to–one correspondence, because there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F}) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}).$$

Definition 3.1 An $\mathcal{O}_X$-module, $\mathcal{F}$, is *generated by a family,* $(s_i)_{i\in I}$, of global sections if the map $s\colon \mathcal{O}_X^{(I)} \to \mathcal{F}$ induced by $(s_i)_{i\in I}$ is a surjective map of sheaves. The sheaf $\mathcal{F}$ is *generated by its sections* if $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F} \longrightarrow 0$ is exact for some index set I.

Definition 3.2 An $\mathcal{O}_X$-module, $\mathcal{F}$, is *quasi-coherent (QC)* if for every $x \in |X|$, there is some open subset, U, with $x \in U$, and some sets I, J so that

$$\mathcal{O}_X^{(I)} \restriction U \longrightarrow \mathcal{O}_X^{(J)} \restriction U \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.}$$

Definition 3.2 means that locally everywhere on $|X|$, the sheaf $\mathcal{F}$ is generated by its local sections, and the sheaf of relations among these generators is also generated by *its* sections. Generation by a family I is testable at each x. For, given a family of sections, $s = (s_i)_{i\in I}$, this family generates $\mathcal{F}$ iff for every $x \in |X|$,

$$\mathcal{O}_{X,x}^{(I)} \xrightarrow{s} \mathcal{F}_x \longrightarrow 0 \quad \text{is exact.}$$

Quasi-coherence is a local property, and surjectivity is testable stalkwise.

Definition 3.3 An $\mathcal{O}_X$-module, $\mathcal{F}$, is *finitely generated (fg)* if for every $x \in |X|$, there is some open subset, U, with $x \in U$, and some integer $p > 0$ so that

$$(\mathcal{O}_X \restriction U)^p \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.}$$

We also say that, $\mathcal{F}$, is *finitely presented (fp)* if for every $x \in |X|$, there is some open subset, U, with $x \in U$, and some integers $p, q > 0$ so that

$$(\mathcal{O}_X \restriction U)^q \longrightarrow (\mathcal{O}_X \restriction U)^p \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.}$$

Note that a finitely presented sheaf is quasi-coherent.

An example of a very bad sheaf is the following: Let $|X| = \mathbb{R}$, and let $\mathcal{O}_X$ be the constant sheaf (the sheaf of locally constant functions on $|X|$). The sheaf, $\mathcal{F}$, is defined by

$$\mathcal{F}(U) = \begin{cases} (0) & \text{if } U \text{ is connected and } 0 \in U \\ \mathbb{Z} & \text{otherwise, } U \text{ connected.} \end{cases}$$

This is a subsheaf of $\mathcal{O}_X$, but it is not quasi-coherent. For,pPick any small interval around 0, the only section of $\mathcal{F} \restriction U$ is 0. Therefore, there is no generation over U by this family (consisting of one point).

Proposition 3.7 *Assume that $X = \operatorname{Spec} A$. Then, for any A-module, M, the sheaf, $\widetilde{M}$, is quasi-coherent.*

Proof. Since M is an A-module, it has some presentation

$$A^{(I)} \longrightarrow A^{(J)} \longrightarrow M \longrightarrow 0.$$

Sheafifying (by applying the operator $\widetilde{\ }$), we get the exact sequence

$$\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)} \longrightarrow \widetilde{M} \longrightarrow 0,$$

and $\widetilde{M}$ is QC. $\square$

Say that $i\colon |V| \to |X|$ is an inclusion map where $|V|$ is open in $|X|$. We claim that $(|V|, \mathcal{O}_X \upharpoonright |V|)$ is a sub–ringed space of $X = (|X|, \mathcal{O}_X)$. We need a sheaf map from $\mathcal{O}_X$ to $i_*\mathcal{O}_X \upharpoonright |V|$. Let U be any open subset of $|X|$, then,

$$\Gamma(U, i_*\mathcal{O}_X \upharpoonright |V|) = \Gamma(U \cap |V|, \mathcal{O}_X \upharpoonright |V|).$$

Since $|V|$ is open, so is $U \cap |V|$, and the righthand side is just $\Gamma(U \cap |V|, \mathcal{O}_X)$. The restriction map $\rho^U_{U\cap|V|}\colon \Gamma(U, \mathcal{O}_X) \to \Gamma(U \cap |V|, \mathcal{O}_X)$, is the required map. So, for any open $|V| \subseteq |X|$, we have the ringed space $(|V|, \mathcal{O}_X \upharpoonright |V|)$. Let us abbreviate $\mathcal{O}_X \upharpoonright |V|$ by $\mathcal{O}_V$.

For ringed spaces, direct images and inverse images are important operations and inverse images require a change from our previous notion. In the case of an open inclusion $i\colon |V| \to |X|$, we have the sheaf $i_*\mathcal{F}$ on X–which, of course, is an $i_*\mathcal{O}_V$-module. Yet, we have a map $i^a\colon \mathcal{O}_X \to i_*\mathcal{O}_V$ of sheaves of rings. Thus, $i_*\mathcal{F}$ can be viewed as an $\mathcal{O}_X$-module. Observe, however, that all this did not depend on the fact that i is an open inclusion. Therefore, given a map of ringed spaces $\varphi\colon Y \to X$, we can view $\varphi_*\mathcal{F}$ as an $\mathcal{O}_X$-module. This is how we define the *push-forward* or *direct image*, $\varphi_*\mathcal{F}$, of the sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules.

Let $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map of ringed spaces, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Then, $|\varphi|^*\mathcal{G}$ is a sheaf of $|\varphi|^*\mathcal{O}_X$-modules. We also have a map $\varphi^a\colon |\varphi|^*\mathcal{O}_X \to \mathcal{O}_Y$. Therefore, $|\varphi|^*\mathcal{G}$ *and* $\mathcal{O}_Y$ are $|\varphi|^*\mathcal{O}_X$-modules. Thus, we can form the tensor product

$$\mathcal{O}_Y \otimes_{|\varphi|^*\mathcal{O}_X} |\varphi|^*\mathcal{G},$$

which is an $\mathcal{O}_Y$-module. This $\mathcal{O}_Y$-module is what we shall mean by $\varphi^*\mathcal{G}$ for a map φ of ringed spaces.

It is instructive to see what $(i_*\mathcal{F}) \upharpoonright |V|$ is in the case that i is the open inclusion $i\colon |V| \to |X|$. Let $U \subseteq |V|$ be an open subset. Because $U = U \cap |V|$ is open in X, we have

$$\Gamma(U \cap |V|, \mathcal{F}) = \Gamma(U, \mathcal{F}).$$

Therefore,

$$(i_*\mathcal{F}) \upharpoonright |V| = \mathcal{F}.$$

The functor, φ^*, is left adjoint to the functor, φ_*, which means that there are isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_Y}(\varphi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \varphi_*\mathcal{F})$$

for all $\mathcal{O}_Y$-modules $\mathcal{F}$ and all $\mathcal{O}_X$-modules $\mathcal{G}$, where $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

Remark: If $X = \mathrm{Spec}\, A$ and M, N are A-modules, then, for any linear map $u\colon M \to N$, we have $\widetilde{\mathrm{Ker}\, u} = \mathrm{Ker}\, \widetilde{u}$, $\widetilde{\mathrm{Im}\, u} = \mathrm{Im}\, \widetilde{u}$, and $\widetilde{\mathrm{Coker}\, u} = \mathrm{Coker}\, \widetilde{u}$.

If $X = (|X|, \mathcal{O}_X)$ is a ringed space, then for every $\mathcal{O}_X$-module $\mathcal{F}$, the module of gobal sections on the underlying topological space, $|X|$, will henceforth be denoted $\Gamma(X, \mathcal{F})$. Similarly, we will write $\Gamma(X_f, \mathcal{F})$ instead of $\Gamma(|X|_f, \mathcal{F})$.

Theorem 3.8 *Suppose $X = \mathrm{Spec}\, A$, and V is a quasi-compact open subset of $|X|$, and further suppose $\mathcal{F}$ is a sheaf on V which is an $\mathcal{O}_V$-module (where $\mathcal{O}_V = \mathcal{O}_X \upharpoonright V$). Then, the following properties are equivalent:*

(1) There is some A-module, M, so that $\widetilde{M} \upharpoonright V \cong \mathcal{F}$.

(2) There is a finite cover $(X_{f_i})_{i=1}^t$ of V so that for every i, $1 \leq i \leq t$, we can find an A_{f_i}-module M_i and we have $\mathcal{F} \upharpoonright X_{f_i} = \widetilde{M_i}$.

(3) The sheaf $\mathcal{F}$ is quasi-coherent.

(4) (Serre's lifting criterion, FAC [47])

(4a) For every $f \in A$ such that $X_f \subseteq V$, for every $s \in \Gamma(X_f, \mathcal{F})$, there is some $n \geq 0$ so that $f^n s$ lifts to a section in $\Gamma(V, \mathcal{F})$,

and

(4b) For every $f \in A$ such that $X_f \subseteq V$, for every $t \in \Gamma(V, \mathcal{F})$, if $t \upharpoonright X_f = 0$, then there is some $n \geq 0$ so that $f^n t = 0$ in $\Gamma(V, \mathcal{F})$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial, since $\widetilde{M} \upharpoonright X_{f_i} = \widetilde{M_{f_i}}$. Thus, in this situation, $M_i = M_{f_i}$.

(2) $\Rightarrow$ (3). Quasi-coherence of $\mathcal{F}$ is local on V. But on X_{f_i}, the sheaf $\mathcal{F}$ is an $\widetilde{M_i}$, which is quasi-coherent.

(3) $\Rightarrow$ (2). Locally everywhere on V, the sheaf $\mathcal{F}$ is the cokernel of some sheaf morphism

$$\varphi\colon \mathcal{O}_V^{(J)} \upharpoonright U \to \mathcal{O}_V^{(I)} \upharpoonright U.$$

We may take opens of the form X_{f_i}, since they are a base of the topology. Then,

$$\mathcal{O}_V^{(I)} \upharpoonright X_{f_i} = \widetilde{A_{f_i}^{(I)}};$$

so that, we have
$$A_{f_i}^{(J)} \longrightarrow A_{f_i}^{(I)} \longrightarrow M_i \longrightarrow 0,$$
where M_i is the cokernel. Therefore, $\mathcal{F} \restriction X_{f_i} = \widetilde{M_i}$. Since V is quasi-compact, it is a finite union of the X_{f_i}'s, and (2) holds.

(2) $\Rightarrow$ (4). First, consider the special case where $V = X_g$ and $\mathcal{F} = \widetilde{N}$ on V.

(4a) Pick $f \in A$ such that $X_f \subseteq X_g$, and $s \in \Gamma(X_f, \mathcal{F})$. Since $\mathcal{F} = \widetilde{N}$, by a previous theorem,
$$\Gamma(X_f, \mathcal{F}) = N_f.$$
Thus, $s = n/f^r$ for some $r \geq 0$ and $n \in N$. Of course, $f^r s = n \in \Gamma(X_g, \mathcal{F}) = \Gamma(V, \widetilde{N}) = N$.

(4b) Pick $f \in A$ such that $X_f \subseteq X_g$, and $t \in \Gamma(X_g, \mathcal{F}) = \Gamma(V, \widetilde{N}) = N$ so that $t \restriction X_f = 0$. Thus $t/1 = 0$ in $\Gamma(X_f, \widetilde{N}) = N_f$. By definition, this means that there is some $l \geq 0$ so that $f^l t = 0$ in N.

Claim: Let $V = X_{g_1} \cup \cdots \cup X_{g_t}$, assume that $\mathcal{F} \restriction X_{g_i}$ satisfies (4a) and (4b), and also that $\mathcal{F} \restriction X_{g_i g_j}$ satisfies (4a) and (4b). Then, $\mathcal{F}$ has the following stronger properties:

(4A) For every $f \in A$, for every $s \in \Gamma(X_f \cap V, \mathcal{F})$, there is some $n \geq 0$ so that $f^n s$ lifts to a section in $\Gamma(V, \mathcal{F})$.

(4B) For every $f \in A$, for every $t \in \Gamma(V, \mathcal{F})$, if $t \restriction X_f \cap V = 0$, then there is some $n \geq 0$ so that $f^n t = 0$ in $\Gamma(V, \mathcal{F})$.

Note that the special case shows that our $\mathcal{F}$ satisfies the hypotheses of the claim. Also, by taking $X_f \subseteq V$, (4A) and (4B) imply (4a) and (4b) for $\mathcal{F}$ on V.

First, we prove (4B). We are given $f \in A$, $t \in \Gamma(V, \mathcal{F})$, and we are assuming that $t \restriction X_f \cap V = 0$. Since V is covered by the X_{g_i}'s, we get
$$t \restriction X_f \cap X_{g_i} = 0.$$
However, $X_f \cap X_{g_i} = X_{fg_i}$, and X_{fg_i} has properties (4a) and (4b), by the special case. As $X_{fg_i} \subseteq X_{g_i}$ we find that there is some $n_i \geq 0$ so that
$$(fg_i)^{n_i} t = 0 \quad \text{on } X_{g_i}.$$
This means that $f^{n_i} g_i^{n_i} t = 0$ on X_{g_i}, but g_i is invertible on X_{g_i}, and thus, $f^{n_i} t = 0$. Since there are finitely many X_{g_i}'s covering V, if we let $n = \max\{n_i\}$, we get
$$f^n t = 0$$
on all the X_{g_i}'s covering V, and (4B) holds on V.

Next, we prove (4A). We are given $f \in A$ and $s \in \Gamma(V \cap X_f, \mathcal{F})$. The restriction, $s \restriction X_{g_i}$, of s to X_{g_i} yields a section on $X_f \cap X_{g_i}$, i.e., a section on $X_{fg_i} \subseteq X_{g_i}$. By (4a) in the special case, there is some $n_i \geq 0$ so that $(fg_i)^{n_i}s$ lifts to a section $s_i' \in \Gamma(X_{g_i}, \mathcal{F})$. But g_i is invertible on X_{g_i}, and thus, $s_i' = g_i^{n_i}s_i$, where $s_i \in \Gamma(X_{g_i}, \mathcal{F})$. Then,

$$g_i^{n_i}s_i = s_i' = f^{n_i}g_i^{n_i}s \quad \text{on } X_f \cap X_{g_i} \subseteq X_{g_i}.$$

Therefore,

$$s_i \upharpoonright X_f \cap X_{g_i} = f^{n_i}s \quad \text{on } X_f \cap X_{g_i} \subseteq X_{g_i},$$

since g_i is invertible on X_{g_i}. As usual, finitely many X_{g_i}'s cover V, and by letting $n = \max\{n_i\}$, we get that for $i = 1, \ldots, t$, there is some $s_i \in \Gamma(X_{g_i}, \mathcal{F})$ so that

$$s_i \upharpoonright X_f \cap X_{g_i} = f^n s.$$

Do the s_i patch on V? In general, they don't, but we can circumvent this problem as explained next.

Observe that $s_i - s_j = 0$ on $X_f \cap X_{g_i} \cap X_{g_j} = X_f \cap X_{g_ig_j}$, since

$$(s_i - s_j) \upharpoonright X_f \cap X_{g_ig_j} = f^n s - f^n s = 0.$$

By the special case of (4b) applied to $\mathcal{F} \upharpoonright X_{g_ig_j}$ and because $X_{fg_ig_j} \subseteq X_{g_ig_j}$, there is some $m_{ij} \geq 0$ so that

$$(fg_ig_j)^{m_{ij}}(s_i - s_j) = 0 \quad \text{on } X_{g_ig_j}.$$

However, g_ig_j is invertible on $X_{g_ig_j}$, and by letting $m = \max\{m_{ij}\}$ (since there are finitely many X_{g_i}'s covering V), we get

$$f^m s_i - f^m s_j = 0 \quad \text{on } X_{g_ig_j}.$$

Thus, the $f^m s_i$ patch on all of V. Therefore, $f^{m+n}s$ lifts to a global section (in $\Gamma(V, \mathcal{F})$), which we get by patching the $f^m s_i$. Thus, (4A) is proved. Since (4A) and (4B) are stronger than (4a) and (4b), we have proved that (2) implies (4).

(4) $\Rightarrow$ (1). First step: We prove that (4a) and (4b), which hold for $\mathcal{F}$ and V, are inherited on the $X_{g_i} \subseteq V$.

Given $X_f \subseteq X_g$ and $s \in \Gamma(X_{fg}, \mathcal{F})$, since $X_{fg} \subseteq X_g \subseteq V$ and (4a) holds for V, there is some $n \geq 0$ so that $(fg)^n s$ lifts to a section in $\Gamma(V, \mathcal{F})$. By restricting this section to X_g, we obtain the fact that $(fg)^n s$ lifts to a section in $\Gamma(X_g, \mathcal{F})$. But g is invertible in X_g, and thus, $f^n s$ lifts to a section in $\Gamma(X_g, \mathcal{F})$, which proves that (4a) holds for X_g and $\mathcal{F} \upharpoonright X_g$.

Given $s \in \Gamma(X_g, \mathcal{F})$ and $f \in A$, such that $X_f \subseteq X_g$, assume that $s \upharpoonright X_{fg} = 0$. Now, $X_g \subseteq V$ and (4a) holds for V. Thus, there is some $m \geq 0$ so that $g^m s$ extends to a section in $\Gamma(V, \mathcal{F})$. Since

$$g^m s \upharpoonright X_{fg} = 0,$$

there is some $p \geq 0$ so that

$$(fg)^p g^m s = 0 \quad \text{on } V,$$

by (4b) applied to V. If we restrict to X_g, we get

$$f^p g^{p+m} s = 0 \quad \text{on } X_g,$$

and, since g is invertible on X_g, we get $f^p s = 0$ on X_g, which is (4b) for X_g.

The claim established in the proof that (2) $\Rightarrow$ (4) now tells us that (4A) and (4B) hold for V and $\mathcal{F}$.

Second step: We need to define the module M. Consider the inclusion $i\colon V \to X$, and form $i_*\mathcal{F}$, a sheaf on X. The sheaf $i_*\mathcal{F}$ is an $\mathcal{O}_X$ module. Let

$$M = \Gamma(X, i_*\mathcal{F}) = \Gamma(V, \mathcal{F}).$$

This is an A-module, since $A = \Gamma(X, \mathcal{O}_X)$. Next, I claim there is a sheaf map

$$\widetilde{M} \mapsto i_*\mathcal{F}.$$

To see this, consider any open $X_f \subseteq X$. We know that $\Gamma(X_f, \widetilde{M}) = M_f$ and

$$\Gamma(X_f, i_*\mathcal{F}) = \Gamma(X_f \cap V, \mathcal{F}).$$

We need a map from M_f to $\Gamma(X_f \cap V, \mathcal{F})$. We have the restriction map $\rho^V_{X_f \cap V}\colon \Gamma(V, \mathcal{F}) \to \Gamma(X_f \cap V, \mathcal{F})$, and as $M = \Gamma(V, \mathcal{F})$. So, we have a map

$$\rho^V_{X_f \cap V}\colon M \to \Gamma(X_f \cap V, \mathcal{F}).$$

But f is invertible on $X_f \supseteq X_f \cap V$; so, by the universal mapping property of localization, the map $\rho^V_{X_f \cap V}\colon M \to \Gamma(X_f \cap V, \mathcal{F})$ factors through M_f, i.e.,

$$M \longrightarrow M_f \longrightarrow \Gamma(X_f \cap V, \mathcal{F}).$$

The second map is the required one. These maps patch together on overlaps $X_f \cap X_g$ (DX). Since the X_f's cover X, we get our sheaf map

$$\theta\colon \widetilde{M} \longrightarrow i_*\mathcal{F}.$$

Now, we claim that θ is an isomorphism.

Pick any $X_f \subseteq X$, and any $\sigma \in \Gamma(X_f, i_*\mathcal{F})$. Since

$$\Gamma(X_f, i_*\mathcal{F}) = \Gamma(X_f \cap V, \mathcal{F})$$

and since (4A) holds, there is some $\tau \in \Gamma(V, \mathcal{F})$ such that τ lifts $f^n\sigma$ for some $n \geq 0$. But $\Gamma(V, \mathcal{F}) = M$; so, $\tau \in M$. In M_f, we get have the element τ/f^n, and $\theta(\tau/f^n) = \sigma$, because

$$\sigma = \frac{f^n\sigma}{f^n} \quad \text{on } X_f \cap V.$$

Thus, θ is surjective.

Assume that $\theta(m/f^r) = 0$. The element $\theta(m/f^r)$ belongs to $\Gamma(X_f \cap V, i_*\mathcal{F}) = \Gamma(X_f \cap V, \mathcal{F})$. Since f is invertible on X_f, we find that $\theta(m/1) = 0$, and (4B) implies that there is some $l \geq 0$ so that $f^l\theta(m/1) = 0$ in $\Gamma(V, \mathcal{F})$. Under the identification $\Gamma(V, \mathcal{F}) = M$, the element $f^l\theta(m/1)$ is identified with $f^l m$; and so $f^l m = 0$ in M. But then, $m/f^r = 0$ in M_f. This proves injectivity, and finishes the proof. $\square$

Corollary 3.9 *Let $X = \operatorname{Spec} A$, and let V be a quasi-compact open subset of $|X|$ and $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_V$-modules. If $i\colon V \to |X|$ is the inclusion map, then the following properties hold:*

(1) The sheaf $i_*\mathcal{F}$ is a QC $\mathcal{O}_X$-module.

(2) Every QC sheaf, $\mathcal{F}$, on V is the restriction of a QC $\mathcal{O}_X$-module.

Corollary 3.10 *Let $X = \operatorname{Spec} A$. An $\mathcal{O}_X$-module, $\mathcal{F}$, is QC iff $\mathcal{F} = \widetilde{M}$ for some A-module M. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ establish an equivalence of the categories of A-modules and QC $\mathcal{O}_X$-modules. The functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is an exact functor on the category of QC $\mathcal{O}_X$-modules (when $X = \operatorname{Spec} A$).*

Since the category of A-modules has enough injectives, when $X = \operatorname{Spec} A$ the category of QC $\mathcal{O}_X$-modules has enough injectives. Thus, we can resolve a QC $\mathcal{O}_X$-module $\mathcal{F}$ by QC injectives on X. The functor $\Gamma(X, -)$ yields the right derived functor

$$H^p_{\mathrm{QC}}(X, \mathcal{F})$$

for every $p \geq 0$. This right derived functor is not the correct object, however. What we really want to do is to consider the category of *all* $\mathcal{O}_X$-modules (which also has enough injectives) and take derived functors there. There is no reason why an injective in QC is injective in the bigger category of all $\mathcal{O}_X$-modules. Also, for the special cohomology $H^p_{\mathrm{QC}}(X, \mathcal{F})$, our results above yield

Corollary 3.11 *Let $X = \operatorname{Spec} A$, an affine scheme. For every QC $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have*

$$H^p_{\mathrm{QC}}(X, \mathcal{F}) = (0)$$

for all $p > 0$.

Corollary 3.12 *Let $X = \operatorname{Spec} A$. Then, each QC $\mathcal{O}_X$-algebra has the form $\widetilde{B}$ for some A-algebra B. Every QC $\widetilde{B}$-module (i.e., QC as $\widetilde{B}$-module) is QC as an $\mathcal{O}_X$-module and has the form $\widetilde{N}$ for some B-module N.*

Proof. Let $\mathcal{B}$ be a QC $\mathcal{O}_X$-algebra. Then, by Theorem 3.8, $\mathcal{B} = \widetilde{B}$ for some A-module B. We claim that B is an A-algebra. The fact that $\mathcal{B}$ is an algebra can be expressed in categorical form by saying that there is an $\mathcal{O}_X$-linear map

$$\mu\colon \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{B} \longrightarrow \mathcal{B},$$

and that some obvious diagrams commute. Thus, we have

$$\mu\colon \widetilde{B} \otimes_{\mathcal{O}_X} \widetilde{B} \longrightarrow \widetilde{B},$$

and since

$$\widetilde{B} \otimes_{\mathcal{O}_X} \widetilde{B} \cong \widetilde{B \otimes_A B},$$

we have

$$\mu\colon \widetilde{B \otimes_A B} \longrightarrow \widetilde{B},$$

from which we get a linear map

$$\mu_1\colon B \otimes_A B \longrightarrow B$$

which makes B an A-algebra, since the required diagrams are still commutative.

Now, let $\mathcal{F}$ be a QC $\widetilde{B}$-module. To check that $\mathcal{F}$ is QC as $\widetilde{A}$-module is local, and thus, we may assume that we have an exact sequence

$$\widetilde{B^{(J)}} \longrightarrow \widetilde{B^{(I)}} \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this, we get an exact sequence

$$B^{(J)} \longrightarrow B^{(I)} \longrightarrow N \longrightarrow 0,$$

where N is the cokernel. Since $B^{(I)}$ and $B^{(J)}$ are A-modules, N is an A-module (and a B-module), and further, $\widetilde{N} = \mathcal{F}$. But $\widetilde{N}$ is QC as $\mathcal{O}_X$-module and N is also a B-module. So, $\mathcal{F} = \widetilde{N}$ for some B-module N. $\square$

Just as finitely generated modules form an interesting and amenable subcategory of all modules, so in the category of $\mathcal{O}_X$-modules we have a distinguished subcategory consisting of teh coherent modules.

Definition 3.4 Given a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-module $\mathcal{F}$, we say that $\mathcal{F}$ is *coherent* if

(1) $\mathcal{F}$ is finitely generated as an $\mathcal{O}_X$-module, and

(2) For each $n > 0$ and for every open subset $U \subseteq X$, homomorphism $\varphi\colon (\mathcal{O}_X \restriction U)^n \to \mathcal{F} \restriction U$, the sheaf Ker φ is finitely generated.

In his studies of several complex variables during the 1940's, the mathematician Oka discovered that the sheaf of germs of holomorphic functions on a complex space is coherent in the above sense. But, while this definition is basically due to him, the actual definition is due to Henri Cartan.

If $\mathcal{F}$ is a coherent sheaf on X, then, by (1), for every $x \in |X|$, there is some open subset U with $x \in U$ and a surjective homomorphism $\varphi_U \colon (\mathcal{O}_X \restriction U)^p \to \mathcal{F} \restriction U$ for some $p > 0$. By (2), the kernel of this map is finitely generated, which means that there is some $q > 0$ and a map from $(\mathcal{O}_X \restriction U)^q$ to $(\mathcal{O}_X \restriction U)^p$ so that

$$(\mathcal{O}_X \restriction U)^q \longrightarrow (\mathcal{O}_X \restriction U)^p \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.}$$

Thus, a coherent sheaf, $\mathcal{F}$, is quasi-coherent, and in fact, finitely presented.

The sheaf $\mathcal{O}_X$ need not be a coherent $\mathcal{O}_X$-module. For example, if $X = \operatorname{Spec} \mathbb{C}[X_1, \ldots]$, with countably many variables, then $\mathcal{O}_X = \widetilde{\mathbb{C}[X_1, \ldots]}$ is not coherent (because the ring $\mathbb{C}[X_1, \ldots]$ is not Noetherian).

Remarks:

(1) The sheaf $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module iff for every open $n > 0$, $U \subseteq X$, and homomorphism $\varphi \colon (\mathcal{O}_X \restriction U)^n \to \mathcal{O}_X \restriction U$, the sheaf, $\operatorname{Ker} \varphi$, is finitely generated.

(2) A sub $\mathcal{O}_X$-module of a coherent sheaf is coherent, provided it is finitely generated.

(3) If $\mathcal{F}, \mathcal{G}$ are coherent and $\varphi \colon \mathcal{F} \to \mathcal{G}$, then $\operatorname{Ker} \varphi$, $\operatorname{Im} \varphi$, $\operatorname{Coker} \varphi$, are coherent.

(4) If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules and two of the sheaves are coherent, then the third one is coherent.

(5) If $\mathcal{F}, \mathcal{G}$ are coherent, then

(a) $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent, and

(b) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent.

Proposition 3.13 *Let X be an LRS, and assume that $\mathcal{O}_X$ is a coherent sheaf. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}$ is coherent iff $\mathcal{F}$ is finitely presented.*

Proof. We know that a coherent sheaf is always f.p. Conversely, since coherence and f.p. are local, we may assume that $U = X$. Assume that

$$\mathcal{O}_X^{(q)} \longrightarrow \mathcal{O}_X^{(p)} \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{is exact.}$$

For finite p and q, the sheaves $\mathcal{O}_X^{(p)}$ and $\mathcal{O}_X^{(q)}$ are coherent iff $\mathcal{O}_X$ is, as is easily shown by induction using the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^{(p)} \longrightarrow \mathcal{O}_X^{(p-1)} \longrightarrow 0.$$

Let $K = \mathrm{Ker}\,(\mathcal{O}_X^{(p)} \to \mathcal{F})$, since

$$\mathcal{O}_X^{(q)} \longrightarrow K \longrightarrow 0 \quad \text{is exact,}$$

we see that K is finitely generated. But then, we have the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_X^{(p)} \longrightarrow \mathcal{F} \longrightarrow 0,$$

and since K is a finitely generated submodule of a coherent $\mathcal{O}_X$-module, K itself is coherent. Then, two of the sheaves in the sequence are coherent, therefore, so is $\mathcal{F}$. $\square$

Theorem 3.14 *Let $X = \mathrm{Spec}\, A$, and assume that A is Noetherian. Then, the following properties hold:*

(1) Every open subset U of $|X|$ is quasi-compact.

(2) $\mathcal{O}_X$ is coherent.

(3) For any open U, the following are equivalent for an $\mathcal{O}_U$-module:

(a) $\mathcal{F}$ is coherent.

(b) $\mathcal{F}$ is QC and finitely generated.

(c) There is some finitely generated A-module M so that $\mathcal{F} \upharpoonright U = \widetilde{M}$.

Proof. (1) is clear.

(2) Since coherence is local, we can check it on a basis of open sets, namely, on the X_f's ($\mathcal{O}_X$ being clearly f.g.). Now, the sequence

$$\mathcal{O}_X^{(n)} \upharpoonright X_f \xrightarrow{\varphi} \mathcal{O}_X \upharpoonright X_f \longrightarrow 0 \quad \text{is exact;}$$

so, we get the exact sequence

$$\widetilde{A_f^n} \longrightarrow \widetilde{A_f} \longrightarrow 0.$$

By taking global sections, we get the exact sequence

$$A_f^n \longrightarrow A_f \longrightarrow 0.$$

Since A_f is Noetherian, the kernel K is f.g., and thus, the fact that we have the exact sequence,

$$A_f^q \longrightarrow K \longrightarrow 0,$$

for some $q > 0$, shows that $\widetilde{K}$ is f.g., where $\widetilde{K} = \mathrm{Ker}\,\varphi$.

(3) $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$. By Theorem 3.8, there is some A-module M such that $\widetilde{M} \upharpoonright U = \mathcal{F}$. We can cover U by opens of the form X_f, and it is always the case that $M = \varinjlim_\lambda M_\lambda$, where M_λ runs over the finitely generated submodules of M. Since $\widetilde{M} \upharpoonright X_f = \mathcal{F} \upharpoonright X_f$, by (b), there is some λ (depending on f) so that

$$\widetilde{M_\lambda} \upharpoonright X_f = \mathcal{F} \upharpoonright X_f.$$

However, U is quasi-compact, and thus, finitely many X_f cover U. This implies that there is some λ so that

$$\widetilde{M_\lambda} \upharpoonright U = \mathcal{F},$$

with M_λ finitely generated.

$(c) \Rightarrow (a)$. We know that $\mathcal{F}$ is QC on U and for any small open $X_f \subseteq U$, we have the exact sequence

$$\widetilde{A_f^n} \longrightarrow \mathcal{F} \upharpoonright X_f \longrightarrow 0.$$

This comes from the module sequence

$$0 \longrightarrow K \longrightarrow A_f^n \longrightarrow M_f \longrightarrow 0,$$

and the kernel K is f.g., as A_f is Noetherian. So, $\widetilde{M_f} = \mathcal{F} \upharpoonright X_f$ is finitely presented, and thus, is coherent, because $\mathcal{O}_X$ is coherent, by (2), and $\mathcal{F}$ itself is coherent. $\square$

Corollary 3.15 *Let $X = \mathrm{Spec}\, A$, where A is Noetherian. If $(U, \mathcal{O}_U)$ is an open in X with inclusion map $i\colon U \to X$, then for every coherent $\mathcal{O}_U$-module $\mathcal{F}$, the $\mathcal{O}_X$-module, $i_*\mathcal{F}$, is coherent on X.*

Corollary 3.16 *Let $X = \mathrm{Spec}\, A$, where A is Noetherian. For any QC $\mathcal{O}_X$-module $\mathcal{F}$, we have*

$$\mathcal{F} = \varinjlim_\lambda \mathcal{F}_\lambda,$$

where $\mathcal{F}_\lambda$ is a coherent submodule of $\mathcal{F}$.

Consider $\varphi_*\mathcal{F}$, where $\varphi\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, and where $\mathcal{F}$ is a QC $\mathcal{O}_Y$-module. Here, $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$. We have a ring map from A to B. For any $U \subseteq |X|$, we have

$$\Gamma(U, \varphi_*\mathcal{F}) = \Gamma(|\varphi|^{-1}(U), \mathcal{F}).$$

Now, assume that $\varphi_*\mathcal{F}$ is QC. Then, $\varphi_*\mathcal{F} = \widetilde{M}$ for some A-module M. We know that

$$M = \Gamma(|X|, \widetilde{M}) = \Gamma(|X|, \varphi_*\mathcal{F}) = \Gamma(|Y|, \mathcal{F}).$$

Let us also assume that $\mathcal{F}$ is coherent on Y and that A and B are Noetherian. Using the ring map from A to B, we see $M = \Gamma(|Y|, \mathcal{F})$ would have to be finitely generated as an A-module, for $\varphi_*\mathcal{F}$ to be coherent. This is generally false, as the following example shows:

Example 3.1 Consider a field k, and let $\mathbb{A}^1_k = \operatorname{Spec} k[T]$. The scheme $\mathbb{A}^0_k = \operatorname{Spec} k$ consists of a single point with k as stalk. We have the inclusion $i\colon k \to k[T]$, and we get a morphism $\varphi\colon \mathbb{A}^1_k \to \mathbb{A}^0_k$. Let $Y = \mathbb{A}^1_k$ and $\mathcal{F} = \mathcal{O}_Y$. The sheaf $\mathcal{O}_Y$ is coherent since $k[T]$ is Noetherian. What is $\varphi_*\mathcal{O}_Y$? We know that $\varphi_*\mathcal{O}_Y = \widetilde{M}$, where $M = \Gamma(|Y|, \mathcal{O}_Y) = k[T]$ as a k-module via $i\colon k \to k[T]$. However, $k[T]$ is *not* a finitely generated k-module, which implies that $\varphi_*\mathcal{O}_Y$ is not a coherent $\mathcal{O}_X$-module.

Consequently, the direct image of a coherent sheaf is not, in general, coherent. Corollary 3.15 is an exception and one needs more restrictive hypotheses on $\varphi\colon Y \to X$ in order that the direct image, $\varphi_*\mathcal{F}$, of the coherent $\mathcal{O}_Y$-module, $\mathcal{F}$, be coherent on X.

3.3 Schemes: Products, Fibres, and Finiteness Properties

Definition 3.5 A *scheme* X is a locally ringed space such that for every $x \in |X|$, there is some open subset U with $x \in U$ and $(U, \mathcal{O}_X \restriction U)$ is isomorphic to Spec A for some ring A (i.e., $(U, \mathcal{O}_X \restriction U)$ is an affine scheme).

Thus, a scheme is an LRS that is locally an affine scheme. We denote the category of schemes by $\mathcal{SCH}$. We can carry over the material on quasi-coherent $\mathcal{O}$-modules for affine schemes to our present level of generality.

Proposition 3.17 *Let X be a scheme. Then, an $\mathcal{O}_X$-module $\mathcal{F}$ is QC iff there is some open cover $(U_i)_{i\in I}$ of $|X|$ such that $\mathcal{F} \restriction U_i$ is QC as an $\mathcal{O}_X \restriction U_i$-module for every $i \in I$, and thus iff for every open affine $(U_i, \mathcal{O}_{U_i}) \cong \operatorname{Spec} A_{U_i}$, we have $\mathcal{F} \restriction U_i = \widetilde{M_{U_i}}$ for some A_{U_i}-module M_{U_i}.*

The notion of a product or a fibred product is an extremely important and convenient notion in studying geometry. The reader need only turn to Chapter 1 to see how often it was used in the classical theory of varieties presented there. In the category of schemes, it turns out that fibred products exist. Assume that we are given some schemes X, Y, Z and some morphisms $p_X\colon X \to Z$ and $p_Y\colon Y \to Z$.

Theorem 3.18 *In the category of schemes over Z, the product $X \prod_Z Y$ exists (and is unique up to unique isomorphism).*

Sketch of proof. We proceed in several steps.

Step 1. First, assume that X, Y, Z are affine, say $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, $Z = \operatorname{Spec} A$. Then, one checks that

$$X \prod_Z Y = \operatorname{Spec}(B \otimes_A C).$$

Step 2. Assume that X, Y are Z-schemes, that Z_0 is an open subscheme of Z and that X, Y are actually Z_0-schemes (which means that p_X and p_Y factor as $X \longrightarrow Z_0 \hookrightarrow Z$ and $Y \longrightarrow Z_0 \hookrightarrow Z$). Then, $X \prod_Z Y$ exists iff $X \prod_{Z_0} Y$ exists, and if they exist, they are equal. The reader should be able to check this without difficulty.

Step 3. Assume that X and Y are arbitrary, but that Z is affine. Then, $X \prod_Z Y$ exists. Indeed, cover X by affine opens, X_α ,and Y by affine opens, Y_β. By (1), $X_\alpha \prod_Z Y_\beta$ exists. Clearly, they patch (DX). (But see the remark and lemma immediately below).

Step 4. Let Z be arbitrary. Cover Z by affine opens, Z_γ, and let $X_\gamma = p_X^{-1}(Z_\gamma)$ and $Y_\gamma = p_Y^{-1}(Z_\gamma)$, which are schemes over Z_γ. By (3), $X_\gamma \prod_{Z_\gamma} Y_\gamma$ exists, and by (2), it is equal to $X_\gamma \prod_Z Y_\gamma$.

Step 5. In general, get an affine open cover, (Z_γ), of Z, make $X_\gamma \prod_Z Y_\gamma$ as in step 4, and patch them together to make $X \prod_Z Y$, as in step 3.

Remark: When we use (1) in step (3), we need to know that the product in the category of affine schemes is the same as the product in the category of schemes. This follows from the lemma:

Lemma 3.19 *Let T be an arbitrary scheme and X an affine scheme, so that $X = \mathrm{Spec}\, A$ and $A = \Gamma(|X|, \mathcal{O}_X)$. Then,*

$$\mathrm{Hom}_{\mathcal{SCH}}(T, X) \cong \mathrm{Hom}_{\mathrm{alg}}(\Gamma(|X|, \mathcal{O}_X), \Gamma(|T|, \mathcal{O}_T)). \tag{$*$}$$

Sketch of proof. If $\varphi \colon T \to X$, we get a map $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(T, \mathcal{O}_T)$. Conversely, cover T by affine opens, T_α. By restriction, the map $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(T, \mathcal{O}_T)$ yields a map $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(T_\alpha, \mathcal{O}_{T_\alpha})$. Since X and T_α are affine, we get $\varphi_\alpha \in \mathrm{Hom}\,(T_\alpha, X)$, and these maps patch (as the reader should check), so that we get a map in $\mathrm{Hom}_{\mathcal{SCH}}(T, X)$. $\square$

Remark: The isomorphism $(*)$ shows that when X is affine, $\mathrm{Hom}_{\mathcal{SCH}}(T, X)$ really depends only on $\mathrm{Spec}\, \Gamma(|T|, \mathcal{O}_T)$ (besides X). This, as described in Chapter 1, is the characteristic property of affines in the category of schemes. That is, X is affine iff the morphisms from an arbitrary scheme into X are exactly the morphisms from the affinization of the arbitrary scheme.

The reader should show that coproducts also exist, and that finite products and coproducts of affines are affine (DX). However, infinite coproducts of affines are *never* affine.

Let us now consider fibres. Given $x \in |X|$, we claim that there is a morphism $i_x \colon \mathrm{Spec}\, \kappa(x) \to X$, where $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, the residue field at x. Clearly, we must have

$$|i_x|(\mathrm{pt}) = x.$$

We need a map of sheaves from $i_x^*\mathcal{O}_X$ to $\mathcal{O}_{\mathrm{Spec}\,\kappa(x)}$. Thus, we need a map from $(i_x^*\mathcal{O}_X)_{\mathrm{pt}}$ to $\kappa(x)$. But

$$(i_x^*\mathcal{O}_X)_{\mathrm{pt}} = \mathcal{O}_{X,|i_x|(\mathrm{pt})} = \mathcal{O}_{X,x},$$

and we need a map from $\mathcal{O}_{X,x}$ to $\kappa(x)$, for which we can use the projection onto the residue field. Now, assume we have a morphism $\pi\colon Y \to X$ and combine it with $i_x\colon \mathrm{Spec}\,\kappa(x) \to X$ to make the fibred product $Y \prod_X \mathrm{Spec}\,\kappa(x)$, with maps $pr_1\colon Y \prod_X \mathrm{Spec}\,\kappa(x) \to Y$ and $pr_2\colon Y \prod_X \mathrm{Spec}\,\kappa(x) \to \mathrm{Spec}\,\kappa(x)$. By definition, we define the scheme $\pi^{-1}(x)$ by

$$\pi^{-1}(x) = Y \prod_X \mathrm{Spec}\,\kappa(x).$$

The scheme $\pi^{-1}(x)$ is always considered as a $\mathrm{Spec}\,\kappa(x)$-scheme.

Remarks:

(1) It is easily checked (DX) that if U is an affine open of X then

$$\pi^{-1}(x) = \pi^{-1}(U) \prod_U \mathrm{Spec}\,\kappa(x),$$

where $\pi^{-1}(U) = Y \prod_X U$. Thus, $\pi^{-1}(x)$ only depends on a local neighborhood of $x \in |X|$.

(2) For such an open U, the scheme $\pi^{-1}(U)$ is covered by open affines Y_α; so, $\pi^{-1}(x)$ is covered by the affines

$$Y_\alpha \prod_U \mathrm{Spec}\,\kappa(x) = \mathrm{Spec}(\Gamma(Y_\alpha) \otimes_{\Gamma(U)} \kappa(x)).$$

Here, we write $\Gamma(U)$ for $\Gamma(U, \mathcal{O}_X)$ and $\Gamma(Y_\alpha)$ for $\Gamma(Y_\alpha, \mathcal{O}_Y)$.

(3) We also claim that there is a canonical morphism from $\mathrm{Spec}(\mathcal{O}_{X,x})$ to X (where $x \in |X|$). Indeed, take any open affine U such that $x \in |U|$, and look at $\Gamma(U, \mathcal{O}_U) = \Gamma(U, \mathcal{O}_X)$. Then, the map

$$\sigma \mapsto \sigma(x)$$

yields a map from $\Gamma(U, \mathcal{O}_X)$ to $\mathcal{O}_{X,x}$, and this is a ring homomorphism. Thus, we get a morphism from $\mathrm{Spec}(\mathcal{O}_{X,x})$ to $\mathrm{Spec}(\Gamma(U, \mathcal{O}_X)) = U \subseteq X$. Clearly, this map does not depend on U, which gives our morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$. (DX). The affine scheme, $\mathrm{Spec}(\mathcal{O}_{X,x})$, is called the *local scheme at x*.

(4) The local scheme $\mathrm{Spec}(\mathcal{O}_{X,x})$ is contained in every affine open subscheme of X which contains x.

(5) If X is a variety, $\mathrm{Spec}(\mathcal{O}_{X,x})$ is not a variety, in general.

(6) The underlying space $|\mathrm{Spec}(\mathcal{O}_{X,x})|$ is neither open nor closed in $|X|$.

Some examples will illustrate the above remarks.

Example 3.2

(1) Observe that every scheme is a scheme over $\mathrm{Spec}\,\mathbb{Z}$. Indeed, having a morphism from X to $\mathrm{Spec}\,\mathbb{Z}$ is equivalent to having a ring morphism from $\mathbb{Z}$ to $\Gamma(X, \mathcal{O}_X)$, and there is always such a canonical ring morphism.

(2) Furthermore, $\mathrm{Spec}\,\mathbb{Z}$ has two kinds of points: Points of the form p, where p is a prime number, these are closed points, and the "fuzzy point" 0, the generic point. The generic point is neither open nor closed, but it is dense. Given $p \in |\mathrm{Spec}\,\mathbb{Z}|$, what's the local scheme at p? The ring is $\mathbb{Z}_{(p)}$, a DVR. The space of $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ has two points, one generic, the other a closed point. The map from $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ to $\mathrm{Spec}\,\mathbb{Z}$ sends the closed point to p, and map generic point to generic point.

(3) For $\xi \in |\mathrm{Spec}\,\mathbb{Z}|$, what are $\kappa(\xi)$ and $\mathrm{Spec}\,\kappa(\xi)$? When $\xi = 0$ (the generic point), then $\kappa(\xi) = \mathbb{Q}$, and we get $\mathrm{Spec}\,\mathbb{Q}$. When $\xi = p$, a prime number, we get $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, and we get $\mathrm{Spec}\,\mathbb{F}_p$.

(4) Given $\xi \in |\mathrm{Spec}\,\mathbb{Z}|$, we have $\pi^{-1}(\xi) = X_\xi$, a scheme over $\mathrm{Spec}\,\mathbb{Q}$ if $\xi = 0$, and a scheme over $\mathbb{F}_p$, if p is a prime number. We call X_p the *reduction of X mod p* (it is a scheme over $\mathbb{F}_p$).

A general scheme can be quite wild and few deep theorems can be proved without some kind of "taming hypotheses." These usually deal with finiteness in one form or another and they are of two types: Conditions on schemes, and conditions on morphisms. We begin with conditions on schemes.

Definition 3.6 A scheme X is *quasi-compact* if $|X|$ is quasi-compact as a topological space. (That is, it can be covered by finitely many affine opens.)

Definition 3.7 Given some property P of schemes and a scheme X, we say that X is *locally P* if for every $x \in |X|$, there is some affine open subset U_x with $x \in |U_x|$, and U_x has the property P. We say that X is *strongly locally P* if for every affine open $U \subseteq X$, the scheme U has P. (N.B., strongly locally P is not a standard locution).

Definition 3.8 (a) A scheme X is *noetherian* if X is quasi-compact and each of the finitely many affines, U_α, covering X is of the form $\mathrm{Spec}\,A_\alpha$, where A_α is a noetherian ring.

(b) If a scheme X is strongly locally noetherian, then it is usually called *locally noetherian*, and there is no confusion in terminology because having an affine open covering by Spec's of noetherian rings is equivalent to being strongly locally noetherian. Observe that in (b), no assumption of quasi-compactness is made.

(c) A scheme X is *artinian* if it is quasi-compact and each of the finitely many affine opens covering X is of the form $\operatorname{Spec} A_\alpha$, where A_α is an artinian ring (i.e., satisfies the DCC condition).

(d) As in (b), strongly locally artinian is usually called *locally artinian.*

To use these finiteness conditions on schemes themselves, it is most convenient to isolate properties of morphisms which allow them to come into play. First, observe that a morphism $\pi\colon Y \to X$ of schemes can be viewed as a "moving algebraic family" of schemes, each one a scheme over a field. Namely, $Y_x = \pi^{-1}(x)$ is a scheme over the field $\kappa(x)$.

Definition 3.9 A morphism $\pi\colon Y \to X$ is *quasi-compact* if X can be covered by affine opens, X_α, so that $\pi^{-1}(X_\alpha) = X_\alpha \prod_X Y = Y_\alpha$ is quasi-compact for every α.

It is easily shown that the condition of Definition 3.9 holds iff the inverse image $\pi^{-1}(U)$ of every affine $U \subseteq X$ is quasi-compact (DX).

Definition 3.10 (a) A morphism $\pi\colon Y \to X$ is a *locally finite-type morphism*, or an LFT-*morphism*, if X can be covered by affine opens, $X_\alpha = \operatorname{Spec} A_\alpha$, so that $\pi^{-1}(X_\alpha) = Y_\alpha$ can be covered by affine opens, $Z_{\alpha\beta}$, where $Z_{\alpha\beta} = \operatorname{Spec} B_{\alpha\beta}$, in which $B_{\alpha\beta}$ is a finitely generated A_α-algebra. Note that in the above definition, it is possible that there are infinitely many X_α and that the covers of the Y_α contain infinitely many schemes. The notion of LFT-morphism is strongly local on both X and Y.

(b) A morphism $\pi\colon Y \to X$ is a *finite-type morphism*, or an FT-*morphism*, if π is quasi-compact and LFT.

(c) A morphism $\pi\colon Y \to X$ is a *locally finite-presentation morphism*, or an LFP-*morphism*, if it is LFT and the algebras $B_{\alpha\beta}$ are finitely presented over A_α. Our morphism $\pi\colon Y \to X$ is a *finite-presentation* morphism, or an FP-*morphism*, if it is FT and the algebras B_α^β are finitely presented over A_α, equivalently if it is LFP and quasi-compact.

(d) A morphism $\pi\colon Y \to X$ is an *affine morphism* if X can be covered by affine opens, X_α, so that $Y_\alpha = \pi^{-1}(X_\alpha)$ is again affine. (The reader should check that π is an affine morphism iff π is strongly affine (DX).)

(e) A morphism $\pi\colon Y \to X$ is a *finite morphism* if it is affine and $\Gamma(Y_\alpha)$ is a finite $\Gamma(X_\alpha)$-module. (Also, in this case, finite is the same as strongly finite (DX).)

(f) A morphism $\pi\colon Y \to X$ is *quasi-finite* if for every $x \in |X|$, the set $|\pi^{-1}(x)|$ is finite.

Beware that finite implies quasi-finite, but the converse is false.

(i) The morphism $\varphi\colon \mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}$ is quasi-finite, but $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module. Thus, φ is not a finite morphism.

(ii) Let $Y = \mathrm{Spec}\,\mathbb{C}[S,T]/(ST-1)$ and $X = \mathrm{Spec}\,\mathbb{C}[S]$. Write φ for the projection map. Every fibre is a single point. This morphism is affine, FT, quasi-finite, but not finite.

To use the above (many!) definitions, we need to investigate how these properties behave w.r.t. base extension and perhaps descent. For this, we need open subschemes, closed subschemes, general immersions, and separation.

Definition 3.11 A scheme $(Y, \mathcal{O}_Y)$ is an *open subscheme* of a scheme $(X, \mathcal{O}_X)$ if the following hold:

(1) The space Y is open in X.

(2) There is an isomorphism $\mathcal{O}_Y \cong \mathcal{O}_X \restriction Y$.

A morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an *open immersion* if there is some open subscheme $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ and our morphism factors through an isomorphism $\varphi\colon (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$. That is,

(1) The space Z is homeomorphic to an open $Y \subseteq X$, and

(2) There is an isomorphism $\mathcal{O}_Z \cong \mathcal{O}_X \restriction Y$.

A scheme $(Y, \mathcal{O}_Y)$ is a *closed subscheme* of $(X, \mathcal{O}_X)$ if the following hold:

(1) The space Y is closed in X.

(2) If $i\colon Y \to X$ is the inclusion map, then the sheaf map $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ is surjective.

A morphism $(Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$ is a *closed immersion* if it factors through an isomorphism of $(Z, \mathcal{O}_Z)$ with a closed subscheme of $(X, \mathcal{O}_X)$.

Let C be a closed subset of X, where $(X, \mathcal{O}_X)$ is a scheme. Then, there exists a unique minimal structure of scheme on C, called the *reduced induced* structure. It is defined as follows: Cover X by affine opens, X_α. Let $C_\alpha = X_\alpha \cap C$. These C_α are closed subsets of the X_α, and we can write $(X_\alpha, \mathcal{O}_\alpha) \cong \mathrm{Spec}\, A_\alpha$. Points of C_α are prime ideals, $\mathfrak{p}$, of A_α. Let

$$\mathfrak{A}_\alpha = \bigcap \{\mathfrak{p} \in |\mathrm{Spec}\, A_\alpha| \mid \mathfrak{p} \in C_\alpha\}.$$

Then, $\mathfrak{A}_\alpha$ is a radical ideal ($A_\alpha/\mathfrak{A}_\alpha$ has no nilpotents). The ideal $\mathfrak{A}_\alpha$ is maximal with respect to the condition $V(\mathfrak{A}_\alpha) = C_\alpha$, so that $V(\mathfrak{A}_\alpha) = C_\alpha$. Write Y_α instead of $\mathrm{Spec}(A_\alpha/\mathfrak{A}_\alpha)$. Of course, $|Y_\alpha| = C_\alpha$, and these schemes patch (DX). Patching them yields a scheme $(Y, \mathcal{O}_Y)$, which is a closed subscheme such that $Y = C$. Let us use the notation, Y_0, for the scheme just constructed; also, let $i_0\colon Y_0 \to X$ be the natural inclusion, a closed immersion.

The scheme Y_0 enjoys a a universal mapping property: Given any scheme, Y, and any closed immersion, $i\colon Y \to X$, so that $C \subseteq |i|(|Y|)$, there is a closed immersion $j\colon Y_0 \to Y$ so that

$$i_0 = i \circ j.$$

The reader should check this universal mapping property.

Let X be a scheme and let (U_α) be a cover of X by affine opens. We can recover $\mathcal{O}_X \upharpoonright U_\alpha$ from the ring $A_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$ as $\widetilde{A_\alpha}$. Let $\mathcal{N}_\alpha$ be the nilradical of A_α, i.e.,

$$\mathcal{N}_\alpha = \{\xi \in A_\alpha \mid \xi^n = 0 \text{ for some } n > 0\}.$$

Clearly, the $\mathcal{N}_\alpha$ patch on overlaps $U_\alpha \cap U_\beta$. We get an $\mathcal{O}_X$-ideal, $\mathcal{N}$, of $\mathcal{O}_X$, and we obtain a scheme $(X, \mathcal{O}_X/\mathcal{N})$. This scheme is denoted by X_{red} and is called the *reduced scheme of* X. It is just the reduced induced scheme structure on the topological space $|X|$. The map $X \mapsto X_{\mathrm{red}}$ is an endfunctor in the category of schemes.

We have defined the notions of open and closed subschemes and open and closed immersions. A combination of them both yields the general notion of subscheme:

Definition 3.12 A *subscheme* $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ is a pair where Y is locally closed in X and the sheaf map $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ is surjective, where $i\colon Y \to X$ is the inclusion, and, as before, a morphism $(Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$ is an *immersion* if there is some subscheme $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ so that our morphism factors through an isomorphism $(Z, \mathcal{O}_Z) \longrightarrow (Y, \mathcal{O}_Y)$.

Proposition 3.20 *Let $f\colon Y \to X$ be a morphism of schemes. Then, f is an immersion, resp. a closed immersion, resp. an open immersion iff*

(1) The map $|f|$ is a homeomorphism onto a locally closed subset of $|X|$, resp. a closed subset of $|X|$, resp. an open subset of $|X|$, and

(2) For every $y \in |Y|$, the map $f_y\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection, resp. a surjection, resp. an isomorphism.

Remark: This is proved by doing the following steps:

(1) Reduce to the case where X is affine.

(2) Show that $f_*\mathcal{O}_X$ is QC as $\mathcal{O}_X$-module.

(3) Finish up. $\square$

Remember, every scheme represents the functor of its points (from $\mathcal{SCH}^o$ to **Sets**),

$$T \mapsto X(T) = \mathrm{Hom}_{\mathcal{SCH}}(T, X).$$

If we have a morphism $\varphi\colon Y \to X$, we get a map of sets $\varphi_T\colon Y(T) \to X(T)$, functorial in T. If for all T, this map is an injection, then we say that $\varphi\colon Y \to X$ is a *monomorphism of schemes*, the notion of monomorphism is obviously a categorical notion. Every closed immersion is a monomorphism.

Just as in Chapter 1, we need a condition to replace the missing Hausdorffness of the Zariski topology. This is the familiar notion of separation.

Definition 3.13 A morphism $\varphi\colon Y \to X$ of schemes is a *separated morphism* (or Y *is separated over* X, or Y *is a separated* X*-scheme*) if the diagonal morphism

$$\Delta_{Y/X}\colon Y \longrightarrow Y \prod_X Y$$

is a closed immersion. We say that Y is a *separated scheme* if Y is separated over $\operatorname{Spec}\mathbb{Z}$.

Remarks:

(1) A morphism $f\colon Y \to X$ is a separated morphism iff X has an affine open cover (U_α) so that, if $Y_\alpha = f^{-1}(U_\alpha)$, then $f \restriction Y_\alpha\colon Y_\alpha \to U_\alpha$ is a separated morphism for every α. That is, separation is a local condition on X.

(2) Every monomorphism of schemes is separated.

(3) Every immersion (of any type) is separated.

Proof of (2) and (3). Assume that $i\colon Y \to X$ is a monomorphism. For every test object T, we have

$$(Y \prod_X Y)(T) = Y(T) \prod_{X(T)} Y(T) = \{(\xi, \eta) \mid \xi, \eta\colon T \to Y,\, i \circ \xi = i \circ \eta\}.$$

Since i is a monomorphism, we get $\xi = \eta$. Thus, we have an isomorphism $Y(T) \longrightarrow Y(T) \prod_{X(T)} Y(T)$ *via*

$$\xi \mapsto (\xi, \xi).$$

It follows that $\Delta_{Y/X}$ is an isomorphism; and in particular, it is a closed immersion.

(4) Every morphism $f\colon Y \to X$ from an affine scheme to a scheme is separated, and thus, every morphism of affine schemes is separated. Every affine scheme is separated.

Proof. To prove (4), we need only prove that an open immersion is separated. But an open immersion is automatically a monomorphism; hence, it is separated.

Assume at first that every morphism between affine schemes is separated. Now, any affine scheme is a scheme over $\operatorname{Spec}\mathbb{Z}$, and hence, the morphism from our affine scheme to $\operatorname{Spec}\mathbb{Z}$ is separated. This means exactly that our affine scheme is separated.

If $f\colon Y \to X$ is an arbitrary morphism but Y is affine, then cover X by affines X_α so that Y is covered by the $Y_\alpha = f^{-1}(X_\alpha)$. Now, each $f^{-1}(X_\alpha)$ is an open subscheme of Y and Y is separated. Hence, each $f^{-1}(X_\alpha)$ is itself separated. Therefore, the morphism

$$f^{-1}(X_\alpha) \longrightarrow f^{-1}(X_\alpha) \prod_{X_\alpha} f^{-1}(X_\alpha)$$

is a closed immersion, and our first remark proves that f is a separated morphism. Finally, we are reduced to the case assumed aboev: X and Y are affine. In this case, let $Y = \mathrm{Spec}\, B$ and $X = \mathrm{Spec}\, A$. Then

$$Y \prod_X Y = \mathrm{Spec}(B \otimes_A B).$$

Our map $\Delta = \Delta_{Y/X}$ is the map

$$\mathrm{Spec}\, B \longrightarrow \mathrm{Spec}(B \otimes_A B),$$

given as the morphism corresponding to the multiplication $m\colon B \otimes_A B \to B$. But the algebra map is surjective, and so we get a closed subscheme of $\mathrm{Spec}(B \otimes_A B)$.

(5) An affine morphism is separated (by Remark (1)).

(6) For every scheme Y over X, the morphism $\Delta = \Delta_{Y/X}$ is an immersion. The scheme Y is separated over X iff $\Delta(|Y|)$ is closed in $|Y \prod_X Y|$.

Proof. Cover X by open affines, U_α, and cover each $f^{-1}(U_\alpha)$ by open affines V^α_β. The products $V^\alpha_\beta \prod_X V^\alpha_\beta$ are all open in $Y \prod_X Y$. Yet,

$$V^\alpha_\beta \prod_X V^\alpha_\beta = V^\alpha_\beta \prod_{U_\alpha} V^\alpha_\beta.$$

So, as $\Delta \restriction V^\alpha_\beta$ takes V^α_β into the product $V^\alpha_\beta \prod_{U_\alpha} V^\alpha_\beta$ and the latter image is closed by Remark (4), we see that the image of Δ is closed in the open subscheme

$$\bigcup_{\alpha,\beta} V^\alpha_\beta \prod_X V^\alpha_\beta.$$

On the ring level we already know by the reduction to affine covers that the morphism is surjective. This proves that Δ is an immersion and, of course, it will be a closed immersion iff its image is closed. $\square$

Here is a useful criterion for separation:

Proposition 3.21 *Let X be an affine scheme and $f\colon Y \to X$ a morphism. Then, f is separated iff Y is covered by affine opens U_α so that*

(1) $U_\alpha \cap U_\beta$ *is again affine.*

(2) $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_Y)$ *is generated by the images of* $\Gamma(U_\alpha, \mathcal{O}_Y)$ *and* $\Gamma(U_\beta, \mathcal{O}_Y)$*.*

Proof. Let $\Delta\colon Y \to Y \prod_X Y$. The schemes $U_\alpha \prod_X U_\beta$ form an affine cover of $Y \prod_X Y$, where

$$U_\alpha \prod_X U_\beta = \mathrm{Spec}(B_\alpha \otimes_A B_\beta),$$

with $B_\alpha = \Gamma(U_\alpha, \mathcal{O}_Y)$ and $B_\beta = \Gamma(U_\beta, \mathcal{O}_Y)$. We have

$$\begin{aligned} \Delta^{-1}(U_\alpha \prod_X U_\beta) &= \Delta^{-1}(pr_1^{-1}(U_\alpha) \cap pr_2^{-1}(U_\beta)) \\ &= \Delta^{-1}(pr_1^{-1}(U_\alpha)) \cap \Delta^{-1}(pr_2^{-1}(U_\beta)) \\ &= U_\alpha \cap U_\beta. \end{aligned}$$

Therefore, we get maps

$$\Delta\colon U_\alpha \cap U_\beta \longrightarrow U_\alpha \prod_X U_\beta. \tag{$\dagger_{\alpha\beta}$}$$

This implies separation iff the map on line $(\dagger_{\alpha\beta})$ is a closed immersion for all α, β. But then, if Δ is a closed immersion, the affineness of $U_\alpha \prod_X U_\beta$ implies that $U_\alpha \cap U_\beta$ is affine and the morphism Δ comes from multiplication. Consequently, its ring satisfies (2), as the multiplication is onto.

Conversely, assume that $U_\alpha \cap U_\beta$ is affine and its ring satisfies (2). Then, the map

$$B_\alpha \otimes_A B_\beta \longrightarrow \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_Y)$$

is surjective. So, Δ is a closed immersion. $\square$

Example 3.3

(1) *The affine line* $\mathbb{A}^1$ *with the origin doubled.* Take two copies of $\mathbb{A}^1$, say $\mathrm{Spec}\,\mathbb{C}[T]$ and $\mathrm{Spec}\,\mathbb{C}[S]$. Let $U_1 = \mathrm{Spec}\,\mathbb{C}[T]_{(T)}$ and $U_2 = \mathrm{Spec}\,\mathbb{C}[S]_{(S)}$, and glue them together by sending T to S. The result is a scheme X, but X is not separated. To see this, let $U = \mathrm{Spec}\,\mathbb{C}[T] \hookrightarrow X$, it is affine open, and let $V = \mathrm{Spec}\,\mathbb{C}[S] \hookrightarrow X$, it is another affine open. We have $U \cap V = U_1 = U_2$, and $U \cap V$ is affine, with ring

$$\mathbb{C}[T]_{(T)} = \mathbb{C}[S]_{(S)}.$$

We have an inclusion $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U \cap V, \mathcal{O}_X)$ which maps $\mathbb{C}[T] \longrightarrow \mathbb{C}[T, 1/T]$, and similarly with S. The ring generated by the images is *not* the whole of $\Gamma(U \cap V, \mathcal{O}_X)$, and criterion (2) fails.

(2) Let Y be the affine plane $\mathbb{A}^2$ with the origin doubled. Check that neither (1) nor (2) is true.

Under good conditions on either a morphism or on the schemes themselves, inverse image and direct image of quasi-coherent sheaves and coherent sheaves are again quasi-coherent and coherent. Here is a standard proposition in this situation:

Proposition 3.22 *Let $f\colon Y \to X$ be a morphism of schemes. Then the following properties hold:*

(1) If $\mathcal{G}$ is a QC $\mathcal{O}_X$-module, then $f^*\mathcal{G}$ is a QC $\mathcal{O}_Y$-module.

(2) Assume that X and Y are locally noetherian and $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module. Then, $f^*\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.

(3) If both X and Y are noetherian, or if f is quasi-compact and separated, and if $\mathcal{F}$ is a QC $\mathcal{O}_Y$ module, then the direct image $f_*\mathcal{F}$ of $\mathcal{F}$ is a QC $\mathcal{O}_X$ module.

Proof. First, note that by Hilbert's basis theorem, Y is locally noetherian if f is a finite type (or LFT) morphism and X is locally noetherian.

(1) The sheaf $\mathcal{G}$ is locally of the form

$$\mathcal{O}_X^{(J)} \longrightarrow \mathcal{O}_X^{(I)} \longrightarrow \mathcal{G} \longrightarrow 0.$$

If we pull back this sequence back using f, we get

$$\mathcal{O}_Y^{(J)} \longrightarrow \mathcal{O}_Y^{(I)} \longrightarrow f^*\mathcal{G} \longrightarrow 0$$

on $f^{-1}(U)$, for some small open U in X. This implies that $f^*\mathcal{G}$ is QC.

(2) Locally on X, $\mathcal{G}$ has the form

$$\mathcal{O}_X^{(q)} \longrightarrow \mathcal{O}_X^{(p)} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where p, q are finite, as $\mathcal{O}_X$ is coherent, because X is locally noetherian. If we pull this sequence back, we get

$$\mathcal{O}_Y^{(q)} \longrightarrow \mathcal{O}_Y^{(p)} \longrightarrow f^*\mathcal{G} \longrightarrow 0,$$

which implies that $f^*\mathcal{G}$ is f.p. on Y. But this implies that $f^*\mathcal{G}$ is coherent, as $\mathcal{O}_Y$ is coherent by the local noetherian nature of Y.

(3) The question is local on X (not Y). Thus, we may assume that X is affine, say $X = \operatorname{Spec} A$, where A is noetherian. Then, $Y = f^{-1}(X)$ is finitely covered by affines U_α, as either Y is noetherian or f is quasi-compact. Look at $U_\alpha \cap U_\beta$.

(a) If Y is noetherian then $U_\alpha \cap U_\beta$ is a finite union of opens, $U_{\alpha\beta j}$, for $j = 1, \ldots, t$.

(b) If, instead, f is separated, then $U_\alpha \cap U_\beta$ is affine, which implies that $U_\alpha \cap U_\beta = U_{\alpha\beta 1}$ (in the notation of (a)).

Claim. There is an exact sequence of sheaves

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \prod_\alpha (f \upharpoonright U_\alpha)_*(\mathcal{F} \upharpoonright U_\alpha) \longrightarrow \prod_{\alpha,\beta,j} (f \upharpoonright U_{\alpha\beta j})_*(\mathcal{F} \upharpoonright U_{\alpha\beta j}). \tag{$*$}$$

To se this, pick some open $V \subseteq X$. Then,

$$\begin{aligned} \Gamma(V, f_*\mathcal{F}) &= \Gamma(f^{-1}(V), \mathcal{F}) \\ \Gamma(V \cap U_\alpha, (f \upharpoonright U_\alpha)_*\mathcal{F}) &= \Gamma(f^{-1}(V) \cap U_\alpha, \mathcal{F}) \\ \Gamma(V \cap U_{\alpha\beta j}, (f \upharpoonright U_{\alpha\beta j})_*\mathcal{F}) &= \Gamma(f^{-1}(V) \cap U_{\alpha\beta j}, \mathcal{F}). \end{aligned}$$

However (U_α) is a cover of Y and the $U_{\alpha\beta j}$ cover $U_\alpha \cap U_\beta$. Thus, $(*)$ is just the exact sequence arising from the fact that $f_*\mathcal{F}$ is a sheaf. Since U_α is affine and $f_*\mathcal{F}$ is locally of the form $\widetilde{M}$ as a $\widetilde{A}$-module (where M is a $\Gamma(U_\alpha, \mathcal{O}_Y)$-module), the direct image $(f \upharpoonright U_\alpha)_*\mathcal{F}$ is QC. Similarly, as the $U_{\alpha\beta j}$ are affine, $(f \upharpoonright U_{\alpha\beta j})_*\mathcal{F}$ is QC. Thus, the two right terms are QC, which implies that $f_*\mathcal{F}$ is QC, as the kernel of a map of QC's. $\square$

Remark: The reader should note that we have not proved a statement to the effect that f_* of a coherent module is coherent. Conditions under which this is true are much more delicate and the theorem itself is quite a bit deeper (see Theorem 7.36).

Given two schemes X and Y, where Y is a closed subscheme of X, with closed immersion $i\colon Y \to X$, let

$$\mathfrak{I}_Y = \mathrm{Ker}\,(\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y).$$

Theorem 3.23 *If X and Y are two schemes, where Y is a closed subscheme of X, with closed immersion $i\colon Y \to X$, the sheaf of ideals, $\mathfrak{I}_Y$, of $\mathcal{O}_X$ is QC. Conversely, if $\mathfrak{I}$ is a QC ideal of $\mathcal{O}_X$, then there exists a unique closed subscheme, $X(\mathfrak{I})$, of X whose ideal is $\mathfrak{I}$ (in the above sense). Therefore, the correspondence*

$$Y \to \mathfrak{I}_Y$$

is a bijection between closed subschemes of X and quasi-coherent ideals of $\mathcal{O}_X$. If X is locally noetherian, then $\mathfrak{I}_Y$ is coherent.

Proof. The morphism $i\colon Y \to X$ is quasi-compact (DX) and separated (from previous work). Proposition 3.22 implies that $i_*\mathcal{O}_Y$ is a QC $\mathcal{O}_X$-module. Then, $\mathfrak{I}_Y$, which is the kernel of a map of QC modules is also QC. Now, assume that $\mathfrak{I}$ is a QC $\mathcal{O}_X$-module. Then, $\mathcal{O}_X/\mathfrak{I}$ is a QC $\mathcal{O}_X$-module, and a sheaf of local rings. Let C be the support of the sheaf $\mathcal{O}_X/\mathfrak{I}$, and look locally on X. We know that $\mathcal{O}_{X_\alpha} = \widetilde{A_\alpha}$, and $\mathfrak{I}_\alpha = \widetilde{\mathfrak{A}_\alpha}$, for some ideal, $\mathfrak{A}_\alpha$, of A_α. Thus,

$$\widetilde{A_\alpha/\mathfrak{A}_\alpha} = \mathcal{O}_{X_\alpha}/\mathfrak{I}_\alpha,$$

and

$$\mathrm{supp}(\widetilde{A_\alpha/\mathfrak{A}_\alpha}) = \mathrm{supp}(A_\alpha/\mathfrak{A}_\alpha),$$

which is just $V(\mathfrak{A}_\alpha)$; hence is closed in X_α. Consequently, C is closed in X. Set $X(\mathfrak{J}) = (C, \mathcal{O}_X/\mathfrak{J})$. We get a closed subscheme of X.

Assume now that X is locally noetherian. Then, on some X_α, the ring A_α is a noetherian ring. Therefore, $(\mathfrak{J}_Y)_\alpha$ is a f.g. ideal, and thus, $\mathfrak{J}_Y$ is a f.g. submodule of $\mathcal{O}_X$. Since X is locally noetherian, $\mathcal{O}_X$ is coherent, which implies that $\mathfrak{J}_Y$ is coherent. □

Sometimes, abstract arguments concerning properties of morphisms can help reduce repetitive proofs in more concrete situations. In the following few pages, we shall use exactly this kind of labor-saving (though abstract) device. Let P be some property of morphisms of schemes. Consider the following statements:

(1) Closed immersions have P.

(1′) Immersions have P.

(2) P is stable by composition of morphisms.

(3) P is stable with respect to fibred products of morphisms.

(4) P is stable under arbitrary base extensions.

(5) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms, and assume that $g \circ f$ has P. If g is separated, then f has P.

(5′) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms, and assume that $g \circ f$ has P, where g is arbitrary. Then, f has P.

(6) If f has P, then f_{red} has P (where, f_{red} is the morphism induced by f on the reduced schemes).

Proposition 3.24 *Assume that (1) and (2) (or (1′) and (2)) hold for P. Then, (3) holds iff (4) holds. Assume that (1), (2), (3) (resp. (1′), (2), (3)) hold for P. Then, (5) and (6) hold (resp. (5′) and (6) hold).*

Proof. (3) ⇒ (4). Assume (3). Now, the identity map, 1, on any scheme is a closed immersion. By (1), 1_S has P for every scheme S. If f has P, then $f \prod_Y 1_S$ has P by (3). But the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X \prod_Y S \arrow[l] \arrow[d, "f \prod 1_S"] \\ Y & S \arrow[l] \end{tikzcd}$$

shows that $f \prod 1_S$ is the base extension of f.

(4) $\Rightarrow$ (3). If f, g are given, then

$$f \prod_S g = (f \prod_S 1) \circ (1 \prod_S g),$$

but $f \prod\limits_S 1$ and $1 \prod\limits_S g$ have P by (4). Now, (2) implies that $f \prod\limits_S g$ has P.

(5) $(1, 2, 3) \Rightarrow$ (5). Look at $T \xrightarrow{\varphi} S \longrightarrow \Sigma$, morphisms of schemes. We have the diagram

$$\begin{tikzcd}
T \arrow[r, "\Gamma_\varphi"] \arrow[d, "\varphi"'] & T \prod_\Sigma S \arrow[d, "\varphi \prod_\Sigma 1_S"] \\
S \arrow[r, "\Delta_{S/\Sigma}"'] & S \prod_\Sigma S.
\end{tikzcd}$$

This is a cartesian diagram (DX). The diagonal morphism Δ is always an immersion and if S is separated over Σ, then Δ is a closed immersion. Let P be the property of being an immersion or a closed immersion. Then, ((1) or (1′)) and (2) hold, and (3) also holds (DX). Thus, Γ_φ is an immersion or a closed immersion when S/Σ is separated.

Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$. Now,

$$f = pr_2 \circ \Gamma_f.$$

Assume that g is separated. Then, pr_2 is a base extension of $g \circ f$ in the diagram

$$\begin{tikzcd}
X \arrow[d, "g \circ f"'] & X \prod_Z Y \arrow[l, "pr_1"'] \arrow[d, "pr_2"] \\
Z & Y. \arrow[l, "g"]
\end{tikzcd}$$

By (4) and the hypothesis that $g \circ f$ has P, we get that pr_2 has P. We know that Γ_φ has P by (1) or (1′). By (2), f has P.

(6) Look at the commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Assume that f has P. The vertical arrrows have P since they are closed immersions. Now, (5) implies that f_{red} has P. $\square$

In the course of the above proof, we have also proved:

Corollary 3.25 *For every morphism, $\varphi\colon X \to Y$, the graph morphism, Γ_φ, is an immersion. If X and Y are Z-schemes and Y/Z is separated, then Γ_φ is a closed immersion as a Z-scheme morphism.*

We can apply our abstract situation by letting P be any of the properties: LFT, FT, quasi-compact, locally noetherian, noetherian, quasi-finite, finite, artinian. In these cases, (1) and (2) hold for P, and (3) also holds. Therefore, (4), (5), and (6) also hold for P. More is true when P is the property of being separated. In this case, (1), (1′), (2), (3) hold, and thus, (4), (5), (5′), (6) also hold. However, the converse of (6) is in fact true in this special case:

Corollary 3.26 *A morphism of schemes f is separated iff f_{red} is separated.*

Proof. If f is separated, we have already showed that f_{red} is separated.

Conversely, assume that f_{red} is separated. Look at the diagram

$$\begin{tikzcd}
X_{\text{red}} \arrow[r, "f_{\text{red}}"] \arrow[d] & Y_{\text{red}} \arrow[d] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Going around the top implies that $f \circ (X_{\text{red}} \hookrightarrow X)$ is separated. But $X_{\text{red}} \hookrightarrow X$ is a homeomorphism of $|X_{\text{red}}|$ and $|X|$. Thus, we get the result. $\square$

Even more is true. Let X and Y be schemes and let $f\colon X \to Y$ be a morphism. Assume that $|X|$ is a finite union of closed subspaces, $|X|_k$, and give $|X|_k$ the reduced induced structure. We get a closed subscheme, X_k, of X, with $|X_k| = |X|_k$. Assume further that $|Y|$ is a finite union of closed subsets, $|Y|_k$, make Y_k similarly, and assume that there are morphisms $f \restriction X_k\colon X_k \to Y_k$ (we are assuming that the number of $|X|_k$ and $|Y_k|$ is the same).

For example, we might take X to be a noetherian scheme, and the $|X|_k$ to be the irreducible components of X, and similarly for Y. Write f_k for $f \restriction X_k$.

A scheme is called *integral* if it is reduced and irreducible. The name arises from the affine case, $X = \operatorname{Spec} A$, for then, X is integral iff A is an integral domain (DX).

Proposition 3.27 *Under the above set-up, f is separated iff the f_k are separated for all k. Hence, separation can be checked for integral schemes.*

Proof. Assume that f is separated. We have the commutative diagram

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[d] & Y_k \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

The lefthand side vertical arrow is a closed immersion. This implies that it is separated and thus, going around the bottom is a separated morphism, which implies that the map $X_k \longrightarrow Y$ *via* the top is separated. But $Y_k \longrightarrow Y$ is a closed immersion, hence is separated, and thus, f_k is separated, by (5).

Conversely, assume that every f_k is separated. Going around the top, since f_k is separated, the map $X_k \longrightarrow Y$ is separated. But,

$$pr_1^{-1}(X_k) \cap \Delta_{X/Y}(|X|) = \Delta_{X_k/Y}(|X_k|),$$

hence

$$\Delta_{X/Y}(|X|) = \bigcup_{k=1}^{t} \Delta_{X_k/Y}(|X_k|).$$

However, $\Delta_{X_k/Y}(|X_k|)$ is closed (as f_k is separated), so we find that $\Delta_{X/Y}(|X|)$ is closed. $\square$

We are now in the position to answer the question: Where are the varieties among the schemes?

Let X be a k-variety (use a field $\Omega \supseteq k$ which is algebraically closed and of infinite transcendence degree over k). Then, the following properties hold:

(1) X is a scheme over the field k, i.e., there is a scheme morphism $X \longrightarrow \operatorname{Spec} k$.

(2) X is locally finite type over $\operatorname{Spec} k$.

(3) X is reduced (which means that the rings $\mathcal{O}_{X,x}$ have no nilpotents for all $x \in |X|$).

We also have the k-scheme, $\operatorname{Spec} \Omega$, and we get

$$X(\Omega) = X(\operatorname{Spec} \Omega) = \mathrm{Hom}_{k-\mathrm{schemes}}(\operatorname{Spec} \Omega, X).$$

The following proposition is not hard to prove and is left as an exercise.

Proposition 3.28 *Let* $\mathrm{FFT}_{\mathrm{red}}(k)$ *be the category of schemes over* $\operatorname{Spec} k$ *satisfying (1), (2), (3), as above. Then, the functor*

$$X \mapsto X(\Omega)$$

is an equivalence between the category $\mathrm{FFT}_{\mathrm{red}}(k)$ *and the category of algebraic varieties over* k*.*

Remark: The only slightly tricky thing is to check the correspondence between morphisms, but here, the definitions given in Chapter 1 were explicitly designed to make this checking easier.

Given a field, k, say that X is a *generalized algebraic variety over* k, if X is obtained by gluing schemes of the form $\mathrm{Spec}(k[X_1, \ldots, X_{n_\alpha}]/\mathfrak{A}_\alpha)$, where $\mathfrak{A}_\alpha$ is not necessarily a radical ideal, i.e., we allow nilpotents in rings of generalized varieties.

In keeping with the above point of view of schemes as generalizations of varieties, we can examine LFT-morphisms. Say $f\colon X \to Y$ is an LFT morphism. For any $y \in |Y|$, look at the fibre

$$X_y = X \prod_Y \mathrm{Spec}\, \kappa(y).$$

Note that X_y is LFT over $\mathrm{Spec}\, \kappa(y)$. Therefore, X_y is a generalized algebraic variety over $\kappa(y)$. Thus, an LFT-morphism is exactly an algebraic moving family of generalized algebraic varieties over fields $\kappa(y)$, parametrized by Y.

3.4 Further Readings

Schemes were invented by A. Grothendieck in the late fifties. The first extensive presentation of the theory of schemes appears in Volume I of the *Elements de Géométrie Algébrique* [22], and then in slightly revised form in [30]. The legendary *Elements de Géométrie Algébrique*, known as the "EGA's," was A. Grothendieck's grand project (with the collaboration of Jean Dieudonné) to rewrite the foundations of algebraic geometry in a monumental treatise in twelve chapters. In fact, only the first four chapters were written over a period of eight years (Grothendieck and Dieudonné [22, 30, 23, 24, 25, 26, 27, 28, 29]), comprising a total of $1914 + 466 = 2380$ pages! The material in our Chapter 3 can be found in EGA I [22], Chapter I, and in its revised and expanded version EGA Ib [30] (Chapter I). This material is also discussed extensively in Hartshorne [33], Chapter II. A more informal presentation of schemes can be found in Mumford [43], and more leisurly treatments are given in Eisenbud and Harris [15], Ueno [56], and Shafarevich [54]. Danilov's survey [11] also contains a nice and intuitive introduction to schemes.

Chapter 4

Affine Schemes: Cohomology and Characterization

4.1 Cohomology and the Koszul Complex

In this section, we begin the study of cohomology over an affine scheme. Most of these results originally appeared in Serre's FAC [47]. On (L)RS's, we have three categories of sheaves, each contained in the next:

1. $\mathrm{QCMod}(X) =$ the category of quasi-coherent $\mathcal{O}_X$-modules.
2. $\mathrm{Mod}(X) =$ the category of $\mathcal{O}_X$-modules.
3. $\mathrm{Ab}(X) =$ the category of sheaves of abelian groups.

Every topological space is a ringed space, with sheaf of rings, $\mathcal{O}_X$, the locally constant sheaf, $\mathbb{Z}$, of integers. For this situation, $\mathrm{Mod}(X) = \mathrm{Ab}(X)$.

Assume that X is a ringed space and that $|X|$ has two topologies $|X|_1$ and $|X|_2$, and further assume that $|X|_2$ is coarser than $|X|_1$ (recall, this means that every open of $|X|_2$ is an open of $|X|_1$). Equivalently, the identity $\mathrm{id}\colon |X|_1 \to |X|_2$ is continuous. We would like a ring map $\mathcal{O}_{X_2} \longrightarrow \mathrm{id}_*\mathcal{O}_{X_1}$. Here are two examples:

Example 4.1

(1) Say that $|X|_2$ is X with the Zariski topology, and $|X|_1$ is X with the norm topology (X is supposed to be a scheme over the complex numbers). Then, $\mathcal{O}_{X_2}$ is the ordinary sheaf of germs of functions, and $\mathcal{O}_{X_1}$ is the sheaf of germs of holomorphic functions (in the analytic sense).

(2) Assume that $|X|_2$ is coarser that $|X|_1$, and that $\mathcal{O}_{X_2} = \mathbb{Z}$. Then, $\mathrm{id}\colon |X_1| \to |X_2|$ is continuous, and we have a map $\mathcal{O}_{X_2} \longrightarrow i_*\mathcal{O}_{X_1}$, no matter the sheaf $\mathcal{O}_{X_1}$. The

categories Mod(X) and Ab(X) have enough injectives for every X. If $\mathrm{id}_*(\mathcal{F})$, where $\mathcal{F}$ is injective, is acyclic, then there exists a spectral sequence (Leray, 1945)

$$H_2^p(X, R^q\mathrm{id}_*\mathcal{F}) \Longrightarrow H_1^\bullet(X, \mathcal{F})$$

converging to $H_1^\bullet(X, \mathcal{F})$, for every sheaf $\mathcal{F}$ on $|X|_1$.

If we are in the first situation of Example 4.1, where X is a scheme, then the Leray spectral sequence exists. If further, our spectral sequence degenerates for $\mathcal{F}$, i.e.,

$$R^q\mathrm{id}_*\mathcal{F} = (0) \quad \text{for all } q > 0,$$

we say that the Zariski topology computes the "correct" cohomology of $\mathcal{F}$ (Recall that $R^q\mathrm{id}_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H_1^q(U, \mathcal{F}),$$

where U is a Zariski open).

When we have degeneration, we get the edge isomorphism

$$H^p_{\mathrm{Zar}}(X, \mathrm{id}_*\mathcal{F}) \cong H^p_{\mathrm{norm}}(X, \mathcal{F}) \quad \text{for all } p \geq 0,$$

i.e., we get the following comparison theorem:

$$H^p_{\mathrm{Zar}}(X, \mathcal{F} \restriction \mathrm{Zar}) \cong H^p_{\mathrm{norm}}(X, \mathcal{F}),$$

where, of course, $\mathcal{F} \restriction \mathrm{Zar}$ is another notation for $\mathrm{id}_*\mathcal{F}$. On the right, we have the "correct" cohomology of $\mathcal{F}$, and so cohomology in the Zariski topology indeed computes the "correct" cohomology of $\mathcal{F}$. This is one of Serre's theorems, from GAGA [48]. For a scheme over $\mathrm{Spec}\, \mathbb{C}$ (perhaps quasi-compact) and $\mathcal{F}$ a *quasi-coherent* analytic $\mathcal{O}_X$-module, the spectral sequence does degenerate.

As in Appendix B, there is also Čech cohomology, which even works for preshaves. Again, as in Appendix B, there is the spectral sequence of Čech cohomology

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^\bullet(X, \mathcal{F}) \quad \text{if } \mathcal{F} \text{ is a sheaf.}$$

Remember that

(a) $\mathcal{H}^q(\mathcal{F})$ is the presheaf given by

$$U \mapsto H^q(U, \mathcal{F}),$$

in the sense of derived functor cohomology.

(b) $\mathcal{H}^0(\mathcal{F}) = \mathcal{F}$, because $\mathcal{F}$ is a sheaf.

Frequently in a short exact sequence, cohomological properties of the lefthand term have a profound effect on the situation. Here is a case in point:

Proposition 4.1 *Let $\mathcal{F}$ be a sheaf on a ringed space X. The following statements are equivalent:*

(1) $H^1(X, \mathcal{F}) = (0)$.

(2) $\check{H}^1(X, \mathcal{F}) = (0)$.

(3) Given $\mathcal{G}$ and $\mathcal{G}''$ in $\mathrm{Ab}(X)$, *suppose that* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ *is exact, then*

$$0 \longrightarrow \Gamma(\mathcal{F}) \longrightarrow \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{G}'') \longrightarrow 0 \quad \textit{is also exact.}$$

(3a) Statement (3), but this is true for $\mathcal{F}$, $\mathcal{G}$, $\mathcal{G}''$ being $\mathcal{O}_X$-modules.

Proof. (1) ⇒ (2). From the spectral sequence of Čech cohomology, we get the edge sequence

$$0 \longrightarrow \check{H}^1(X, \mathcal{H}^0(\mathcal{F})) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \ldots$$

However, $\mathcal{H}^0(\mathcal{F}) = \mathcal{F}$ and (1) implies that $H^1(X, \mathcal{F}) = (0)$, and thus, $\check{H}^1(X, \mathcal{F}) = (0)$.

(2) ⇒ (3). We are given an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

in $\mathrm{Ab}(X)$. Cover X by opens U_α, so that $s \in \mathcal{G}''(X) = \Gamma(X, \mathcal{G}'')$, when restricted to each U_α lifts to a section $t_\alpha \in \Gamma(U_\alpha, \mathcal{G})$, which is possible, by exactness. On $U_\alpha \cap U_\beta$, we have the cochain $g_{\alpha\beta} = t_\alpha - t_\beta$. In fact, $g_{\alpha\beta}$ is a cocycle in $\mathcal{F}$, as $g_{\alpha\beta}$ goes to 0 in $\mathcal{G}''$. Refining the cover, we may assume by (2) that $g_{\alpha\beta} = u_\alpha - u_\beta$, where $u_\alpha \in \Gamma(U_\alpha, \mathcal{F})$ and $u_\beta \in \Gamma(U_\beta, \mathcal{F})$. Thus,

$$t_\alpha - t_\beta = u_\alpha - u_\beta \quad \text{on } U_\alpha \cap U_\beta,$$

and it follows that

$$t_\alpha - u_\alpha = t_\beta - u_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

Therefore, $t_\alpha - u_\alpha$ patch to a global section $t \in \mathcal{G}(X)$. Moreover, t goes to s in $\mathcal{G}''$, as u_α goes to 0 for all α. Thus, $\mathcal{G}(X) \longrightarrow \mathcal{G}''(X)$ is surjective, and since Γ is left exact, (3) holds.

(3) ⇒ (1). Take Q to be some injective sheaf containing $\mathcal{F}$ (which exists, since $\mathrm{Ab}(X)$ has enough injectives). Let $\mathcal{G} = Q$ and $\mathcal{G}'' = \mathrm{Coker}\,(\mathcal{F} \to Q)$; we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0.$$

Make the (long) cohomology sequence

$$0 \longrightarrow \Gamma(\mathcal{F}) \longrightarrow \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{G}'') \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) = (0),$$

with $H^1(X, \mathcal{G}) = (0)$, since $\mathcal{G}$ is injective. By (3), we get (1), namely $H^1(X, \mathcal{F}) = (0)$.

Of course, (3) always implies (3a). Now, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, we can repeat the argument that (3) implies (1) with Q an injective $\mathcal{O}_X$-module containing $\mathcal{F}$, and thus, (3a) also implies (1). $\square$

We can apply Proposition 4.1 immediately *viz*:

Proposition 4.2 *Let X be an affine scheme and $\mathcal{F}$ a QC $\mathcal{O}_X$-module. Then $H^1(X, \mathcal{F}) = (0)$, and thus, all of (1)-(3a) of Proposition 4.1 hold.*

Proof. By Yoneda's lemma, we know that $H^1(X, \mathcal{F})$ is isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$, i.e., isomorphic to extension classes of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{$\dagger$}$$

Given $s \in \Gamma(X, \mathcal{O}_X)$, there is some open cover (U_α) where $s_\alpha = s \upharpoonright U_\alpha$ lifts to a section $t_\alpha \in \Gamma(U_\alpha, \mathcal{G})$. Pick $s = 1$. Then, we have $t_\alpha \in \Gamma(U_\alpha, \mathcal{G})$, and t_α goes to 1 in $\Gamma(U_\alpha, \mathcal{O}_X)$. But t_α corresponds to a map $\varphi_\alpha \colon \mathcal{O}_X \upharpoonright U_\alpha \to \mathcal{G} \upharpoonright U_\alpha$ so that

$$\varphi_\alpha(1) = t_\alpha.$$

Therefore, the sequence

$$0 \longrightarrow \mathcal{F} \upharpoonright U_\alpha \longrightarrow \mathcal{G} \upharpoonright U_\alpha \longrightarrow \mathcal{O}_X \upharpoonright U_\alpha \longrightarrow 0 \quad \text{is exact,}$$

and our remarks imply that it splits. Thus,

$$\mathcal{G} \upharpoonright U_\alpha = \mathcal{F} \upharpoonright U_\alpha \coprod \mathcal{O}_X \upharpoonright U_\alpha.$$

Now, $\mathcal{F} = \widetilde{M}$ and $\mathcal{O}_X = \widetilde{A}$ for some module M and some ring A, which implies that

$$\mathcal{G} \upharpoonright U_\alpha = (\widetilde{M \coprod A}) \upharpoonright U_\alpha,$$

and thus, that $\mathcal{G}$ is quasi-coherent, since X is affine. Consequently, $\mathcal{G} = \widetilde{N}$ for some module N, and ($\dagger$) implies that

$$0 \longrightarrow M \longrightarrow N \longrightarrow A \longrightarrow 0 \quad \text{is exact.}$$

Since A is free, this last exact sequence splits, which implies that ($\dagger$) splits. Therefore, the cohomology class of ($\dagger$) is null, and $H^1(X, \mathcal{F}) = (0)$, as desired. $\square$

Remark: We have *not* proved that $H^p(X, \mathcal{F}) = (0)$ for all $p > 0$ and for every QC $\mathcal{O}_X$-module $\mathcal{F}$ (X being affine), because to do so is not a purely categorical matter. It mixes resolving $\mathcal{F}$ by arbitrary injective $\mathcal{O}_X$-modules and the quasi-coherence of $\mathcal{F}$ itself. A module which is quasi-coherent and injective *in the category of QC-modules* need not be injective in the larger category of all modules. Since we wish to prove that $H^p(X, \mathcal{F}) = (0)$ when X is affine, $p > 0$, and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module, we must go around this difficulty. There are several methods available and we choose to use the Koszul resolution because that complex is important in its own right and because the method is perfectly general. As so, we digress to matters of pure algebra:

Let A be a ring and M a module over this ring. The Koszul complex is defined with respect to any given sequence $(f_1, \ldots, f_r)$ of elements of A. We write

$$\overrightarrow{f} = (f_1, \ldots, f_r).$$

Form the graded exterior power $\bigwedge^\bullet A^r$. We make $\bigwedge^\bullet A^r$ into a complex according to the following prescription: Since

$$\bigwedge^\bullet A^r = \coprod_{k=0}^{r} \bigwedge^k A^r,$$

it is a graded module, and we just have to define differentiation. Let $(e_1, \ldots, e_r)$ be the canonical basis of A^r, and set

$$de_j = f_j \in \bigwedge^0 A^r = A,$$

then extend d to be an antiderivation. That is, extend d *via*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

For example,

$$d(e_i \wedge e_j) = f_i e_j - f_j e_i,$$

and

$$\begin{aligned} d(e_i \wedge e_j \wedge e_k) &= d(e_i \wedge e_j) \wedge e_k + (e_i \wedge e_j) \wedge de_k \\ &= (f_i e_j - f_j e_i) \wedge e_k + f_k (e_i \wedge e_j) \\ &= f_i \widehat{e_i} \wedge e_j \wedge e_k - f_j e_i \wedge \widehat{e_j} \wedge e_k + f_k e_i \wedge e_j \wedge \widehat{e_k}, \end{aligned}$$

where, as usual, the hat above a symbol means that this symbol is omitted. By an easy induction, we get the formula:

$$d(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{t} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_t}.$$

We denote this complex by $K_\bullet(\overrightarrow{f})$, i.e., it is the graded module $\bigwedge^\bullet A^r$ with the antiderivation d that we just defined. This is the *Koszul complex*.

Given an A-module M, we can make two Koszul complexes for the module M, namely:

$$\begin{aligned} K_\bullet(\overrightarrow{f}, M) &= K_\bullet(\overrightarrow{f}) \otimes_A M, \\ K^\bullet(\overrightarrow{f}, M) &= \mathrm{Hom}_A(K_\bullet(\overrightarrow{f}), M). \end{aligned}$$

We can take the homology and the cohomology respectively of these complexes, and we get the modules
$$H_\bullet(\overrightarrow{f}, M) \quad \text{and} \quad H^\bullet(\overrightarrow{f}, M).$$
For the cohomology complex, we need the explicit form of δ. Now,
$$K^t(\overrightarrow{f}, M) = \mathrm{Hom}_A(\bigwedge^t A^r, M),$$
and the family of elements of the form
$$e_{i_1} \wedge \cdots \wedge e_{i_t} \quad \text{with } 1 \leq i_1 < i_2 < \ldots < i_t \leq r,$$
is a basis of $\bigwedge^t A^r$; thus, $\mathrm{Hom}_A(\bigwedge^t A^r, M)$ is isomorphic to the set of alternating functions, g, from the set of ordered increasing sequences $(i_1, \ldots, i_t)$ of length t in $\{1, \ldots, r\}$ to M. Thus, the coboundary δ is given (on elements $g \in \mathrm{Hom}_A(\bigwedge^t A^r, M)$) by
$$(\delta g)(i_1, \ldots i_{t+1}) = \sum_{j=1}^{t+1} (-1)^{j-1} f_{i_j} g(i_1, \ldots, \widehat{i_j}, \ldots, i_{t+1}).$$

We have $H^0(\overrightarrow{f}, M) = Z^0(\overrightarrow{f}, M) = \mathrm{Ker}\,\delta$. (Note that $K^0(\overrightarrow{f}, M) = M$, *via* the map $g \mapsto g(1)$.) Then,
$$\delta g(e_i) = f_i g(1) = f_i m,$$
so $\delta f = 0$ implies that $f_i m = 0$ for all i. We find that
$$H^0(\overrightarrow{f}, M) = \{m \in M \mid \mathfrak{A} m = 0\},$$
where $\mathfrak{A}$ is the ideal generated by $\{f_1, \ldots, f_r\}$. Also, it is clear that
$$H^t(\overrightarrow{f}, M) = 0 \quad \text{if } t < 0 \text{ or } t > r.$$
Let us compute the top cohomology group $H^r(\overrightarrow{f}, M)$. We have
$$Z^r(\overrightarrow{f}, M) = K^r(\overrightarrow{f}, M) = \mathrm{Hom}_A(\bigwedge^r A^r, M) = M,$$
via the map $g \mapsto g(e_1 \wedge \cdots \wedge e_r)$. Now, $\mathrm{Im}\,\delta_{r-1} = B^r(\overrightarrow{f}, M)$, but what is $B^r(\overrightarrow{f}, M)$? If $g \in K^{r-1}(\overrightarrow{f}, M)$ is an alternating function on $i_1, \ldots, i_{r-1}$, then
$$\delta_{r-1} g(1, \ldots, r) = (\delta_{r-1} g)(e_1 \wedge \cdots \wedge e_r) = \sum_{j=1}^{r} (-1)^{j-1} f_j g(1, \ldots, \widehat{j}, \ldots, r).$$
Therefore,
$$B^r = f_1 M + \cdots + f_r M,$$

and we find that

$$H^r(\overrightarrow{f}, M) = M/(f_1M + \cdots + f_rM) = M/\mathfrak{A}M.$$

It is important to connect the Koszul homology and cohomology *via* the notion of *Koszul duality*. This is the following: Consider $K_t(\overrightarrow{f}, M)$, an element of $K_t(\overrightarrow{f}, M)$ has the form

$$h = \sum e_{i_1} \wedge \cdots \wedge e_{i_t} \otimes z_{i_1 \ldots i_t}, \quad \text{where } 1 \leq i_1 < i_2 < \ldots < i_t \leq r.$$

We define a map (the duality map)

$$\Theta\colon K_t(\overrightarrow{f}, M) \longrightarrow K^{r-t}(\overrightarrow{f}, M)$$

as follows: Pick $j_1 < j_2 < \cdots < j_{r-t}$, and set

$$\Theta(h)(j_1, \ldots, j_{r-t}) = \epsilon z_{i_1 \ldots i_t},$$

where

(α) $i_1, \ldots, i_t$ is the set of complementary indices of $j_1, \ldots, j_{r-t}$ taken in ascending order,

(β) ϵ is the sign of the permutation

$$(1, 2, \ldots, r) \mapsto (i_1, \ldots, i_t, j_1, \ldots, j_{r-t}),$$

where both $i_1, \ldots, i_t$ and $j_1, \ldots, j_{r-t}$ are in ascending order.

We find (DX) that

$$\Theta(\partial h) = \delta\Theta(h),$$

where ∂ is the obvious map induced on $H_\bullet(\overrightarrow{f}, M)$ by d on $H_\bullet(\overrightarrow{f})$. So, the isomorphism, Θ, induces an isomorphism

$$H_t(\overrightarrow{f}, M) \cong H^{r-t}(\overrightarrow{f}, M) \quad \text{for all } t \geq 0,$$

which is called *Koszul duality*.

We need one more definition to exhibit the main algebraic property of the Koszul complex.

Definition 4.1 The sequence $\overrightarrow{f} = (f_1, \ldots, f_r)$ is *regular* for M if for every i, with $1 \leq i \leq r$, the map

$$z \mapsto f_i z$$

is an injection of $M/(f_1M + \cdots + f_{i-1}M)$ to itself.

Proposition 4.3 *(Koszul) Let M be an A-module and let $\overrightarrow{f}$ be a regular sequence of length r for M. Then, $H^i(\overrightarrow{f}, M) = (0)$ if $i \neq r$.*

Proof. By Koszul duality, we have to prove that $H_t(\overrightarrow{f}, M) = (0)$, for all $t > 0$. We proceed by induction on r. For $r = 0$, there is nothing to prove and the proposition holds trivially. Let $\overrightarrow{f'} = (f_1, \ldots, f_{r-1})$, and write $L_\bullet = K_\bullet(\overrightarrow{f'}, M)$. Note that $\overrightarrow{f'}$ is regular for M. By the induction hypothesis,

$$H_p(\overrightarrow{f'}, M) = H_p(L_\bullet) = (0) \quad \text{for all } p > 0.$$

Let

$$K_\bullet = (K_t(f_r)) = \begin{cases} A & \text{if } t = 0, 1, \\ (0) & \text{otherwise,} \end{cases}$$

a complex with two terms. The differentiation, d, is given by

$$de = f_r,$$

where $e = 1$ in $A = A^1$. Now, make the complex

$$K_\bullet \otimes_A L_\bullet.$$

(Recall that if $C_\bullet$ and $D_\bullet$ are two complexes of A-modules bounded below by 0, then $C_\bullet \otimes D_\bullet$ is the complex defined by

$$(C_\bullet \otimes D_\bullet)_t = \coprod_{i+j=t} C_i \otimes D_j,$$

and in which differentiation is given by

$$d(\alpha \otimes \beta) = d_{C_\bullet}(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes d_{D_\bullet}(\beta)).$$

The reader should check that

$$L_\bullet \otimes_A K_\bullet = K_\bullet(\overrightarrow{f}, M).$$

(The reader should also check that, in general,

$$K_\bullet(\overrightarrow{f}) = K_\bullet(f_1) \otimes \cdots \otimes K_\bullet(f_r).)$$

Claim. For every $p \geq 0$, there is an exact sequence

$$0 \longrightarrow H_0(K_\bullet \otimes H_p(L_\bullet)) \longrightarrow H_p(K_\bullet \otimes L_\bullet) \longrightarrow H_1(K_\bullet \otimes H_{p-1}(L_\bullet)) \longrightarrow 0. \qquad (*)$$

First, assume the claim. If $p \geq 2$, then $p - 1 \geq 1$, and so,

$$H_{p-1}(L_\bullet) = H_p(L_\bullet) = (0).$$

Thus,

$$H_p(K_\bullet \otimes L_\bullet) = H_p(\overrightarrow{f}, M) = (0) \quad \text{for } p \geq 2.$$

When $p = 1$, we have $H_1(L_\bullet) = (0)$, and the exact sequence $(*)$ yields

$$H_1(\overrightarrow{f}, M) = H_1(K_\bullet \otimes H_0(L_\bullet)).$$

By Koszul duality,

$$H_0(L_\bullet) = H^{r-1}(L_\bullet) = M/(f_1M + \cdots + f_{r-1}M),$$

and

$$H_1(K_\bullet \otimes H_0(L_\bullet)) = H^0(K_\bullet \otimes H_0(L_\bullet)).$$

Now, the latter module is the kernel of multiplication by f_r on $M/(f_1M + \cdots + f_{r-1}M)$, which, by the assumption of regularity, is zero. We obtain

$$H_1(\overrightarrow{f}, M) = (0).$$

It only remains to prove our claim. There are two proofs of a general cohomological lemma establishing that $(*)$ is exact.

Lemma 4.4 *Let $K_\bullet$ be a complex of A-modules, and assume that*

(a) $K_l = (0)$ if $l \neq 0$ or $l \neq 1$, and

(b) K_0 and K_1 are free A-modules.

Then, for any complex, $L_\bullet$, of A-modules, we have $(*)$ for all p.

The first proof uses the general homological Künneth formula, since the modules are free (see Godement [18], Chapter 5, Section 5): There is a spectral sequence with E^2-term

$$E^2_{pq} = H_p(K_\bullet \otimes H_q(L_\bullet))$$

which converges to $H_\bullet(K_\bullet \otimes L_\bullet)$. On its lines of lowest degree, this spectral sequence gives the "zipper sequence" $(*)$.

The second proof proceeds as follows: Make the complexes K_0 and K_1, in which K_i has one term of degree i and d is the trivial differentiation. By freeness, we have the exact sequence of complexes

$$0 \longrightarrow K_0 \otimes L_\bullet \longrightarrow K_\bullet \otimes L_\bullet \longrightarrow K_1 \otimes L_\bullet \longrightarrow 0.$$

From this, we get the long exact homology sequence:

$$\longrightarrow H_{p+1}(K_1 \otimes L_\bullet) \xrightarrow{\partial} H_p(K_0 \otimes L_\bullet) \longrightarrow H_p(K_\bullet \otimes L_\bullet) \longrightarrow H_p(K_1 \otimes L_\bullet) \longrightarrow \cdots.$$

However, we have:

$$H_p(K_0 \otimes L_\bullet) = K_0 \otimes H_p(L_\bullet)$$

and

$$H_p(K_1 \otimes L_\bullet) = K_1 \otimes H_{p-1}(L_\bullet),$$

and $\partial = d_K \otimes 1$. Therefore, we get $(*)$. $\square$

4.2 Connection With Geometry; Cartan's Isomorphism Theorem

Having understood the Koszul complex in the abstract, let us apply it to the computation of cohomology in geometric situations. To do this, first take another sequence $(g_1, \ldots, g_r)$ and make

$$\overrightarrow{fg} = (f_1 g_1, \ldots, f_r g_r).$$

I claim that the map

$$K_\bullet(\overrightarrow{fg}) \longrightarrow K_\bullet(\overrightarrow{f})$$

induced by the map

$$\varphi_{\overrightarrow{g}} \colon (\xi_1, \ldots, \xi_r) \mapsto (g_1\xi_1, \ldots, g_r\xi_r)$$

is a chain map. From $\varphi_{\overrightarrow{g}}$, we obtain the map

$$\wedge^\bullet \varphi_{\overrightarrow{g}} \colon \bigwedge^\bullet A^r \longrightarrow \bigwedge^\bullet A^r,$$

namely,

$$e_{i_1} \wedge \cdots \wedge e_{i_t} \mapsto g_{i_1} \cdots g_{i_t} e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Now,

$$d_{\overrightarrow{fg}}(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{t} (-1)^{j-1} f_{i_j} g_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_t}$$

and

$$\begin{aligned}
\varphi_{\overrightarrow{g}}(d_{\overrightarrow{fg}}(e_{i_1} \wedge \cdots \wedge e_{i_t})) &= \sum_{j=1}^{t} (-1)^{j-1} f_{i_j} (g_{i_1} \cdots g_{i_t}) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_t} \\
&= (g_{i_1} \cdots g_{i_t}) \sum_{j=1}^{t} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_t} \\
&= (g_{i_1} \cdots g_{i_t}) d_{\overrightarrow{f}}(e_{i_1} \wedge \cdots \wedge e_{i_t}) \\
&= d_{\overrightarrow{f}}(\varphi_{\overrightarrow{g}}(e_{i_1} \wedge \cdots \wedge e_{i_t})),
\end{aligned}$$

as contended.

Pick s, t such that $0 < s < t$, if we write $\overrightarrow{f^p} = (f_1^p, \ldots, f_r^p)$, then we get a map

$$\varphi_{\overrightarrow{f^{t-s}}} \colon K_\bullet(\overrightarrow{f^t}) \longrightarrow K_\bullet(\overrightarrow{f^s}).$$

However, what we really need is the cochain complex map

$$\varphi^\bullet_{\overrightarrow{f^{t-s}}}(M) \colon K^\bullet(\overrightarrow{f^s}, M) \longrightarrow K^\bullet(\overrightarrow{f^t}, M),$$

which gives us an inductive system of cochain complexes. Therefore, we can take the inductive limit

$$\varinjlim_t K^\bullet(\overrightarrow{f^t}, M),$$

which we denote by $C^\bullet((\overrightarrow{f}), M)$. Now, taking an inductive limit is an exact functor, and therefore, $\varinjlim$ commutes with cohomology, which implies that if we denote by $H^\bullet((\overrightarrow{f}), M)$ the cohomology of $C^\bullet((\overrightarrow{f}), M)$, then

$$\begin{aligned} H^\bullet((\overrightarrow{f}), M) &= \varinjlim_t \text{ cohomology of } K^\bullet(\overrightarrow{f^t}, M) \\ &= \varinjlim_t H^\bullet(\overrightarrow{f^t}, M). \end{aligned}$$

For the applications to geometry, the acyclicity of our complexes is an essential feature. This acyclicity follows most readily from the construction of a "contracting homotopy." It is to this construction that we now turn: Pick some g_i, with $i = 1, \ldots, r$, and consider the map

$$E_{g_\bullet} \colon K_\bullet(\overrightarrow{f}) \longrightarrow K_\bullet(\overrightarrow{f}),$$

defined *via*

$$E_{g_\bullet}(z) = \left(\sum_{j=1}^r g_j e_j\right) \wedge z.$$

In particular, we have

$$E_g(e_p) = \left(\sum_{j=1}^r g_j e_j\right) \wedge e_p = \sum_{j=1}^r g_j(e_j \wedge e_p).$$

Observe that in each degree, E_g raises degrees by one. Look at $d \circ E_g + E_g \circ d$, where d is one of the differentials $d_{\overrightarrow{f}}$ for some sequence $\overrightarrow{f}$. For example, look at the effect of this map on e_p. We have

$$(d \circ E_g)(e_p) = d\left(\sum_{j=1}^r g_j(e_j \wedge e_p)\right) = \sum_{j=1}^r g_j(f_j e_p - f_p e_j),$$

and

$$(E_g \circ d)(e_p) = E_g(f_p \cdot 1) = f_p E_g(1) = f_p \left(\sum_{j=1}^r g_j e_j\right).$$

Thus, we have

$$(d \circ E_g + E_g \circ d)(e_p) = \left(\sum_{j=1}^r g_j f_j\right) e_p,$$

which means that

$$d \circ E_g + E_g \circ d = \left(\sum_{j=1}^{r} g_j f_j\right) \mathrm{id} \quad \text{on } K_1(\overrightarrow{f}).$$

Of course, the reader should now realize that

$$d \circ E_g + E_g \circ d = \left(\sum_{j=1}^{r} g_j f_j\right) \mathrm{id} \quad \text{on } K_t(\overrightarrow{f}) \text{ for all } t \geq 0.$$

Consequently, if there exist $g_1, \ldots, g_r$ so that

$$\sum_{j=1}^{r} g_j f_j = 1,$$

then,

$$d \circ E_g + E_g \circ d = \mathrm{id},$$

and so, E_g is the required contracting homotopy. This yields the following proposition:

Proposition 4.5 *If $(f_1, \ldots, f_r)$ generate the unit ideal of A, then for all modules M, the complexes*

$$K_\bullet(\overrightarrow{f^t}), \quad K_\bullet(\overrightarrow{f^t}, M), \quad K^\bullet(\overrightarrow{f^t}, M), \quad C^\bullet((\overrightarrow{f^t}), M)$$

have trivial (co) homology in all dimensions, even 0.

(Note that if $f_1, \ldots f_r$ generate the unit ideal, then $f_1^t, \ldots f_r^t$ also generate the unit ideal.)

The applications to geometry of the Koszul complex follows by its connection with Čech cohomology. The set-up is as follows: We have a scheme, X, and a sheaf, $\mathcal{F}$, of $\mathcal{O}_X$-modules which is QC, and we let $A = \Gamma(X, \mathcal{O}_X)$ and $M = \Gamma(X, \mathcal{F})$.

Pick $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X) = A$, and write X_{f_i} for the open in X where $f_i \neq 0$.

Beware that X_{f_i} need not be affine.

Let

$$U = \bigcup_{j=1}^{r} X_{f_j},$$

and write $\{U_j \to U\}$ for the covering of U by the $U_j = X_{f_j}$. We also need a finiteness hypothesis on X:

Definition 4.2 A scheme X is *nerve-finite* if the following conditions hold:

(1) For all affine opens U, V of X, the open $U \cap V$ is quasi-compact.

(2) The scheme X is covered by finitely many affine opens U_α.

Important nerve-finite schemes are:

(1) Those whose underlying space, $|X|$, is noetherian: i.e., $|X|$ has the DCC on closed subspaces.

(2) Those whose underlying space, $|X|$, is quasi-compact and for which X is separated.

We can augment Theorem 3.8 (the characterization of quasi-compact sheaves on affine schemes) to those schemes which are nerve-finite by effectively repeating the relevant parts of its proof–since the proof of Serre's conditions in that theorem use only nerve-finiteness. This yields the following proposition whose proof we leave to the reader:

Proposition 4.6 *Let X be a nerve-finite scheme and let $\mathcal{F}$ be a QC $\mathcal{O}_X$-module. Then, for every $g \in \Gamma(X, \mathcal{O}_X) = A$, the following properties hold:*

(a) If $\sigma \in \Gamma(X, \mathcal{F})$ and $\sigma \restriction X_g = 0$ in $\Gamma(X_g, \mathcal{F})$, there is some $n > 0$ so that $g^n \sigma = 0$.

(b) Given $\sigma \in \Gamma(X_g, \mathcal{F})$, there is some $n > 0$ so that $g^n\sigma$ lifts to a section $s \in \Gamma(X, \mathcal{F})$.

(c) $\Gamma(X_g, \mathcal{F}) = M_g$ (recall, $M = \Gamma(X, \mathcal{F})$).

Take a nerve-finite scheme, X, with a quasi-coherent sheaf, $\mathcal{F}$, and let A and M be as above. Define

$$U_{i_0 \ldots i_t} = X_{f_{i_0} \cdots f_{i_t}} = \bigcap_{j=1}^{t} X_{f_{i_j}},$$

by Proposition 4.6(c),

$$\Gamma(U_{i_0 \ldots i_t}, \mathcal{F}) = M_{f_{i_0} \cdots f_{i_t}}.$$

Observe that we can define $M_{f_{i_0} \cdots f_{i_t}}$ as an inductive limit. Namely, set $M^{(n)} = M$ for all $n \geq 0$, and write

$$\varphi_m^n \colon M^{(m)} \longrightarrow M^{(n)}$$

for the map

$$\xi \mapsto (f_{i_0} f_{i_1} \ldots f_{i_t})^{n-m} \xi.$$

We map $M^{(n)}$ to $M_{f_{i_0} \cdots f_{i_t}}$ *via*

$$\xi \mapsto \frac{\xi}{(f_{i_0} f_{i_1} \ldots f_{i_t})^n},$$

then, one easily sees that

$$M_{f_{i_0} \cdots f_{i_t}} = \varinjlim_n (M^{(n)}, \varphi_m^n).$$

Let $C_n^p(M)$ denote the set of all alternating maps, g, such that

$$g \colon (i_0, \ldots, i_p) \mapsto M^{(n)}, \quad \text{where } 1 \leq i_0 < \ldots < i_p \leq r.$$

We have the isomorphism

$$C^p(\{U_j \to U\}, \mathcal{F}) \cong \varinjlim_n C_n^p(M),$$

since the lefthand side consists of alternating maps to $\Gamma(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}) = M_{f_{i_0} \cdots f_{i_t}}$, which is just the righthand side. However, there is a bijection between the collection of alternating maps from $(p+1)$-tuples, $(i_0, \ldots, i_p)$ to $M^{(n)}$, and maps from the wedges $e_{i_0} \wedge \cdots \wedge e_{i_p}$ to the same module $M^{(n)}$. Consequently, we find a bijection

$$C_n^p(M) \longrightarrow K^{p+1}(\overrightarrow{f^n}, M),$$

which takes the map φ_m^n to multiplication by $\overrightarrow{f^{n-m}}$. Thus, we get the isomorphism

$$C^p(\{U_j \to U\}, \mathcal{F}) \cong \varinjlim_n C_n^p(M) \cong \varinjlim_n K^{p+1}(\overrightarrow{f^n}, M) \cong K^{p+1}((\overrightarrow{f}), M).$$

This is a chain map. From this map, we obtain the following proposition due to Serre (see FAC [47]):

Proposition 4.7 *If X is a nerve-finite scheme and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module, and if we choose $f_1, \ldots, f_r$ in $\Gamma(X, \mathcal{O}_X)$ and write $U_j = X_{f_j}$ and $U = \bigcup_j^t U_j$, then:*

(1) There is a chain isomorphism

$$C^p(\{U_j \to U\}, \mathcal{F}) \cong K^{p+1}((\overrightarrow{f}), M) \quad \textit{functorial in } \mathcal{F},$$

where $M = \Gamma(X, \mathcal{F})$,

(2) There is an isomorphism

$$H^p(\{U_j \to U\}, \mathcal{F}) \cong H^{p+1}((\overrightarrow{f}), M) \quad \textit{for all } p \geq 1, \textit{ functorial in } \mathcal{F},$$

and finally,

(3) There is a functorial exact sequence

$$0 \longrightarrow H^0((\overrightarrow{f}), M) \longrightarrow M \longrightarrow H^0(\{U_j \to U\}, \mathcal{F}) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0.$$

Proof. Assertion (1) is exactly what was proved above.

For (2), consider the following diagram (for $p \geq 1$)

$$\begin{tikzcd}
C^{p-1}(\{U_j \to U\} \arrow[r, "\sim"] \arrow[d, "\delta"'] & K^p((\overrightarrow{f}), M) \arrow[d, "d_p"] \\
C^p(\{U_j \to U\}, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\delta"'] & K^{p+1}((\overrightarrow{f}), M) \arrow[d, "d_{p+1}"] \\
C^{p+1}(\{U_j \to U\}, \mathcal{F}) \arrow[r, "\sim"] & K^{p+2}((\overrightarrow{f}), M).
\end{tikzcd}$$

By (1), it is commutative, and so, (2) follows.

(3) For $p = 0$, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K^0((\overrightarrow{f}), M) \arrow[d, "d_0"] \\
C^0(\{U_j \to U\}, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\delta"'] & K^1((\overrightarrow{f}), M) \arrow[d, "d_1"] \\
C^1(\{U_j \to U\}, \mathcal{F}) \arrow[r, "\sim"] & K^2((\overrightarrow{f}), M).
\end{tikzcd}$$

However,

$$H^0(\{U_j \to U\}, \mathcal{F}) = \mathrm{Ker}\, \delta \cong \mathrm{Ker}\, d_1 = Z^1((\overrightarrow{f}), M),$$

and we have the exact sequence

$$0 \longrightarrow B^1((\overrightarrow{f}), M) \longrightarrow Z^1((\overrightarrow{f}), M) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0,$$

by definition of cohomology. Thus,

$$0 \longrightarrow B^1((\overrightarrow{f}), M) \longrightarrow H^0(\{U_j \to U\}, \mathcal{F}) \longrightarrow H^1((\overrightarrow{f}), \mathcal{F}) \longrightarrow 0 \quad \text{is exact.}$$

Now,

$$0 \longrightarrow \mathrm{Ker}\, d_0 \longrightarrow K^0((\overrightarrow{f}), M) \xrightarrow{d_0} B^1((\overrightarrow{f}), M) \longrightarrow 0, \quad \text{is exact,}$$

and $\mathrm{Ker}\, d_0 = Z^0((\overrightarrow{f}), M) = H^0((\overrightarrow{f}), M)$, while $K^0((\overrightarrow{f}), M) = M$. The desired result is obtained by splicing the two exact sequences. $\square$

Remarks:

1. Later on, we will see that $H^p((\overrightarrow{f}), M)$ depends only on the ideal, $\mathfrak{A}$, generated by $f_1, \ldots, f_r$. If X is affine and Y is the subscheme $V(\mathfrak{A}) = \mathrm{Spec}(A/\mathfrak{A})$, then this cohomology is just $H^p_Y(X, \mathcal{F})$, the so-called *local cohomology of X in $\mathcal{F}$ along Y*.

2. All the functors $H_\bullet(\overrightarrow{f^t}, M)$, $H^\bullet(\overrightarrow{f^t}, M)$, $H^\bullet((\overrightarrow{f}), M)$, are δ-functors.

Having established that the limiting Koszul cohomology is just the Čech cohomology of the covering $\{U_j \to U\}$, we now need a result due to Henri Cartan to make the final application to the (derived functor) cohomology of affine schemes. This comes about by relating Čech cohomology to the derived functor cohomology. As usual, we have the Čech cohomology spectral sequence, denoted $\check{\text{S}}\text{S}$:

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^\bullet(X, \mathcal{F}),$$

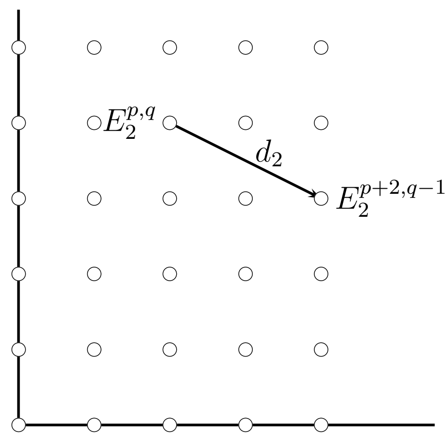

Figure 4.1: The second level of a spectral sequence

where X is a ringed spaced and $\mathcal{F}$ is a sheaf of abelian groups. Recall that $\mathcal{H}^q(\mathcal{F})$ is the presheaf defined by

$$\Gamma(U, \mathcal{H}^q(\mathcal{F})) = H^q(U, \mathcal{F}),$$

where $H^q(U, \mathcal{F})$ is the cohomology of $\mathcal{F}$ on the open set U defined via derived functors, resolutions, etc.

Consider our spectral sequence whose second level is illustrated in Figure 4.1. Observe that $\mathrm{Ker}\, d_2^{p,0} = Z_3^{p,0} = E_2^{p,0}$. Thus, we get the surjection

$$H(E_2^{p,0}) \longrightarrow E_3^{p,0}.$$

If we repeat this argument with d_3, d_4, etc., we deduce that there is a surjection

$$E_2^{p,0} \longrightarrow E_\infty^{p,0} \longrightarrow 0.$$

We know that $H^r(X, \mathcal{F})$ is filtered by subgroups $H^r(X, \mathcal{F})_p$, and

$$H^r(X, \mathcal{F}) = H^r(X, \mathcal{F})_0 \supseteq H^r(X, \mathcal{F})_1 \supseteq \cdots \supseteq H^r(X, \mathcal{F})_r \supseteq (0),$$

because $E_\infty^{p,r-p}$ is isomorphic to $H^r(X, \mathcal{F})_p / H^r(X, \mathcal{F})_{p+1}$ and $E_\infty^{p,r-p} = (0)$ if $p > r$. If we set $p = r$, we get

$$E_\infty^{r,0} \cong H^r(X, \mathcal{F})_r \hookrightarrow H^r(X, \mathcal{F}).$$

Thus, we get the canonical map from $\check{H}^r(X, \mathcal{F})$ to $H^r(X, \mathcal{F})$ given by the composition

$$\check{H}^r(X, \mathcal{F}) = \check{H}^r(X, \mathcal{H}^0(\mathcal{F})) = E_2^{r,0} \longrightarrow E_\infty^{r,0} \hookrightarrow H^r(X, \mathcal{F}).$$

This map is neither injective nor surjective in general.

Lemma 4.8 *Let X be a ringed space and $\mathcal{G}$ be a presheaf of abelian groups on X. If the sheaf, $\mathcal{G}^\sharp$, generated by $\mathcal{G}$ is zero, then*

$$\check{C}^0(X, \mathcal{G}) = \check{H}^0(X, \mathcal{G}) = (0).$$

If $\mathcal{F}$ is any presheaf of abelian groups, then the presheaf $\mathcal{H}^q(\mathcal{F})$ generates the zero sheaf if $q > 0$. Thus,

$$E_2^{0,q} = H^0(X, \mathcal{H}^q(\mathcal{F})) = (0) \quad \textit{for all } q > 0.$$

Proof. It suffices (for the first statement) to prove that $\check{C}^0(X, \mathcal{G}) = (0)$. Pick $\xi \in \check{C}^0(X, \mathcal{G})$. Then, there is an open cover $\{U_\alpha \to X\}$ and ξ arises from this cover. For each $x \in X$, we have

$$\mathcal{G}_x = \mathcal{G}_x^\sharp = (0),$$

by hypothesis. Hence, if $x \in U_\alpha$, there is some open, $V_x \subseteq U_\alpha$, with $x \in V_x$ such that $\xi \restriction V_x = 0$. Therefore, as the cover $\{V_x \to X\}$ refines $\{U_\alpha \to X\}$, and as $\xi = 0$ in $\{V_x \to X\}$, we find that $\xi = 0$ in $\check{C}^0(X, \mathcal{G})$.

Recall that $\mathcal{H}^q(\mathcal{F})$ is the presheaf given by

$$U \mapsto H^q(U, \mathcal{F}),$$

where U is any open subset of X. Now, we have the Godement canonical resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(\mathcal{F}) \longrightarrow C^1(\mathcal{F}) \longrightarrow \cdots, \tag{$*$}$$

which is obtained by an inductive construction where we begin with the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(\mathcal{F}) \longrightarrow \mathrm{Coker}^{\,0}(\mathcal{F}) \longrightarrow 0.$$

Here, $C^0(\mathcal{F})$ is the sheaf whose sections over an open U consists in $\prod_{x \in U} \mathcal{F}_x$. Of course, $\mathrm{Coker}^{\,0}(\mathcal{F})$ is the cokernel presheaf arising from of the injection $\mathcal{F} \longrightarrow C^0(\mathcal{F})$. Next, one repeat the above with $\mathcal{F}$ replaced by $\mathrm{Coker}^{\,0}(\mathcal{F})$ and sets $C^1(\mathcal{F}) = C^0(\mathrm{Coker}^{\,0}(\mathcal{F}))$, and so on. Therefore, for any $p > 0$, we have

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dr] & & 0 \\
& \mathrm{Coker}^{\,p-1}(\mathcal{F}) \arrow[ur] \arrow[dr] & & \mathrm{Coker}^{\,p}(\mathcal{F}) \arrow[ur] \arrow[dr] & \\
C^{p-1}(\mathcal{F}) \arrow[ur] \arrow[rr, "d^{p-1}"] & & C^p(\mathcal{F}) \arrow[ur] \arrow[rr, "d^p"] & & C^{p+1}(\mathcal{F}).
\end{tikzcd}$$

Applying the functor $\Gamma(U, -)$ to the Godement resolution $(*)$, we get a complex whose cohomology is $H^\bullet(U, \mathcal{F})$. The diagram

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dr] & & \\
& \mathrm{Coker}^{\,p-1}\mathcal{F}(U) \arrow[dr] & & \mathrm{Coker}^{\,p}\mathcal{F}(U) \arrow[dr] & \\
C^{p-1}\mathcal{F}(U) \arrow[ur] \arrow[rr, "d^{p-1}(U)"] & & C^p\mathcal{F}(U) \arrow[ur] \arrow[rr, "d^p(U)"] & & C^{p+1}\mathcal{F}(U)
\end{tikzcd}$$

and the left-exactness of $\Gamma(U, -)$ shows that

$$Z^p(U, \mathcal{F}) = \mathrm{Coker}^{\,p-1}\mathcal{F}(U).$$

But the map

$$C^{p-1}(\mathcal{F}) \longrightarrow \mathrm{Coker}^{\,p-1}(\mathcal{F})$$

is a surjection of sheaves, which means that if $z \in Z^p(U, \mathcal{F})$, there is a covering of U by smaller opens, V_α, and $z \upharpoonright V_\alpha$ comes from $C^{p-1}(\mathcal{F})(V_\alpha)$, for all α. Hence, $z \upharpoonright V_\alpha$ is a coboundary: And so, each $\xi \in H^p(U, \mathcal{F})$ goes to zero in some $H^p(V_\alpha, \mathcal{F})$, for suitably small V_α. It follows that $\mathcal{H}^p(\mathcal{F}) = (0)$ for all $p > 0$ (note the analogy with the Poincaré lemma). The last statement of the lemma follows from the first. $\square$

We are now ready to prove the *Cartan isomorphism theorem* (1951/1952).

Theorem 4.9 *(Cartan) Let X be a ringed space and $\mathcal{F}$ be a sheaf of abelian groups on X. Assume that X has a family, $\mathcal{U}$, of open subsets, which cover X, and which satisfy the following conditions:*

(1) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

(2) The family, $\mathcal{U}$, contains arbitrarily small opens. That is, for every $x \in X$ and every open subset, U, with $x \in U$, there is some open $V \in \mathcal{U}$ so that $x \in V \subseteq U$.

(3) For all $U \in \mathcal{U}$ and all $p > 0$, $\check{H}^p(U, \mathcal{F}) = (0)$.

Then, the canonical map

$$\check{H}^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$$

is an isomorphism for all $n \geq 0$.

Proof. We begin with the following claim:

Claim. If $U \in \mathcal{U}$, then for all $p > 0$, the canonical map

$$\check{H}^p(U, \mathcal{F}) \longrightarrow H^p(U, \mathcal{F}) \quad \text{is an isomorphism.}$$

Assume that the claim holds. Then, condition (3) implies that for every $U \in \mathcal{U}$,

$$H^q(U, \mathcal{F}) = (0) \quad \text{for all } q > 0.$$

By (2), the groups $\check{C}^p(X, \mathcal{G})$ can be computed by using coverings and taking direct limits chosen from $\mathcal{U}$. However, by (1) and (3),

$$\check{C}^p(X, \mathcal{H}^q(\mathcal{F})) = (0) \quad \text{for all } p \geq 0 \text{ and all } q > 0.$$

This is because

$$\begin{aligned} C^p(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F})) &= \prod_{\alpha_0, \ldots, \alpha_p} \mathcal{H}^q(\mathcal{F})(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}) \\ &= \prod_{\alpha_0, \ldots, \alpha_p} H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{F}), \end{aligned}$$

where $\alpha_0, \ldots, \alpha_p$ are distinct, and by (1), $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \in \mathcal{U}$, so by (3),

$$H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{F}) = (0).$$

Thus, we get in the spectral sequence ($\check{\text{S}}\text{S}$),

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{F})) = (0) \quad \text{for all } p \geq 0 \text{ and } q > 0.$$

The spectral sequence degenerates, and

$$E_2^{n,0} = \check{H}^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}) \quad \text{is an isomorphism for all } n \geq 0.$$

It remains to prove the claim.

Note that as a consequence of Lemma 4.8, we get the isomorphism

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F})$$

for $p = 0, 1$, and

$$\check{H}^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F})$$

is injective. Indeed, for $p = 0$, this is a tautology. Write the edge sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2,$$

and observe that $E_2^{0,1} = (0)$, by Lemma 4.8; so, $E_2^{1,0} \cong H^1$ and $E_2^{2,0} \longrightarrow H^2$ is injective.

We now prove the claim by induction on p. The cases $p = 0$, and $p = 1$ have just been verified. Assume that

$$\check{H}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F}) \quad \text{for } 0 \leq q < n \text{ and for all } U \in \mathcal{U}.$$

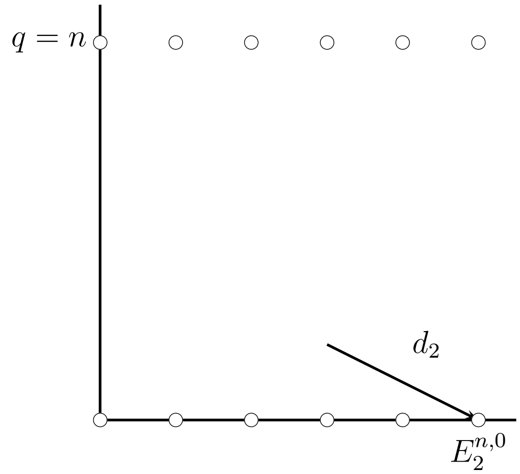

Figure 4.2: The E_2 terms of a lacunary spectral sequence

We may also assume that $n \geq 2$. Pick $U \in \mathcal{U}$, and apply the spectral sequence for $X = U$. The open set U inherits a family $\mathcal{U}'$, whose opens are the form $U \cap V$, with V ranging over $\mathcal{U}$. Look at $\check{C}^p(U, \mathcal{H}^q(\mathcal{F}))$ for $0 < q < n$. This is the direct limit over covers (U_α) from $\mathcal{U}$:

$$\varinjlim C^p(\{U_\alpha \cap U \to U\}, \mathcal{H}^q(\mathcal{F}))$$

i.e.,

$$\varinjlim \prod_{\alpha_0, \ldots, \alpha_p} H^p(U_{\alpha_0} \cap \cdots U_{\alpha_p} \cap U, \mathcal{F}).$$

By (1), $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \cap U \in \mathcal{U}'$, and

$$\check{H}^q(V, \mathcal{F}) \cong H^q(V, \mathcal{F}) \quad \text{for all } q, \text{ with } 0 < q < n,$$

where $V = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \cap U \in \mathcal{U}'$. By (3), the group $H^q(V, \mathcal{F})$ vanishes. Thus,

$$\check{C}^p(U, \mathcal{H}^q(\mathcal{F})) = (0) \quad \text{for all } p \geq 0 \text{ and } 0 < q < n.$$

Therefore,

$$E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathcal{F})) = (0) \quad \text{for all } p \geq 0 \text{ and } 0 < q < n.$$

Consider the lacunary spectral sequence whose second level is illustrated in Figure 4.2. The hypotheses imply that

$$E_3^{n,0} = H(E_2^{n,0}) = E_2^{n,0}.$$

The same argument shows that

$$E_4^{n,0} = H(E_3^{n,0}) = E_3^{n,0},$$

and this holds up to level $r = n$. Thus,

$$E_n^{n,0} = E_2^{n,0}.$$

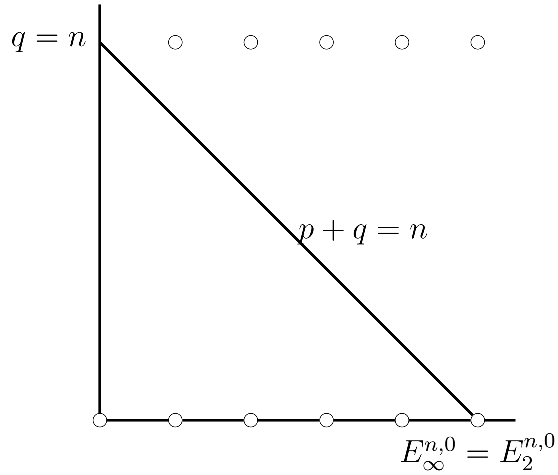

Figure 4.3: The E_∞ terms of a lacunary spectral sequence

Now look at the E_{n+1} terms. Note that d_{n+1} operates on $E_{n+1}^{-1,n}$ to get to $E_{n+1}^{n,0}$. However $E_{n+1}^{-1,n} = (0)$, and thus

$$E_{n+1}^{n,0} = E_2^{n,0},$$

which implies that

$$E_\infty^{n,0} = E_2^{n,0}.$$

Look at the E_∞ terms. We know that $E_2^{0,q} = (0)$ if $q > 0$, by Lemma 4.8. Thus,

$$E_\infty^{0,q} = (0) \quad \text{if } q > 0.$$

The ∞ level of our spectral sequence is illustrated in Figure 4.3. On the line $p + q = n$, we get

$$E_\infty^{p,n-p} = (0) \quad \text{if } 0 \leq p < n,$$

and $E_\infty^{n,0} = E_2^{n,0}$, by what we already proved. But, $H_p^n / H_{p+1}^n \cong E_\infty^{p,n-p} = (0)$, so

$$H^n = H_0^n = H_1^n = \cdots = H_n^n = E_\infty^{n,0} = E_2^{n,0} \supseteq (0) = H_{n+1}^n.$$

Therefore, $E_2^{n,0} \cong H^n$, that is,

$$\check{H}^n(U, \mathcal{F}) \cong H^n(U, \mathcal{F}).$$

The induction is completed and with it, the theorem. $\square$

4.3 Cohomology of Affine Schemes

Having investigated both homological methods and their applications to geometry in the previous sections, we can now reap the consequences in the important case of affine schemes. First, we use Serre's proposition (Proposition 4.7) to get the following:

Proposition 4.10 *Let X be a nerve-finite scheme and let $\mathcal{F}$ a QC sheaf of $\mathcal{O}_X$-modules, with global sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$, and write $M = \Gamma(X, \mathcal{F})$. Assume the following conditions:*

(1) Each X_{f_i} is a quasi-compact open $(1 \leq i \leq r)$.

(2) There exist $g_1, \ldots, g_r \in \Gamma(U, \mathcal{O}_X)$ so that

$$\sum_{i=1}^{r} g_i(f_i \upharpoonright U) = 1 \upharpoonright U, \quad \textit{where } U = \textstyle\bigcup_{j=1}^{r} X_{f_j}.$$

Then,

(1) $H^p(\{X_{f_j} \to U\}, \mathcal{F}) = (0)$ if $p > 0$,
and

(2) If $X = U$, then the map $M = \Gamma(X, \mathcal{F}) \longrightarrow H^0(\{U_i \to U\}, \mathcal{F})$ is an isomorphism.

Proof. Since the X_{f_i} are quasi-compact, the open $U = \bigcup_{j=1}^{r} X_{f_j}$ is also quasi-compact. Thus, we can reduce to the case where $X = U$, and where (2) is trivial. By Proposition 4.7,

$$H^q(\{X_{f_j} \to U\}, \mathcal{F}) = H^{q+1}((\overrightarrow{f}), M), \quad \text{for } q \geq 1.$$

Then, by hypothesis (2) and the Koszul complex, the righthand side is (0). Thus,

$$H^p(\{X_{f_j} \to U\}, \mathcal{F}) = (0) \quad \text{for all } p > 0,$$

as claimed. $\square$

Here the first main result on the cohomology of an affine scheme.

Theorem 4.11 *(Vanishing theorem for affines) Let X be an affine scheme and $\mathcal{F}$ be a QC sheaf of $\mathcal{O}_X$-modules. Then,*

$$H^q(X, \mathcal{F}) = (0) \quad \textit{for all } q > 0.$$

Proof. In Cartan's theorem (Theorem 4.9), take $\mathcal{U}$ to be the family of all affine opens of the form X_f, where $f \in \Gamma(X, \mathcal{O}_X)$. These open sets form a basis of the topology, and $X_f \cap X_g = X_{fg}$, so that conditions (1) and (2) of that theorem are satisfied. Now, if we knew that

$$\check{H}^p(X_f, \mathcal{F}) = (0) \quad \text{for } p > 0 \text{ and all } f,$$

then, by Cartan's theorem, we would get the isomorphism

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F}).$$

But, when $f = 1$, Cartan's condition (3) is exactly that the lefthand side vanishes, and the theorem would hold. Therefore, we have to show that if X is affine then

$$\check{H}^p(X, \mathcal{F}) = (0) \quad \text{for } p > 0.$$

Cover X by the X_{f_i}, with only finitely many, since X is quasi-compact. As the X_{f_i} form a cover, the elements $f_1, \ldots, f_r$ generate the unit ideal. By Proposition 4.10, we have

$$H^p(\{X_{f_j} \to X\}, \mathcal{F}) = (0) \quad \text{for all } p > 0.$$

However, the X_{f_i} are arbitrarily fine. Consequenlty, we get

$$\begin{aligned} \check{H}^p(X, \mathcal{F}) &= \varinjlim \check{H}^p(\{X_{f_j} \to X\}, \mathcal{F}) \\ &= \varinjlim H^p(\{X_{f_j} \to X\}, \mathcal{F}) = (0), \end{aligned}$$

which completes the proof. $\square$

Corollary 4.12 *Let $\pi \colon X \to Y$ be an affine morphism of schemes. Then, the following facts hold:*

(1) For all $q > 0$, we have $R^q\pi_*\mathcal{F} = (0)$.

(2) The canonical morphism

$$H^n(Y, \pi_*\mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$$

is an isomorphism for every $n \geq 0$.

Proof. We already know from before that $R^q\pi_*\mathcal{F}$ is the sheaf associated with the presheaf

$$\mathcal{R}^q\pi_*\mathcal{F} \colon U \mapsto H^q(\pi^{-1}(U), \mathcal{F}),$$

where U is any open of Y. (Thus, $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$.) But we have the Leray spectral sequence of a morphism (see Appendix B)

$$E_2^{p,q} = H^p(Y, R^q\pi_*\mathcal{F}) \Longrightarrow H^\bullet(X, \mathcal{F}).$$

Thus, if $R^q\pi_*\mathcal{F} = (0)$, then $E_2^{p,q} = 0$ if $q > 0$, for all $p \geq 0$. Consequently, (2) follows from (1), by degeneration of the spectral sequence. We need to prove (1). However, for every $y \in Y$,

$$(R^q\pi_*\mathcal{F})_y = \lim_{\overrightarrow{U}} (\mathcal{R}^q\pi_*\mathcal{F})(U), \quad \text{where } y \in U,$$

and the righthand side is

$$\lim_{\overrightarrow{U}} (\mathcal{R}^q\pi_*\mathcal{F})(U) = \lim_{\overrightarrow{U}} H^q(\pi^{-1}(U), \mathcal{F}).$$

Take the neighborhood basis at y to consist of affines U. Then, $\pi^{-1}(U)$ is affine, since π is an affine morphism. By the vanishing theorem (Theorem 4.11), we get

$$H^q(\pi^{-1}(U), \mathcal{F}) = (0). \qquad \square$$

Corollary 4.13 *Let $\pi\colon X \to Y$ be an affine morphism and $\theta\colon Y \to Z$ be any morphism. Then, the canonical map*
$$R^p\theta_*(\pi_*\mathcal{F}) \longrightarrow R^p(\theta\circ\pi)_*(\mathcal{F})$$
is an isomorphism for all $p \geq 0$ and all QC $\mathcal{O}_X$-modules $\mathcal{F}$.

Proof. We have the spectral sequence of composed functors (see Appendix B)
$$E_2^{p,q} = (R^p\theta_* \circ R^q\pi_*)(\mathcal{F}) \Longrightarrow R^\bullet(\theta\circ\pi)_*(\mathcal{F}).$$
However, π is affine, and therefore, $R^q\pi_*(\mathcal{F}) = (0)$ for all $q > 0$. The spectral sequence degenerates and gives the edge isomorphism
$$E_2^{p,0} = R^p\theta_*(\pi_*\mathcal{F}) \cong R^p(\theta\circ\pi)_*(\mathcal{F}). \qquad \square$$

Corollary 4.14 *Let X be a scheme, $\{U_\alpha \to X\}$ be an open cover by affines (not necessarily finite) so that $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is again affine for all $p \geq 0$ (e.g., if X is separated). Then, the canonical homomorphisms*
$$H^p(\{U_\alpha \to X\}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$
are isomorphisms for all $p \geq 0$ and all QC $\mathcal{O}_X$-modules $\mathcal{F}$. Therefore, the cohomology of a "good cover" computes the "real" cohomology.

Proof. We have the spectral sequence of a cover (see Appendix B)
$$E_2^{p,q} = H^p(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^\bullet(X, \mathcal{F}).$$
Look at $C^p(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F}))$. We have
$$C^p(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F})) = \prod_{\alpha_0,\ldots,\alpha_p} \mathcal{H}^q(\mathcal{F})(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}) = \prod_{\alpha_0,\ldots,\alpha_p} H^q(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}, \mathcal{F}) = (0)$$
for all $q > 0$, as $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ ia affine and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module (where $\alpha_0, \ldots, \alpha_p$ are distinct). Therefore, the cohomology of the complex $C^\bullet(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F}))$ is zero, i.e.,
$$H^p(\{U_\alpha \to X\}, \mathcal{H}^q(\mathcal{F})) = (0), \quad \text{for all } q > 0 \text{ and all } p \geq 0.$$
Thus, the spectral sequence degenerates and we get the edge isomorphism
$$E_2^{p,0} = H^p(\{U_\alpha \to X\}, \mathcal{F}) \cong H^p(X, \mathcal{F}) \quad \text{for all } p \geq 0. \qquad \square$$

Remark: For a quasi-compact, separated scheme, if r is the minimum number of affine opens in a cover, we have
$$H^q(X, \mathcal{F}) = (0) \quad \text{if } q \geq r.$$

Indeed, we can can compute $H^q(X,\mathcal{F})$ using $H^q(\{U_j\to X\},\mathcal{F})$. The maximum number of opens is r, and we get (0) at the cochiain level from $C^r(X,\mathcal{F})$ on, by the requirement of alternation in our cochains. Thus, the top level nontrivial cohomology group is at most $H^{r-1}(X,\mathcal{F})$.

We can apply this immediately to projective space of dimension r over a field. For, here, the minimum number of affines is $r+1$; hence, beyond H^r, the cohomology vanishes.

Corollary 4.15 *Let X be a separated quasi-compact scheme, and let $f_1,\ldots,f_r\in\Gamma(X,\mathcal{O}_X)$ be some global sections of $\mathcal{O}_X$. Assume that X_{f_j} is affine for $j=1,\ldots,r$. Letting $U=\bigcup_{j=1}^r X_{f_j}$, we have the isomorphisms*

$$H^p(U,\mathcal{F})\cong H^{p+1}((\overrightarrow{f}),M),\quad \textit{for all } p\geq 1.$$

Here, $M=\Gamma(X,\mathcal{F})$ and $\mathcal{F}$ is any QC $\mathcal{O}_X$-module. Furthermore, we have the exact sequence

$$0\longrightarrow H^0((\overrightarrow{f}),M)\longrightarrow M\longrightarrow \Gamma(U,\mathcal{F})\longrightarrow H^1((\overrightarrow{f}),M)\longrightarrow 0. \tag{$*$}$$

Proof. Let $U_j=X_{f_j}$. Since X is separated and quasi-compact, we know by previous work that

$$H^p(\{U_j\to U\},\mathcal{F})\cong H^{p+1}((\overrightarrow{f}),M)\quad \text{for all } p\geq 1,$$

and we have the sequence $(*)$, as stated. However, corollary 4.14 shows that

$$H^{p+1}((\overrightarrow{f}),M)\cong H^p(\{U_j\to U\},\mathcal{F})\cong H^p(U,\mathcal{F}).$$

$\square$

Example 4.2 Let $X=\mathbb{A}^r_k$, where k is an algebraically closed field. (This example can be generalized to $\mathbb{A}^r_B$, where B is a ring, and even further (DX).) Take for the f_i the T_i in $A=\Gamma(X,\mathcal{O}_X)=k[T_1,\ldots,T_r]$ (the T_i are indeterminates), and let $U_j=\mathbb{A}^r_{T_j}$, the localization of $\mathbb{A}^r$ at T_j. Then,

$$U=\bigcup_{j=1}^r U_j=\mathbb{A}^r-\{(0,\ldots,0)\},$$

the complement of the origin in $\mathbb{A}^r$. We have

$$H^p(U,\mathcal{F})\cong H^{p+1}((\overrightarrow{T}),M),\quad \text{for all } p\geq 1,$$

and the exact sequence

$$0\longrightarrow H^0((\overrightarrow{T}),M)\longrightarrow M\longrightarrow \Gamma(U,\mathcal{F})\longrightarrow H^1((\overrightarrow{T}),M)\longrightarrow 0. \tag{$**$}$$

Say $r\geq 2$, and let $p=r-1$. Then, we get

$$H^{r-1}(U,\mathcal{F})\cong H^r((\overrightarrow{T}),M)=M/\mathfrak{M}_0M,$$

where $\mathfrak{M}_0 = (T_1, \ldots, T_r)$, the maximal ideal defining $\{0\}$. In our exact sequence $(**)$,

$$H^0((\overrightarrow{T}), M) = \{m \in M \mid \mathfrak{M}_0 m = (0)\}.$$

Consider the case where $\mathcal{F} = \mathcal{O}_X$, i.e., $M = A = \Gamma(X, \mathcal{O}_X)$. In this case, $(\overrightarrow{T})$ is a regular sequence for M. Then, we know that

$$H^r((\overrightarrow{T}), M) = (0)$$

except for $p = r$, where $A/\mathfrak{M}_0 A \cong k \neq (0)$. We conclude that for $r \geq 2$ and $\mathcal{F} = \mathcal{O}_X$, we get

$$H^p(U, \mathcal{O}_X) = \begin{cases} A & \text{if } p = 0 \\ (0) & \text{if } 1 \leq p \leq r-2 \\ k & \text{if } p = r-1 \\ (0) & \text{if } p \geq r. \end{cases}$$

Hence, we get:

(1) If $r \geq 2$, $U = \mathbb{A}^r - \{0\}$ is not affine.

(2) (Hartogs) Given a global section $f \in \Gamma(U, \mathcal{O}_X)$ (i.e., a holomorphic function on U), this section extends uniquely to a global section in $\Gamma(X, \mathcal{O}_X)$ (i.e., a holomorphic function on X).

Corollary 4.16 *Let X be affine and $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$. Then, $H^p((\overrightarrow{f}), M)$ depends only on the ideal, $\mathfrak{A}$, generated by $(f_1, \ldots, f_r)$.*

Proof. Let $U = \bigcup_{j=1}^r X_{f_j}$. We know that

$$U = V(\mathfrak{A})^c,$$

and

$$H^p(U, \widetilde{M}) = H^{p+1}((\overrightarrow{f}), M), \quad \text{for all } p \geq 1.$$

We also have the exact sequence

$$0 \longrightarrow H^0((\overrightarrow{f}), M) \longrightarrow M \longrightarrow \Gamma(U, \widetilde{M}) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0.$$

Clearly, $H^p(U, \widetilde{M})$ and the various modules in the sequence depend only on U. $\square$

We will need two more cohomological results of the same kind as Theorem 4.11.

Proposition 4.17 *Let X be a scheme, and suppose*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules for which $\mathcal{F}$ and $\mathcal{H}$ are QC. Then, $\mathcal{G}$ is also QC.

Proof. Since the question is local on X, we may assume that X is affine. We have to verify Serre's conditions (a) and (b) for $\mathcal{G}$. Since the cohomology of affines is trivial, we have the following commutative diagram for every $f \in \Gamma(X, \mathcal{O}_X)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r, "v"] \arrow[d, "res"] & \mathcal{G}(X) \arrow[r] \arrow[d, "res"] & \mathcal{H}(X) \arrow[r] \arrow[d, "res"] & 0 \\
0 \arrow[r] & \mathcal{F}(X_f) \arrow[r, "u"] & \mathcal{G}(X_f) \arrow[r] & \mathcal{H}(X_f) \arrow[r] & 0
\end{tikzcd}$$

(a) Assume that $s \in \mathcal{G}(X)$ and $res(s) = 0$ in $\mathcal{G}(X_f)$. Thus, $\overline{s}$, the image of s in $\mathcal{H}(X)$, has $res(\overline{s}) = 0$. By Serre (a) for $\mathcal{H}$, there is some $n > 0$ so that $f^n\overline{s} = 0$ in $\mathcal{H}(X)$, and thus, $f^n s$ goes to 0 in $\mathcal{H}(X)$; so, $f^n s$ comes from some $t \in \mathcal{F}(X)$. Now,

$$u \circ res(t) = res(f^n s) = f^n res(s) = 0.$$

By injectivity, we have $res(t) = 0$ in $\mathcal{F}(X_f)$, and so, there is some $m > 0$ so that $f^m t = 0$ in $\mathcal{F}(X)$, by Serre (a) for $\mathcal{F}$. Finally,

$$f^{m+n}s = 0 \quad \text{in } \mathcal{G}(X),$$

and Serre's (a) holds for $\mathcal{G}$.

(b) Assume that $\sigma \in \mathcal{G}(X_f)$. Then, $\overline{\sigma} \in \mathcal{H}(X_f)$, so, by Serre (b) for $\mathcal{H}$, there is some $n > 0$ with

$$f^n\overline{\sigma} = res(s), \quad \text{for some } s \in \mathcal{H}(X).$$

Now, by affineness, $\mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ is onto, and so, there is some $t \in \mathcal{G}(X)$ having $s \in \mathcal{H}(X)$ as its image. Observe that

$$\overline{res(t)} = res(\overline{t}) = res(s) = f^n\overline{\sigma} = \overline{f^n\sigma}.$$

Thus,

$$res(t) - f^n\sigma = u(\tau),$$

for some $\tau \in \mathcal{F}(X_f)$. Serre's (b), applied to $\mathcal{F}$, yields an $m > 0$ so that

$$f^m\tau = res(\widetilde{t})$$

for some $\widetilde{t} \in \mathcal{F}(X)$. We have

$$res(f^m t) - f^{m+n}\sigma = u(f^m\tau) = u(res(\widetilde{t})) = res(v(\widetilde{t})).$$

Thus,

$$res(f^m t - v(\widetilde{t})) = f^{m+n}\sigma,$$

and (b) holds for $\mathcal{F}$. $\square$

To investigate the higher direct images of a quasi-coherent $\mathcal{O}_X$-module, we will need to sheafify the Čech complex of a cover. In this investigation, the cohomology of intersection of affines will enter, and so, to guarantee an application of nerve-finiteness, we will need X to be a quasi-compact and separated scheme.

Let $\{U_\alpha \to X\}$ be an affine open cover of X, and let

$$\varphi_{\alpha_0\ldots\alpha_p} \colon U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \to X$$

be the open immersion associated with $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. We define $\mathcal{C}^p(\{U_\alpha \to X\}, \mathcal{F})$, abbreviated by $\mathcal{C}^p(\mathcal{F})$, by

$$\mathcal{C}^p(\mathcal{F}) = \prod_{\alpha_0,\ldots,\alpha_p} (\varphi_{\alpha_0\ldots\alpha_p})_*(\mathcal{F} \upharpoonright (U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})),$$

where $\alpha_0, \ldots, \alpha_p$ are distinct (and there are only finitely many, since X is quasi-compact). We know the following facts:

(1) $\mathcal{C}^p(\mathcal{F}) = \mathcal{C}^p(\{U_\alpha \to X\}, \mathcal{F})$ is a sheaf.

(2) $(\varphi_{\alpha_0\ldots\alpha_p})_*(\mathcal{F} \upharpoonright (U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}))$ is a QC $\mathcal{O}_X$-module.

(3) $\mathcal{C}^p(\{U_\alpha \to X\}, \mathcal{F})$ is a QC $\mathcal{O}_X$-module.

(4) $\mathcal{C}^\bullet(\{U_\alpha \to X\}, \mathcal{F})$, which by definition is the coproduct $\coprod_{p\geq 0} \mathcal{C}^p(\mathcal{F})$ of the $\mathcal{C}^p(\mathcal{F})$, is a QC $\mathcal{O}_X$-module.

(5) $H^\bullet(\mathcal{C}^\bullet(\{U_\alpha \to X\}, \mathcal{F}))$ is a QC $\mathcal{O}_X$-module, since the cohomology is computed as a quotient of QC $\mathcal{O}_X$-modules.

Take V open in X. Then, we have

$$\begin{aligned}
\Gamma(V, \mathcal{C}^p(\{U_\alpha \to X\}, \mathcal{F})) &= \prod_{\alpha_0,\ldots,\alpha_p} \Gamma(V, (\varphi_{\alpha_0,\ldots,\alpha_p})_*\mathcal{F}) \\
&= \prod_{\alpha_0,\ldots,\alpha_p} \mathcal{F}(V \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}) \\
&= C^p(\{U_\alpha \cap V \to V\}, \mathcal{F}).
\end{aligned}$$

Consequently, we get

$$H^\bullet(\Gamma(V, \mathcal{C}^\bullet(\{U_\alpha \to X\}, \mathcal{F}))) = H^\bullet(\{U_\alpha \cap V \to V\}, \mathcal{F}),$$

for every open subset V. Now, if V is affine, $\Gamma(V, -)$ is exact on the category of QC $\mathcal{O}_X$-modules, which implies that cohomology commutes with $\Gamma(V, -)$. Hence, for V affine open, we get

$$H^\bullet(\{U_\alpha \cap V \to V\}, \mathcal{F}) = H^\bullet(\Gamma(V, \mathcal{C}^\bullet(\mathcal{F}))) = \Gamma(V, H^\bullet(\mathcal{C}^\bullet(\mathcal{F}))).$$

We can apply the above computation to prove that higher direct images of QC modules are again QC in good cases.

Theorem 4.18 *Let X, Y be schemes and let $\pi\colon X \to Y$ be a quasi-compact, separated morphism. If $\mathcal{F}$ is a QC $\mathcal{O}_X$-module, then $R^q\pi_*\mathcal{F}$ is a QC $\mathcal{O}_Y$-module for every $q \geq 0$.*

Proof. As the question is local on Y, we may assume that Y is affine. Now if V is affine open and $V \subseteq$Y, we claim that $\pi^{-1}(V) \longrightarrow X$ is an affine morphism. This is an important first step in the proof. It means that if W is an affine open in X, then $\pi^{-1}(V) \cap W$ is again affine. Look at $X \prod Y$. Note that $W \prod V$ is an affine open in $X \prod Y$. Also, observe that $\Gamma_\pi(X) \cap pr_1^{-1}(W) \cap pr_2^{-1}(V)$ has as image under pr_1 our set $\pi^{-1}(V) \cap W$, where Γ_π is the graph morphism corresponding to π. As usual, this can be checked by viewing products as representing their functors and using a test object, so that we can pretend that the objects are indeed sets. Then

$$\Gamma_\pi(X) \cap pr_1^{-1}(W) \cap pr_2^{-1}(V) = \{(a, b) \mid b = \pi(a),\, a \in W,\, b \in V\}.$$

Clearly, $a \in W \cap \pi^{-1}(V)$, so we get our observation. Since $pr_1^{-1}(W) \cap pr_2^{-1}(V) \cong W \prod V$, it is affine. Since Y is affine, Γ_π is a closed immersion, and thus, $\Gamma_\pi(X)$ is closed. Hence, $\Gamma_\pi(X) \cap pr_1^{-1}(W) \cap pr_2^{-1}(V)$ is a closed subset of an affine, and thus, is affine. But the same object is also open in $\Gamma_\pi(X)$; hence, an affine open in $\Gamma_\pi(X)$. On $\Gamma_\pi(X)$, the morphisms π and pr_1 are inverse isomorphisms. Therefore, $\pi^{-1}(V) \cap W$ is affine, as claimed.

Since π is quasi-compact, X is covered by finitely many affine opens $U_1, \ldots, U_r$, and if V is an affine open in Y, then each $\pi^{-1}(V) \cap U_j$ is again affine, by the claim that we just established. Thus,

$$\{\pi^{-1}(V) \cap U_j \to \pi^{-1}(V)\}$$

is an affine open cover of $\pi^{-1}(V)$. Previous work implies that

$$H^\bullet(\pi^{-1}(V), \mathcal{F}) \cong H^\bullet(\{\pi^{-1}(V) \cap U_j \to \pi^{-1}(V)\}, \mathcal{F}) \cong H^\bullet(\Gamma(V, \pi_*\mathcal{C}^\bullet(\mathcal{F}))),$$

and we just proved that since V is affine

$$H^\bullet(\Gamma(V, \pi_*\mathcal{C}^\bullet(\mathcal{F}))) \cong \Gamma(V, H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))).$$

Thus, we have

$$H^\bullet(\pi^{-1}(V), \mathcal{F}) \cong \Gamma(V, H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))). \tag{$*$}$$

However, $\pi_*\mathcal{C}^\bullet(\mathcal{F})$ is a QC $\mathcal{O}_X$-module (by previous work), and thus, $H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))$ is again a QC $\mathcal{O}_X$-module. It is also easily checked that for any two affine opens $V' \subseteq V$, we have the commutative diagram

$$\begin{tikzcd}
H^\bullet(\pi^{-1}(V), \mathcal{F}) \arrow[r, "\sim"] \arrow[d] & \Gamma(V, H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))) \arrow[d] \\
H^\bullet(\pi^{-1}(V'), \mathcal{F}) \arrow[r, "\sim"] & \Gamma(V', H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))).
\end{tikzcd}$$

This implies by passing to the inductive limit that

$$(R^q\pi_*\mathcal{F})_y \cong H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))_y \quad \text{for all } y \in Y$$

(isomorphism of stalks). However, isomorpphism $(*)$ show that we have a map of sheaves $R^\bullet\pi_*\mathcal{F} \longrightarrow H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F}))$ which we have just seen is an isomorphism on stalks. Therefore,

$$R^\bullet\pi_*\mathcal{F} \cong H^\bullet(\pi_*\mathcal{C}^\bullet(\mathcal{F})),$$

and thus, $R^\bullet\pi_*\mathcal{F}$ is QC. $\square$

Corollary 4.19 *Let X, Y be schemes and let $\pi\colon X \to Y$ be a quasi-compact, separated morphism. Assume $\mathcal{F}$ is a QC $\mathcal{O}_X$-module. Then, for every affine open V in Y, we have*

$$\Gamma(V, R^q\pi_*\mathcal{F}) = H^q(\pi^{-1}(V), \mathcal{F}).$$

If Y itself is affine, this gives

$$\Gamma(Y, R^q\pi_*\mathcal{F}) = H^q(X, \mathcal{F}),$$

and thus, $R^q\pi_*\mathcal{F} = \widetilde{H^q(X, \mathcal{F})}$.

Corollary 4.20 *Let X, Y be schemes, $\pi\colon X \to Y$ be a quasi-compact, separated morphism, and assume that Y is also quasi-compact. If $\mathcal{F}$ is a QC $\mathcal{O}_X$-module, there is some $r > 0$ (independent of $\mathcal{F}$) so that*

$$R^q\pi_*(\mathcal{F}) = (0) \quad \textit{for all } q \geq r.$$

In particular, when Y is affine, r may be taken to be the minimum number of affine opens to cover X.

Proof. Since Y is quasi-compact, we may assume that Y is affine. Then,

$$R^q\pi_*\mathcal{F} = \widetilde{H^q(X, \mathcal{F})} = (0)$$

beyond the number of elements in an open affine cover of X, by Čech theory. $\square$

Remark: While we have proved under fairly general hypotheses that higher direct images of QC sheaves are themselves QC, it is a far different matter for the same question as concerns coherent sheaves. Coherence is a kind of finiteness property and may easily be lost if the morphism π does not have strong finiteness properties itself. In particular, finite presentation or finite type, are inadequate to guarantee the analog of Theorem 4.18. One really needs that the fibres of the morphism π behave as do compact spaces in the norm topology–in our case, this translates into the assumption that π is a proper morphism. We will return to the analog of Theorem 4.18 in Section 7.1 but we will find that the proof is far more subtle. However, there are two cases where we can say right now that the push-forward of a coherent sheaf is again coherent. The first is the case where Y is locally noetherian and $f\colon X \to Y$ is a finite surjective morphism. The question of coherence of $f_*\mathcal{F}$ is of course local on Y, so we may and do assume Y is an affine noetherian scheme. Then, by Corollary 4.20, as f is an affine morphism, $R^q f_*\mathcal{F} = (0)$ if $q > 0$ and $f_*\mathcal{F}$ is just $\widetilde{\Gamma(X, \mathcal{F})}$ as A-module, where $Y = \mathrm{Spec}\, A$. Now, $X = \mathrm{Spec}\, B$ for some finite A-module, B, and $\Gamma(X, \mathcal{F})$ is a finitely generated B-module as $\mathcal{F}$ is coherent (Theorem 3.14–recall B *is* noetherian). Consequently, $\Gamma(X, \mathcal{F})$ is a finitely generated A-module and Theorem 3.14, again, shows that $f_*\mathcal{F}$ is coherent.

The second case is when $i\colon X \to Y$ is a closed immersion. Here, i is an affine morphism; so, $R^q i_*\mathcal{F} = (0)$, for $q > 0$ and all we need to prove is that $i_*\mathcal{F}$ itself is coherent when $\mathcal{F}$ is coherent. This is the content of:

Proposition 4.21 *If $i\colon X \to Y$ is a closed immersion and Y is locally noetherian, then $R^q i_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module whenever X is a coherent $\mathcal{O}_X$-module.*

Proof. The case $q > 0$ has already been proved and does not require the hypothesis of local noetherianess on Y. Since the question of coherence is local on Y we may assume that Y is affine, say $Y = \mathrm{Spec}\, A$ (and A is noetherian), and then X will be $\mathrm{Spec}\, \overline{A}$ where $\overline{A} = A/\mathfrak{A}$, for some ideal, $\mathfrak{A}$, of A. Moreover, the sheaf $\mathcal{F}$ has the form $\widetilde{\overline{M}}$ for an $\overline{A}$-module, $\overline{M}$. Under these conditions, the coherence of $\mathcal{F}$ is just the finite generation of $\overline{M}$ because A (and hence $\overline{A}$) is noetherian. But, $i_*\mathcal{F}$ is just the $\overline{A}$-module $\overline{M}$ considered as A-module *via* the homomorphism $A \longrightarrow \overline{A}$. Hence, $\overline{M}$ as A-module is finitely generated; it follows that $i_*\mathcal{F}$ is coherent. $\square$

4.4 Cohomological Characterization of Affine Schemes

One of the interesting uses of cohomology is that by viewing the cohomology of a scheme with coefficients in a restricted class of QC sheaves, we can decide on the affiness or nonaffineness of the given scheme. This fact was discovered by Serre [50] (1957), here is his theorem.

Theorem 4.22 *(Serre) Let X be a quasi-compact scheme. Then, the following properties are equivalent:*

(1) The scheme, X, is affine.

(2) For all QC $\mathcal{O}_X$-modules, $\mathcal{F}$, for all $q > 0$, we have $H^q(X, \mathcal{F}) = (0)$.

(3) For all QC $\mathcal{O}_X$-modules, $\mathcal{F}$, we have $H^1(X, \mathcal{F}) = (0)$.

(4) For all QC $\mathcal{O}_X$-ideals, $\mathfrak{I} \subseteq \mathcal{O}_X$, we have $H^1(X, \mathfrak{I}) = (0)$.

(5) There exist $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ so that

(a) Each X_{f_i} is affine, for $i = 1, \ldots, r$.

(b) The ideal $(f_1, \ldots, f_r)$ is the unit ideal of $\Gamma(X, \mathcal{O}_X)$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 4.11. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4), are trivial.

(4) $\Rightarrow$ (5) (Serre's argument). Let P be any closed point of X, and let U be any affine open containing P. Since X is quasi-compact and a T_0-space, such a point exists, by Proposition 3.1. Let $Y = U^c$, which is a closed set in X, and let

$$\widetilde{Y} = Y \cup \{P\}.$$

There exist QC ideals $\mathfrak{I}$ and $\widetilde{\mathfrak{I}}$ defining the reduced, induced, structure of scheme for Y and $\widetilde{Y}$. We have $\widetilde{\mathfrak{I}} \subseteq \mathfrak{I}$. Consider the following exact sequence of QC $\mathcal{O}_X$-modules:

$$0 \longrightarrow \widetilde{\mathfrak{I}} \longrightarrow \mathfrak{I} \longrightarrow \mathrm{Cok} \longrightarrow 0, \tag{$*$}$$

where Cok denotes the cokernel of $\widetilde{\mathfrak{I}} \longrightarrow \mathfrak{I}$. If $Q \in Y$, then the localization of $\mathfrak{I}$ at Q, namely $\mathfrak{I}_Q$, is $\mathfrak{I}\mathcal{O}_{X,Q}$. Since $Y \subseteq \widetilde{Y}$, we also have

$$\widetilde{\mathfrak{I}}_Q = \mathfrak{I}\mathcal{O}_{X,Q} = \mathfrak{I}_Q,$$

and thus,

$$\mathrm{Cok}_Q = (0).$$

If $Q \notin \widetilde{Y}$, then

$$\widetilde{\mathfrak{I}}_Q = \mathcal{O}_{X,Q} = \mathfrak{I}_Q,$$

and again,

$$\mathrm{Cok}_Q = (0).$$

Therefore, Cok is a sheaf supported exactly at P, and at P, we have $\widetilde{\mathfrak{I}}_P = \mathfrak{m}_P$, a maximal ideal in $\mathcal{O}_{X,P}$. But P is not on Y, so $\mathfrak{I}_P = \mathcal{O}_{X,P}$. We thus find that

$$\mathrm{Cok}_P = \kappa(P),$$

that is, the sheaf Cok is a "skyscraper sheaf." If we apply cohomology to $(*)$, we get

$$H^0(X, \mathfrak{I}) \longrightarrow H^0(X, \mathrm{Cok}) \longrightarrow H^1(X, \widetilde{\mathfrak{I}}),$$

and since (4) is assumed to hold, we get $H^1(X, \widetilde{\mathfrak{I}}) = (0)$. Therefore, the map $\Gamma(X, \mathfrak{I}) \longrightarrow \kappa(P)$ is surjective. Thus, we can find there is some $f \in \Gamma(X, \mathfrak{I})$ so that $f = 1 \bmod \mathfrak{m}_P$; hence, $f(P) \neq 0$. Consider X_f and $\xi \in X_f$. We know that $f(\xi) \neq 0$, and so, if we had $\xi \in Y$, we would get $f(\xi) = 0$, a contradiction. Therefore, $\xi \in U$, and $X_f \subseteq U$. Since $f(P) \neq 0$, we have $P \in X_f$. Let $\overline{f} = f \restriction U$. Then, $X_f = U_{\overline{f}}$, and $U_{\overline{f}}$ is affine because U is affine. Thus, X_f is affine. Now,

$$V = \bigcup \{X_f \quad \text{all such } f\}$$

is an open subset of X that contains all closed points of X, by construction. By Proposition 3.1, we must have $V = X$, since otherwise, $X - V$ would be a closed nonempty subset of X without any closed point. Thus, we can cover X by the X_f's, and since X is quasi-compact, there is a finite subcover, say $X_{f_1}, \ldots, X_{f_r}$, which yields (a).

We now prove that the elements $f_1, \ldots, f_r$ generate the whole of $\Gamma(X, \mathcal{O}_X)$. The f_i's induce a map

$$\varphi\colon \mathcal{O}_X^r \longrightarrow \mathcal{O}_X$$

defined as follows: For any open subset U, and for every vector $(a_1, \ldots, a_r) \in \mathcal{O}_X^r(U)$,

$$(\varphi \restriction U)(a_1, \ldots, a_r) = \sum_{i=1}^{r} f_i a_i.$$

Given any open U, observe that U is covered by the $U \cap X_{f_i}$'s, and if $\xi \in \mathcal{O}_X(U)$ is given, let

$$b = \frac{\xi}{f_i} \quad \text{on } U \cap X_{f_i}.$$

Then,

$$\varphi(0, \ldots, 0, b, 0, \ldots, 0) = \xi \quad \text{on } U \cap X_{f_i}.$$

Therefore, φ is surjective as a map of sheaves. If $K = \operatorname{Ker} \varphi$, we have the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{$*$}$$

and K is quasi-coherent. Now, $H^1(X, K) = (0)$; to see this, proceed by induction on r. For $r = 1$, since K is quasi-coherent, we have $H^1(X, K) = (0)$, by the hypothesis. Next, consider the QC subsheaf, $K \cap \mathcal{O}_X^{r-1}$, of $\mathcal{O}_X^{r-1}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^{r-1} \arrow[r] & \mathcal{O}_X^r \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & K \cap \mathcal{O}_X^{r-1} \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where Q is the cokernel in the lower sequence and the two left vertical arrows are injections. It is clear that the map $Q \longrightarrow \mathcal{O}_X$ is also an inclusion, and so, Q is a QC ideal of $\mathcal{O}_X$. Applying cohomology to the lower row, we get

$$(0) = H^1(X, K \cap \mathcal{O}_X^{r-1}) \longrightarrow H^1(X, K) \longrightarrow H^1(X, Q) = (0),$$

since $H^1(X, K \cap \mathcal{O}_X^{r-1}) = (0)$, by the induction hypothesis, and $H^1(X, Q) = (0)$. Thus, $H^1(X, K) = (0)$, as claimed. Applying cohomology to $(*)$, we get

$$\Gamma(X, \mathcal{O}_X^r) \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow H^1(X, K) = (0).$$

Therefore, globally, $\Gamma(X, \mathcal{O}_X)$ is generated by the f_i's, and (b) holds.

(5) $\Rightarrow$ (1). First, we note that for any scheme X, there is a map

$$\mathrm{can}\colon X \to X_{\mathrm{aff}},$$

where $X_{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)$. This is because X_{aff} being affine, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{SCH}}(X, X_{\mathrm{aff}}) \cong \mathrm{Hom}_{\mathrm{rings}}(\Gamma(X_{\mathrm{aff}}), \Gamma(X)),$$

and $\Gamma(X_{\mathrm{aff}}) = \Gamma(X)$; so that we can take 1 on the righthand side and get the map can on the lefthand side. We know that the X_{f_i}'s are affine and cover X. Also, $X_{f_i} \cap X_{f_j} (= X_{f_i f_j})$ is the localization of the affine X_{f_i}, and thus, $X_{f_i} \cap X_{f_j}$ is affine. This cover by the X_{f_i} is consequently nerve-finite. By the usual reasoning, we have an isomorphism

$$\Gamma(X_{f_i}, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)_{f_i}.$$

Let $Y = X_{\mathrm{aff}}$. Then,

$$Y_{f_i} = \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)_{f_i} \cong \mathrm{Spec}\,\Gamma(X_{f_i}, \mathcal{O}_X),$$

and (as X_{f_i} is affine),

$$\mathrm{can}_X \upharpoonright X_{f_i} = \mathrm{can}_{X_{f_i}} \colon X_{f_i} \longrightarrow Y_{f_i}$$

is an isomorphism. Therefore,

$$\mathrm{can}_X \colon X \longrightarrow \bigcup_{i=1}^{r} Y_{f_i}$$

is an isomorphism. However, the ideal $(f_1, \ldots, f_r)$ is all of $\Gamma(X, \mathcal{O}_X)$, which implies that

$$Y = \bigcup_{i=1}^{r} Y_{f_i}.$$

This proves that

$$\mathrm{can}_X \colon X \longrightarrow Y = \bigcup_{i=1}^{r} Y_{f_i}$$

is an isomorphism, and X is affine. $\square$

Corollary 4.23 *If X is noetherian, then conditions (2), (3), (4) may be replaced by their equivalent conditions:*

(2a) For all coherent $\mathcal{O}_X$-modules, $\mathcal{F}$, for all $q > 0$, we have $H^q(X, \mathcal{F}) = (0)$.

(3a) For all coherent $\mathcal{O}_X$-modules, $\mathcal{F}$, we have $H^1(X, \mathcal{F}) = (0)$.

(4a) For all coherent $\mathcal{O}_X$-ideals, $\mathfrak{I} \subseteq \mathcal{O}_X$, we have $H^1(X, \mathfrak{I}) = (0)$.

Proof. The fact that (x) implies (xa) is trivial for x = 2, 3, 4. The converse implications hold because every QC object is the direct limit of its coherent subobjects, and

$$H^\bullet(X, \mathcal{F}) = \varinjlim_{\alpha} H^\bullet(X, \mathcal{F}_\alpha),$$

with $\mathcal{F}_\alpha$ coherent, as X is Noetherian. $\square$

Remark: Consider $\mathbb{A}^r$, with $r \geq 2$, and elements $T_1, \ldots, T_r \in k[T_1, \ldots, T_r]$. Let $X = \mathbb{A}^r - (0)$. We know that each X_{T_i} is affine and that the X_{T_i}'s cover X. Yet, X is not affine. This is because $T_1, \ldots, T_r$ do not generate the unit ideal. Therefore, Serre's condition (5b) is essential.

Theorem 4.24 *Let X be a scheme and assume that $\mathfrak{I}$ is a QC $\mathcal{O}_X$-ideal such that $\mathfrak{I}^n = (0)$ for some $n \geq 1$. Let X_0 be the scheme given by $\mathfrak{I}$. Then, X is affine iff X_0 is affine.*

Proof. If X is affine, then X_0 is also affine, since it is a closed subscheme of an affine scheme.

Let us now assume that X_0 is affine. As $\mathfrak{I}$ is nilpotent, $|X_0| = |X|$. Let X_k be the subscheme determined by $\mathfrak{I}^{k+1}$. We prove by induction on k that X_k is affine. Observe that the passage from X_k to X_{k+1} involves the ideal $\widetilde{\mathfrak{I}} = \mathfrak{I}^{k+1}/\mathfrak{I}^{k+2}$ of $\mathcal{O}_X/\mathfrak{I}^{k+2}$, and $\widetilde{\mathfrak{I}}$ satisfies the equation

$$\widetilde{\mathfrak{I}}^2 = (0).$$

Therefore, we are reduced to the case $\mathfrak{I}^2 = (0)$. Since $\mathfrak{I}^2 = (0)$, the ideal $\mathfrak{I}$ is also an $\mathcal{O}_{X/\mathfrak{I}}$-module, and since X_0 is determined by $\mathfrak{I}$, the sheaf $\mathcal{O}_{X_0}$ is $\mathcal{O}_{X/\mathfrak{I}}$, and $\mathfrak{I}$ is a QC $\mathcal{O}_{X_0}$-module. By Serre's criterion (4), we have $H^1(X_0, \mathfrak{I}) = (0)$, and since $|X| = |X_0|$, we get

$$H^1(X, \mathfrak{I}) = (0).$$

Let $\mathfrak{A}$ be any $\mathcal{O}_X$-ideal. We have the exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} + \mathfrak{I} \longrightarrow (\mathfrak{A} + \mathfrak{I})/\mathfrak{I} \longrightarrow 0, \tag{$\dagger_1$}$$

where $(\mathfrak{A} + \mathfrak{I})/\mathfrak{I}$ is an $\mathcal{O}_X/\mathfrak{I} = \mathcal{O}_{X_0}$-module. Applying cohomology to $(\dagger_1)$, we get

$$H^1(X, \mathfrak{I}) \longrightarrow H^1(X, \mathfrak{A} + \mathfrak{I}) \longrightarrow H^1(X, (\mathfrak{A} + \mathfrak{I})/\mathfrak{I}),$$

but we showed that $H^1(X, \mathfrak{I}) = (0)$ and

$$H^1(X, (\mathfrak{A} + \mathfrak{I})/\mathfrak{I}) \cong H^1(X_0, (\mathfrak{A} + \mathfrak{I})/\mathfrak{I}) = (0),$$

since X_0 is affine. Therefore,

$$H^1(X, \mathfrak{A} + \mathfrak{I}) = (0).$$

We also have the exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A} + \mathfrak{I} \longrightarrow (\mathfrak{A} + \mathfrak{I})/\mathfrak{A} \longrightarrow 0. \tag{$\dagger_2$}$$

However, we have the isomorphism

$$(\mathfrak{A} + \mathfrak{I})/\mathfrak{A} \cong \mathfrak{I}/\mathfrak{I} \cap \mathfrak{A},$$

and $\mathfrak{I}/\mathfrak{I} \cap \mathfrak{A}$ is an $\mathcal{O}_{X_0}$-ideal. However, in the exact sequence

$$0 \longrightarrow \mathfrak{I} \cap \mathfrak{A} \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{I}/\mathfrak{I} \cap \mathfrak{A} \longrightarrow 0, \tag{$\dagger_3$}$$

all ideals are $\mathcal{O}_{X_0}$-ideals. As X_0 is affine and $\Gamma(X_0, -)$ is thus exact, we get that

$$\Gamma(X_0, \mathfrak{I}) \longrightarrow \Gamma(X_0, \mathfrak{I}/\mathfrak{I} \cap \mathfrak{A}) \longrightarrow$$

is surjective. Since $|X| = |X_0|$, the sequence

$$\Gamma(X, \mathfrak{I}) \longrightarrow \Gamma(X, \mathfrak{I}/\mathfrak{I} \cap \mathfrak{A}) \longrightarrow 0$$

is exact. We have the commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathfrak{I} + \mathfrak{A}) \arrow[r] & \Gamma(X, (\mathfrak{I} + \mathfrak{A})/\mathfrak{A}) & \\
\Gamma(X, \mathfrak{I}) \arrow[r] \arrow[u] & \Gamma(X, \mathfrak{I}/\mathfrak{I} \cap \mathfrak{A}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the second vertical arrow is an isomorphism. Therefore, the top horizontal map $\Gamma(X, \mathfrak{I} + \mathfrak{A}) \longrightarrow \Gamma(X, (\mathfrak{I} + \mathfrak{A})/\mathfrak{A})$ is surjective. By taking the cohomology of ($\dagger_2$), we get that

$$H^1(X, \mathfrak{A}) \longrightarrow H^1(X, \mathfrak{A} + \mathfrak{I})$$

is injective. Yet, $H^1(X, \mathfrak{A} + \mathfrak{I}) = (0)$. So, $H^1(X, \mathfrak{A}) = (0)$, and we conclude by Serre's criterion that X is affine. $\square$

Corollary 4.25 *Let X be a noetherian scheme. Then, X is affine iff X_{red} is affine.*

Proof. Let $\mathcal{N} = \mathcal{N}(X)$ be the nil-ideal of $\mathcal{O}_X$. Then, $\mathcal{N}$ is nilpotent as X is noetherian, and we apply Theorem 4.24. $\square$

Remark: If X is given and $\mathfrak{I}$ is a QC ideal of $\mathcal{O}_X$ such that $\mathfrak{I}^2 = (0)$, then, for X_0 determined by $\mathfrak{I}$, we say that X is an *infinitesimal extension of X_0 by $\mathfrak{I}$*. We proved that an infinitesimal extension of an affine scheme is affine.

How does one make infinitesimal extensions of X_0? How can one classify them? The answers to these two questions turn out to be very important–of great use in making examples of nonintuitive phenomena in algebraic geometry. In fact, these phenomena were exactly the sort of phenomena which elluded (in the main) the efforts of the classical Italian geometers of the early twentieth century. *They* did not possess a full cohomology theory, and *we do*. We will return to these questions in the next chapter.

4.5 Further Readings

The material on the cohomology of quasi-coherent sheaves over an affine scheme and the Koszul Complex (Section 4.1) can be found in EGA IIIa [24], Section 1 (see also Serre [52], Chapter IV, for the Koszul Complex). Cartan's isomorphism theorem and most related material (Section 4.2) is covered in Godement [18], Chapter V, and in Hartshorne [33], Chapter III. The cohomology of affine schemes (Section 4.3) is discussed EGA IIIa [24], Section 1, Hartshorne [33], Chapter III, and Ueno [57]. The cohomological characterization of affine schemes (Section 4.4) is due to Serre in the case of coherent sheaves over algebraic varieties (Serre [50]). The generalization to quasi-coherent sheaves appears in EGA II [23], Section 5.2 (Le critère de Serre), page 97–99, and Hartshorne [33], Chapter III. In both of these references, the proof is a bit sketchy. Danilov's chapter in [10] contains an excellent informal introduction to cohomological methods in algebraic geometry, and Dieudonné [12] gives a fascinating account of the history of algebraic geometry up to 1970.

Chapter 5

Bundles and Geometry

5.1 Locally Free Sheaves and Bundles

This chapter and the next two form the heart of the material in these notes. The language of bundles and locally free sheaves together with the theorems one can prove about them strike directly at the center of geometric questions in algebraic and complex analytic geomety. In particular, the structure of the set of codimension one subspaces of a scheme is intimately connected with the collection of line bundles on the scheme. Unfortunately, for higher dimension, the connection to vector bundles is much weaker–but it is a beginning. In the literature, constant use of the concepts and theorems of this and the next two chapters is the norm.

Throughout this section, $(X, \mathcal{O}_X)$ denotes a ringed space.

Definition 5.1 An $\mathcal{O}_X$-module, $\mathcal{F}$, is *locally free* if for every $x \in X$, there is some open subset, U, with $x \in U$, so that

$$\mathcal{F} \upharpoonright U \cong (\mathcal{O}_X \upharpoonright U)^{(I)} = \mathcal{O}_X^{(I)} \upharpoonright U$$

for some set I (possibly dependent on U). If for some covering family, (U_α), the sets, I, are all finite, we say that $\mathcal{F}$ is *locally free of finite rank*.

We have the following basic proposition:

Proposition 5.1 *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{F}$, $\mathcal{G}$ be two finitely presented $\mathcal{O}_X$-modules, and $\mathcal{M}$ any $\mathcal{O}_X$-module. The following properties hold:*

(1) If $x \in X$, the canonical map

$$\theta\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M})_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{M}_x)$$

is an isomorphism.

(2) If $x \in X$ and $\mathcal{F}_x$ is isomorphic to $\mathcal{G}_x$, then there is some open subset, U, with $x \in U$, so that $\mathcal{F} \restriction U \longrightarrow \mathcal{G} \restriction U$ is an isomorphism.

Proof. (1) The question is local, even punctual, on X. Thus, we may assume that

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{is exact.}$$

We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M})_x \arrow[r] \arrow[d, "\theta"'] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{M})_x \arrow[r] \arrow[d, "\alpha"'] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{M})_x \arrow[d, "\beta"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_x, \mathcal{M}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{X,x}^q, \mathcal{M}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{X,x}^p, \mathcal{M}_x),
\end{tikzcd}$$

and α and β are isomorphisms, since

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{M})_x \cong \mathcal{M}_x^q \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{X,x}^q, \mathcal{M}_x) \cong \mathcal{M}_x^q.$$

By the five lemma, θ is an isomorphism.

Next, let $\varphi\colon \mathcal{F}_x \to \mathcal{G}_x$ and $\psi\colon \mathcal{G}_x \to \mathcal{F}_x$ be inverse isomophisms at x. By (1), there exist some opens $W, \widetilde{W}$, with $x \in W, \widetilde{W}$, $u \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \restriction W, \mathcal{G} \restriction W)$, and $v \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \restriction \widetilde{W}, \mathcal{G} \restriction \widetilde{W})$, such that

$$u_x = \varphi \quad \text{and} \quad v_x = \psi.$$

Look at $u \circ v$ and $v \circ u$ on $V = W \cap \widetilde{W}$. Both become the identity at x. Then, $u \circ v - \mathrm{id}$ and $v \circ u - \mathrm{id}$ both go to 0 at x. Therefore, there is some open $U \subseteq V$ such that

$$u \circ v = \mathrm{id} \quad \text{and} \quad v \circ u = \mathrm{id} \quad \text{on } U, \text{ by (1).}$$ ☐

Corollary 5.2 *Given $(X, \mathcal{O}_X)$, assume that $\mathcal{O}_X$ is coherent. For every $x \in X$, for every finitely presented $\mathcal{O}_{X,x}$-module M, there exists an open subset, U, with $x \in U$, and a coherent $\mathcal{O} \restriction U$-module $\mathcal{G}$ so that $\mathcal{G}_x = M$.*

Proof. Since M is finitely presented, we have an exact sequence

$$\mathcal{O}_{X,x}^p \xrightarrow{\varphi} \mathcal{O}_{X,x}^q \longrightarrow M \longrightarrow 0.$$

Since $\mathcal{O}_X$ is coherent, it is finitely presented as $\mathcal{O}_X$-module, and so is $\mathcal{O}_X^p$. By Proposition 5.1 part (1), there is some open subset, U, with $x \in U$, and a map

$$u\colon \mathcal{O}_X^p \restriction U \to \mathcal{O}_X^q \restriction U$$

so that $u_x = \varphi$. Letting $\mathcal{G} = \mathrm{Coker}\, u$, we have a coherent $\mathcal{O}_X \restriction U$-module such that $\mathcal{G}_x = M$. ☐

Corollary 5.3 *If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of finite rank and $\mathcal{F}$ has rank $n(x)$ at some $x \in X$, then there is some open subset, U, with $x \in U$, so that $\mathcal{F} \upharpoonright U$ is locally free and of rank $n(x)$ on all of U. Hence, $\mathrm{rk}\, \mathcal{F} = n$ is an open condition on X, and if X is connected and $\mathcal{F}$ has finite rank, then $\mathcal{F}$ has constant rank.*

Proof. Since $\mathcal{F}$ is locally free of finite rank, it is finitely presented, and at x,

$$\mathcal{F}_x \cong \mathcal{O}_{X,x}^{n(x)}.$$

By Proposition 5.1 part (2), there is some open subset, U, with $x \in U$, so that $\mathcal{F} \upharpoonright U$ is locally free and of rank $n(x)$ on all of U. $\square$

If $\mathcal{G}$ is locally free, but not necessarily of finite rank, then the functor

$$\mathcal{M} \mapsto \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{M}$$

is exact on $\mathcal{O}_X$-modules. For any such $\mathcal{G}$, we let

$$\mathcal{G}^D = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X),$$

and call $\mathcal{G}^D$ the *dual* of $\mathcal{G}$.

Remark: Having finite rank is the most important case for a locally free sheaf. Thus, from now on, locally free means locally free of finite rank, *unless otherwise specified*. Of course, $\mathcal{F}$ has finite rank iff $\mathcal{F}^D$ has finite rank.

Proposition 5.4 *If $\mathcal{G}$ is an $\mathcal{O}_X$-module, then for any $\mathcal{O}_X$-module $\mathcal{M}$, there is a homomorphism*

$$\mathcal{G}^D \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}). \qquad (*)$$

Furthermore, if $\mathcal{G}$ is locally free, then $(*)$ is an isomorphism.

Proof. Let U be some open subset of X. For all $(\xi, \eta) \in \mathcal{G}^D(U) \times \mathcal{M}(U)$, we have $\xi \in \mathcal{G}^D(U) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G} \upharpoonright U, \mathcal{O}_X \upharpoonright U)$, and we define the map

$$(\xi, \eta) \mapsto \xi \cdot \eta \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G} \upharpoonright U, \mathcal{M} \upharpoonright U)$$

by associating to (ξ, η) the element $\xi \cdot \eta \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G} \upharpoonright U, \mathcal{M} \upharpoonright U)$ defined so that for every $x \in U$ and every $s_x \in \mathcal{G}_x$, the element s_x is mapped to $\xi_x(s_x)\eta_x \in \mathcal{M}_x$. This map is bilinear, and thus, it is equivalent to a linear map

$$\mathcal{G}^D(U) \otimes \mathcal{M}(U) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})(U).$$

However, $\mathcal{G}^D(U) \otimes \mathcal{M}(U)$ is just a presheaf, and when we sheafify, our map factors through $\mathcal{G}^D \otimes \mathcal{M}$. We get the homomorphism $(*)$. If we assume that $\mathcal{G}$ is locally free, then whether or not $(*)$ is an isomorphism is a purely local question. Thus, we may assume that $\mathcal{G} \cong \mathcal{O}_X^{(I)}$, in which case both sides are isomorphic to $\mathcal{M}^{(I)}$ (remember, I is finite). $\square$

As a consequence, if $\mathcal{G}$ is locally free, then the functor

$$\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})$$

is exact, and thus, its right derived functors vanish, i.e.,

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}) = (0) \quad \text{for all } r > 0.$$

However, we have the local-global $\mathcal{E}xt$-spectral sequence (see Appendix B)

$$H^p(X, \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})) \Longrightarrow \mathrm{Ext}^\bullet_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}).$$

If $\mathcal{G}$ is locally free, we have just shown that the spectral sequence degenerates, which implies that

$$H^p(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})) \cong \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}) \quad \text{for all } p \geq 0.$$

This proves the following proposition:

Proposition 5.5 *Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{G}$ be a locally free sheaf. Then, we have the isomorphisms*

$$H^p(X, \mathcal{G}^D \otimes \mathcal{M}) \cong H^p(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M})) \cong \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}).$$

If X is an affine scheme, then for QC $\mathcal{O}_X$-modules, $\mathcal{M}$, we deduce that

$$\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}) = (0) \quad \textit{for all } p > 0.$$

As a consequence of Proposition 5.5, when X is affine, all multiextensions

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}_p \longrightarrow \cdots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{G} \longrightarrow 0$$

split (as well as the ordinary extensions, i.e., when $p = 1$).

We can also prove the following proposition, which is more general in some sense, and less general in another sense:

Proposition 5.6 *If $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules, with $\mathcal{G}$ locally free of finite rank, and if there is a given extension*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 0,$$

then, for every $x \in X$, there is some open subset, U, with $x \in U$, so that the sequence

$$0 \longrightarrow \mathcal{F} \upharpoonright U \longrightarrow \mathcal{H} \upharpoonright U \longrightarrow \mathcal{G} \upharpoonright U \longrightarrow 0$$

splits.

Remark: Here, $(X, \mathcal{O}_X)$ need not be a scheme; so, vanishing of cohomology on sufficiently fine opens may not hold.

Proof. The question is local on X. Thus, we may assume that $\mathcal{G} \cong \mathcal{O}_X^n$ for some n. Let $s_1, \ldots, s_n$ be the canonical sections. Then, there is some open subset, U, so that the sections $s_1, \ldots, s_n$ lift to sections $t_1, \ldots, t_n \in \mathcal{H}(U)$. However, each section $t \in \mathcal{H}(U)$ defines a map $\mathcal{O}_X(U) \longrightarrow \mathcal{H}(U)$, and thus, we get a map

$$\mathcal{O}_X^n \restriction U = \mathcal{G} \restriction U \longrightarrow \mathcal{H} \restriction U$$

which splits the sequence. $\square$

Definition 5.2 An $\mathcal{O}_X$-module $\mathcal{L}$ on a ringed space $(X, \mathcal{O}_X)$ is *invertible* if $\mathcal{L}$ is locally free of rank 1.

Let $\mathcal{L}$ be any locally free $\mathcal{O}_X$-module (remember: finite rank). We have an isomorphism

$$\mathcal{L}^D \otimes \mathcal{L} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}).$$

If $\mathcal{L}$ is invertible, more is true:

Proposition 5.7 *For any invertible sheaf $\mathcal{L}$, on the ringed space, $(X, \mathcal{O}_X)$, there is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X.$$

Proof. For any sheaf $\mathcal{F}$, there is a canonical map

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}).$$

Indeed, for any open subset U and any $\sigma \in \mathcal{O}_X(U)$, multiplication by σ gives a map $\mathcal{F} \restriction U \longrightarrow \mathcal{F} \restriction U$, and all these maps patch to yield the desired map. If $\mathcal{F}$ is invertible, then our map

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. For this, it is enough to check locally; and we may assume that $\mathcal{F} = \mathcal{O}_X$. But then, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X$, which concludes the proof. $\square$

If $\mathcal{L}$ is an invertible sheaf, then by taking global sections, we get an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \Gamma(X, \mathcal{O}_X).$$

When X is compact (or proper), connected, and complex analytic, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Therefore, in this case,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) = \mathrm{End}(\mathcal{L}) \cong \mathbb{C}.$$

Also observe that if $\mathcal{L}$ is invertible, then

$$\mathcal{L}^D \otimes \mathcal{L} \cong \mathcal{O}_X.$$

From this, we get immediately

$$\mathcal{L}^{\otimes p} \otimes \mathcal{L}^{\otimes q} \cong \mathcal{L}^{\otimes (p+q)}, \quad \text{for all } p, q \in \mathbb{Z}$$

and

$$\mathcal{L}^{\otimes p} \otimes \mathcal{L}^{\otimes q} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{\otimes -p}, \mathcal{L}^{\otimes q}) \quad \text{for all } p, q \in \mathbb{Z}.$$

We can check as well that if $\mathcal{L}$ and $\mathcal{M}$ are invertible, then $\mathcal{L} \otimes \mathcal{M}$ is invertible. Therefore, the isomorphism classes of invertible sheaves on $(X, \mathcal{O}_X)$ form a group under $\otimes$. This group if called the *Picard group* of X, and is denoted by $\mathrm{Pic}(X)$. It is a fundamental invariant of X.

Proposition 5.8 *Let A be a local ring and M be a finitely generated A-module. For any A-module, M', if $M \otimes_A M' \cong A$, then $M \cong A$. In particular, M' must then also be isomorphic to A.*

Proof. Reduce mod $\mathfrak{m}$ (the maximal ideal of A). We get

$$M/\mathfrak{m}M \otimes_{\kappa(A)} M'/\mathfrak{m}M' \cong \kappa(A).$$

Since these are vector spaces over the field $\kappa(A)$, we must have $\mathrm{rk}(M/\mathfrak{m}M) = 1$. By Nakayama's lemma, since M is finitely generated, it is generated by a single element ξ. We have a surjective map $A \longrightarrow M$ *via* $1 \mapsto \xi$. The kernel of this map is the annihilator of ξ: $\mathfrak{A} = \mathrm{Ann}(\xi)$. Now, $\mathfrak{A}$ kills $M \otimes_A M' \cong A$, and thus, $\mathfrak{A}$ kills A. This implies that $\mathfrak{A} = (0)$, and $M \cong A$. $\square$

Proposition 5.9 *Let $(X, \mathcal{O}_X)$ be a* **local** *ringed space and assume that $\mathcal{O}_X$ is coherent. If $\mathcal{L}$ is a coherent $\mathcal{O}_X$-module, then the following are equivalent:*

(1) $\mathcal{L}$ is invertible.

(2) There is some $\mathcal{O}_X$-module, $\mathcal{M}$, so that

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X.$$

(3) For every $x \in X$, there is some open subset, U, with $x \in U$, and an $\mathcal{O}_X$-module, $\mathcal{M}$, so that

$$\mathcal{M} \upharpoonright U \otimes_{\mathcal{O}_X \upharpoonright U} \mathcal{L} \upharpoonright U \cong \mathcal{O}_X \upharpoonright U.$$

(4) For every $x \in X$, there is an $\mathcal{O}_{X,x}$-module, $\mathcal{M}_x$, so that

$$\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x \cong \mathcal{O}_{X,x}.$$

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), are clear. Assume (4). Since $\mathcal{L}$ is coherent, it is finitely generated. Therefore, we can apply Proposition 5.8 (with $M = \mathcal{L}_x$), and we get

$$\mathcal{L}_x \cong \mathcal{O}_{X,x}.$$

But $\mathcal{L}$ is finitely presented because $\mathcal{L}$ is coherent, and so is $\mathcal{O}_X$, as $\mathcal{O}_X$ is also coherent. Apply Proposition 5.1 part (2). We find that there is some open subset, U, with $x \in U$, and an isomorphism

$$u\colon \mathcal{L} \upharpoonright U \to \mathcal{O}_X \upharpoonright U.$$

However, this means that $\mathcal{L}$ is invertible. $\square$

Corollary 5.10 *Proposition 5.9 also holds when X is a locally noetherian scheme or a complex analytic space of finite dimension.*

Proof. In the first case, we know that $\mathcal{O}_X$ is coherent. In the second, the coherence of $\mathcal{O}_X$ is a fundamental theorem of Oka as reformulated by Cartan and Serre.

Given a ringed space $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X^*$, also denoted $\mathbb{G}_m$, is defined by setting

$$\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_X)^*$$

for every open subset, U, of X. Here, $\Gamma(U, \mathcal{O}_X)^*$ is the group of units in the ring $\Gamma(U, \mathcal{O}_X)$. The following important theorem shows the relationship between the Picard group and the cohomology of $\mathbb{G}_m$:

Theorem 5.11 *Let $(X, \mathcal{O}_X)$ be a ringed space. There is a canonical isomorphism*

$$H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X).$$

Proof. The presheaf $\mathcal{H}^1(\mathbb{G}_m)$ is defined by

$$\mathcal{H}^1(\mathbb{G}_m)(U) = H^1(U, \mathbb{G}_m).$$

By a familiar argument (Poincaré's lemma), the associated sheaf is trivial, and thus,

$$\check{H}^0(X, \mathcal{H}^1(\mathbb{G}_m)) = (0).$$

Using the Čech spectral sequence (see Appendix B), we get the exact sequence

$$0 \longrightarrow \check{H}^1(X, \mathbb{G}_m) \longrightarrow H^1(X, \mathbb{G}_m) \longrightarrow \check{H}^0(X, \mathcal{H}^1(\mathbb{G}_m)) = (0).$$

Therefore, we have the isomorphism

$$\check{H}^1(X, \mathbb{G}_m) \cong H^1(X, \mathbb{G}_m).$$

For any U open in X, let us denote $\mathcal{O}_X \restriction U$ by $\mathcal{O}_U$. Observe that the automorphism group, $\mathrm{Aut}_{\mathcal{O}_U}(\mathcal{O}_U)$, of the $\mathcal{O}_U$-module $\mathcal{O}_U$ is isomorphic to the (multiplicative) group $\Gamma(U, \mathcal{O}_X^*)$, via the map sending any section $t \in \Gamma(U, \mathcal{O}_X^*)$ to the automorphism $u \in \mathrm{Aut}_{\mathcal{O}_U}(\mathcal{O}_U)$ defined by

$$u_x(s_x) = t_x s_x, \quad \text{for all } x \in U \text{ and all } s_x \in \mathcal{O}_{X,x}.$$

Let $\mathcal{U} = \{U_\alpha \to X\}$ be a cover of X. First, we define a map

$$\varphi_{\mathcal{U}} \colon H^1(\{U_\alpha \to X\}, \mathcal{O}_X^*) \longrightarrow \mathrm{Pic}(X).$$

Observe that a Čech cochain θ is defined by a family, (θ_β^α), of sections $\theta_\beta^\alpha \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$, that is, in view of the isomorphism

$$\Gamma(U, \mathcal{O}_X^*) \cong \mathrm{Aut}_{\mathcal{O}_U}(\mathcal{O}_U),$$

by a family of automorphisms θ_β^α of $\mathcal{O}_X \restriction U_\alpha \cap U_\beta$. Also, the cochain θ is a cocycle iff

(1) $\theta_\alpha^\beta = (\theta_\beta^\alpha)^{-1}$.

(2) $\theta_\beta^\alpha \cdot \theta_\gamma^\beta = \theta_\gamma^\alpha$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

However, these are precisely the gluing conditions, and we can define the invertible sheaf $\mathcal{L}_\theta$ by gluing the free sheaves $\mathcal{O}_{U_\alpha}$ together, via the gluing maps

$$\theta_\beta^\alpha \colon \mathcal{O}_{U_\beta} \restriction U_\alpha \cap U_\beta \longrightarrow \mathcal{O}_{U_\alpha} \restriction U_\alpha \cap U_\beta.$$

For later use, let

$$\psi_\alpha(\theta) \colon \mathcal{O}_{U_\alpha} \to \mathcal{L}_\theta \restriction U_\alpha$$

be the isomorphism on U_α, for every index α, and note that

$$\theta_\beta^\alpha = \psi_\alpha(\theta)^{-1}\psi_\beta(\theta) \quad \text{on } U_\alpha \cap U_\beta.$$

Thus $\theta = (\theta_\beta^\alpha)$ in $Z^1(\{U_\alpha \to X\}, \mathcal{O}_X^*)$ gives us an element of $\mathrm{Pic}(X)$. If $\eta = (\eta_\beta^\alpha)$ is another cocycle in $Z^1(\{U_\alpha \to X\}, \mathcal{O}_X^*)$, then, θ and η are cohomologous iff

$$\eta_\beta^\alpha = \omega_\alpha \cdot \theta_\beta^\alpha \cdot \omega_\beta^{-1} \tag{$*$}$$

for some 0-cochain, ω, in $C^0(\{U_\alpha \to X\}, \mathcal{O}_X^*)$. But such a cochain is a family $\omega = (\omega_\alpha)$ of automorphisms of $\mathcal{O}_{U_\alpha}$, and the condition $(*)$ means that the invertible sheaves $\mathcal{L}_\theta$ and $\mathcal{L}_\eta$ are isomorphic. Here, the isomorphism is given by the local isomorphisms

$$\psi_\alpha(\eta)\omega_\alpha\psi_\alpha(\theta)^{-1} \colon \mathcal{L}_\theta \to \mathcal{L}_\eta \quad \text{on } U_\alpha,$$

which patch on $U_\alpha \cap U_\beta$, by condition $(*)$ (DX). Therefore, we get a map

$$\varphi_{\mathcal{U}} \colon H^1(\{U_\alpha \to X\}, \mathcal{O}_X^*) \longrightarrow \mathrm{Pic}(X).$$

If $\mathcal{V} = \{V_\alpha \to X\}$ is another cover of X refining $\mathcal{U}$, the naturality of the gluing implies that the following diagram commutes:

$$\begin{tikzcd}
H^1(\{U_\alpha \to X\}, \mathcal{O}_X^*) \arrow[dd] \arrow[dr, "\varphi_{\mathcal{U}}"] & \\
& \mathrm{Pic}(X) \\
H^1(\{V_\alpha \to X\}, \mathcal{O}_X^*) \arrow[ur, "\varphi_{\mathcal{V}}"'] &
\end{tikzcd}$$

By passing to the inductive limit, we get a map

$$\varphi_X\colon \check{H}^1(X, \mathcal{O}_X^*) \longrightarrow \mathrm{Pic}(X).$$

Since an inductive limit of injective maps is injective, to prove that the map φ_X is injective, it is enough to prove that each $\varphi_{\mathcal{U}}$ is injective. However, if $\varphi_{\mathcal{U}}(\theta) = 0$, then the invertible sheaf $\mathcal{L}_\theta$ constructed from θ is trivial, i.e., $\mathcal{L}_\theta \cong \mathcal{O}_X$. The image $\psi_\alpha(1)$ of the section $1 \in \Gamma(X, \mathcal{O}_X^*)$ is some section $\omega_\alpha \in \Gamma(U_\alpha, \mathcal{L}_\theta)$ (where $\psi_\alpha\colon \mathcal{O}_{U_\alpha} \to \mathcal{L}_\theta \restriction U_\alpha$ is the isomorphism.) Now, since the automorphisms θ^α_β of $\mathcal{O}_X \restriction U_\alpha \cap U_\beta$ are just multiplication by θ^α_β, we have

$$\omega_\alpha = \theta^\alpha_\beta \cdot \omega_\beta \quad \text{on } U_\alpha \cap U_\beta, \text{ for all } \alpha, \beta,$$

which implies that θ is the coboundary of ω. Thus, $\varphi_{\mathcal{U}}$ is indeed injective. The map φ_X is surjective because every invertible sheaf comes from a cover where it is locally trivial, and thus, is obtained by gluing, i.e., from a cocycle.

It remains to prove that φ_X is a group homomorphism. It is enough to prove this for $\varphi_{\mathcal{U}}$, since being a homomorphism is preserved by taking the inductive limit. Let θ and η be two cocycles in $Z^1(\{U_\alpha \to X\}, \mathcal{O}_X^*)$ (by refining covers, we may assume that θ and η are defined over the same cover). For each α, if $a_\alpha = \psi_\alpha(1)$, where $\psi_\alpha\colon \mathcal{O}_{U_\alpha} \to \mathcal{L}_\theta \restriction U_\alpha$ is an isomorphism, we have

$$\Gamma(U_\alpha, \mathcal{L}_\theta) = \Gamma(U_\alpha, \mathcal{O}_{U_\alpha})a_\alpha,$$

and similarly

$$\Gamma(U_\alpha, \mathcal{L}_\eta) = \Gamma(U_\alpha, \mathcal{O}_{U_\alpha})b_\alpha.$$

But the automorphisms θ^α_β and η^α_β act by multiplication, and so,

$$a_\alpha = \theta^\alpha_\beta \cdot a_\beta \quad \text{and} \quad b_\alpha = \eta^\alpha_\beta \cdot b_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

Now, $\Gamma(U_\alpha, \mathcal{L}_\theta \otimes \mathcal{L}_\eta)$ consists of linear combinations of the $a_\alpha \otimes b_\alpha$'s, and since

$$a_\alpha \otimes b_\alpha = (\theta^\alpha_\beta \cdot a_\beta) \otimes (\eta^\alpha_\beta \cdot b_\beta) = \theta^\alpha_\beta \cdot \eta^\alpha_\beta (a_\beta \otimes b_\beta),$$

we see that $\mathcal{L}_\theta \otimes \mathcal{L}_\eta$ does correspond to $(\theta^\alpha_\beta \cdot \eta^\alpha_\beta)$. $\square$

Remark: Observe that the above proof holds for any arbitrary ringed space. Nowhere did we use the fact that X is an LRS, that X is affine, etc. In particular, the result holds for X a topological space and $\mathcal{O}_X$ the sheaf of germs of C^k-functions ($0 \leq k \leq \omega$), real or complex. Moreover, because we were careful in the proof to use the proper definitions which do not depend upon the commutativity of $\mathbb{G}_m$, the same proof applies immediately to $\mathbb{GL}(n)$. Here, we denote by $\mathbb{GL}(X, n)$ the sheaf of (nonabelian) groups defined so that for every open subset $U \subseteq X$,

$$\Gamma(U, \mathbb{GL}(X, n)) = \mathrm{GL}(\Gamma(U, \mathcal{O}_X)^n),$$

the group of linear invertible maps of the free module $\Gamma(U, \mathcal{O}_X)^n \cong \Gamma(U, \mathcal{O}_X^n)$. It should also be noted that $\check{H}^1(X, \mathbb{GL}(n))$ is generally only a set, and not a group, since $\mathbb{GL}(n)$ is not abelian for $n \geq 2$. The reader will find the definitions relative to nonabelian Čech cohomology in Appendix B. We obtain the

Corollary 5.12 *Let $(X, \mathcal{O}_X)$ be a ringed space. There is a canonical isomorphism*

$$\check{H}^1(X, \mathbb{GL}(n)) \approx \mathcal{LF}_n(\mathcal{O}_X),$$

where $\mathcal{LF}_n(\mathcal{O}_X)$ is the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank n on X.

Let us consider the complex analytic case. Let X be a complex analytic space, for example, X^{an}, where X is a finitely generated scheme over $\mathbb{C}$. If $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions in the *norm topology*, we have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*.$$

If X is a manifold, the local existence of a logarithm implies that the exponential map is surjective. We can apply cohomology, and we get

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathbb{Z}) \arrow[r] & H^0(X,\mathcal{O}_X) \arrow[r, "\exp"] & H^0(X,\mathcal{O}_X^*) \arrow[dll] \\ & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & \mathrm{Pic}(X) \arrow[dll, "c"] \\ & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathcal{O}_X) \arrow[r] & \cdots, \end{tikzcd}$$

where the map $c\colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ plays a special role. If X is a complex manifold, we can examine the two cases:

(1) X is compact connected, for example, $\mathbb{P}^m_{\mathbb{C}}$.

(2) X is connected and simply connected.

In the first case, $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$, and exp is onto.

In the second case, as X is simply connected, the map $\exp\colon H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*)$ is surjective, as a single valued branch of the logarithm exists on X.

In both cases, the sequence

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X)$$

is exact. The map c takes an invertible sheaf of $\mathcal{O}_X$-modules to its *first Chern class*. The kernel of c consists of the invertible sheaves with trivial Chern class, and $\mathrm{Ker}\, c$ is isomorphic to $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. The kernel of c is usually denoted, $\mathrm{Pic}^0(X)$. For $X = \mathbb{P}^m_{\mathbb{C}}$, we will prove later that $H^1(X, \mathcal{O}_X) = (0)$. In this case, $c\colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an injection. We will also find that $H^2(X, \mathcal{O}_X) = (0)$.

If X is a curve of genus g, then

$$H^1(X, \mathcal{O}_X) = \mathbb{C}^g,$$

and

$$H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g},$$

a lattice in $\mathbb{C}^g$. So, $\mathrm{Pic}^0(X)$ is a torus.

To return to the general case, let $f\colon Y \to X$ be a morphism. If $\mathcal{L}$ is locally free of finite rank, then $f^*\mathcal{L}$ is also locally free as an $\mathcal{O}_Y$-module and the rank is preserved. We claim that there is a canonical map

$$f^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{B})$$

for any two $\mathcal{O}_X$-modules $\mathcal{A}, \mathcal{B}$, not necessarily locally free. By adjointness, we need only give a map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}) \longrightarrow f_*\mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{B}).$$

Let U be an open subset of X. We have

$$\begin{aligned}
\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A} \restriction U, \mathcal{B} \restriction U) \\
\Gamma(U, f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{A}, f^*\mathcal{B})) &= \Gamma(f^{-1}(U), \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{A}, f^*\mathcal{B})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{A} \restriction f^{-1}(U), f^*\mathcal{B} \restriction f^{-1}(U)).
\end{aligned}$$

Our map is

$$\xi \mapsto f^*(\xi),$$

and it patches on overlaps.

If $\mathcal{A}$ is locally free of finite rank, and f taking $\mathcal{A}$ to $\mathcal{B}$ is a map of $\mathcal{O}_X$-modules, then injectivity, surjectivity, and isomorphism, are local properties on X; so, we may assume that that $\mathcal{A} = \mathcal{O}_X^r$. For $\mathcal{A} = \mathcal{O}_X^r$, we see that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^r, \mathcal{B}) = \mathcal{B}^r$ and $f^*\mathcal{A} = \mathcal{O}_Y^r$. Consequently,

$$\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{O}_X^r, f^*\mathcal{B}) = (f^*\mathcal{B})^r,$$

and our map above is

$$f^*(\mathcal{B}^r) \longrightarrow (f^*\mathcal{B})^r,$$

which is an isomorphism, since r is finite. Therefore, if $\mathcal{A}$ is locally free of finite rank, the map

$$f^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{B})$$

is an isomorphism. If $\mathcal{L}$ is locally free and we let $\mathcal{B} = \mathcal{O}_X$, then

$$f^*(\mathcal{L}^D) = (f^*\mathcal{L})^D.$$

If $\mathcal{L}$ and $\mathcal{M}$ are locally free, we also have

$$f^*(\mathcal{L} \otimes \mathcal{M}) = f^*\mathcal{L} \otimes f^*\mathcal{M}.$$

Moreover, if $\mathcal{L}$ and $\mathcal{M}$ are invertible, then

$$f^*(\mathcal{L}^{\otimes p} \otimes \mathcal{M}^{\otimes q}) = (f^*\mathcal{L})^{\otimes p} \otimes (f^*\mathcal{M})^{\otimes q}. \qquad \text{(DX)}$$

The reader should show that the diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d, "H^1(f)"'] & \mathrm{Pic}(X) \arrow[d, "f^*"] \\ H^1(Y, \mathcal{O}_Y^*) \arrow[r] & \mathrm{Pic}(Y) \end{tikzcd}$$

commutes.

Proposition 5.13 *(Projection formula) Let X, Y be ringed spaces and let $\mathcal{F}$ be any $\mathcal{O}_X$-module, and $\mathcal{L}$ be any locally free sheaf of finite rank. For any morphism $f\colon Y \to X$, there is an isomorphism*

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}).$$

Proof. We always have a canonical map

$$\mathcal{L} \longrightarrow f_*(f^*\mathcal{L}).$$

Thus, we get

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} f_*(f^*\mathcal{L}).$$

However, we always have the map

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} f_*(f^*\mathcal{L}) \longrightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}) \qquad \text{(DX)},$$

and by composition, we get our map. As isomorphism is a local property, we may assume that $\mathcal{L} = \mathcal{O}_X^p$. Then, the lefthand side is $(f_*(\mathcal{F}))^p$ and the righthand side is $f_*(\mathcal{F}^p)$. However, f_* commutes with finite coproducts, and this finishes the proof. $\square$

Invertible sheaves on local ringed spaces have special properties. So we now assume that $(X, \mathcal{O}_X)$ is an LRS. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then for any $x \in X$, the stalk, $\mathcal{F}_x$, is an $\mathcal{O}_{X,x}$-module. We set

$$\mathcal{F}(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x,$$

a $\kappa(x)$-vector space, and call it the *fibre of $\mathcal{F}$ at x*. If $u \in \Gamma(U, \mathcal{F})$, then u_x, the image of u in $\mathcal{F}_x$ gives us

$$u(x) = u_x \bmod \mathfrak{m}_x\mathcal{F}_x.$$

We call $u(x) \in \mathcal{F}(x)$, the *value* of u at x.

Assume that $\mathcal{L}$ is an invertible sheaf and that $u \in \Gamma(U, \mathcal{L})$ and $x \in U$, then, $u(x) \in \mathcal{L}(x)$. Now, $\mathcal{L}(x) \cong \kappa(x)$ noncanonically, and different isomorphisms are connected to one another by multiplication by a nonzero element of $\kappa(x)$. Thus, $u(x) \in \kappa(x)$ *does not make sense*. However, the statements, $u(x) = 0$ and $u(x) \neq 0$, do make sense.

Proposition 5.14 *Let X be an LRS and $\mathcal{L}$ be an invertible sheaf on X. If $x \in X$ and $u \in \Gamma(U, \mathcal{L})$ (where $x \in U$), then, the following facts are equivalent:*

(1) u_x generates the stalk $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module.

(2) $u(x) \neq 0$.

(3) There is some open $V \subseteq U$ with $x \in V$ and some section $v \in \Gamma(V, \mathcal{L}^{-1})$ so that $v \otimes u$ is mapped to 1 in $\Gamma(V, \mathcal{O}_X)$.

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follows from Nakayama's lemma, and (3) $\Longrightarrow$ (2) is trivial. We now prove (2) $\Longrightarrow$ (3). If $u(x) \neq 0$, then u_x is a unit of $\mathcal{O}_{X,x}$. Since $\mathcal{L}$ is locally trivial, near x, we may assume that $\mathcal{L} = \mathcal{O}_X$. Then, there is some $v_x \in \mathcal{O}_{X,x}$ so that

$$v_x u_x = 1.$$

By definition, there is some open set, W, with $x \in W$, and there is some $v \in \Gamma(W, \mathcal{O}_X)$, so that $v_x = v$ at x. Then,

$$(vu - 1)_x = 0,$$

and so, there is an open subset $V \subseteq U \cap W$ with $x \in V$, and

$$vu - 1 = 0, \quad \text{on } V. \qquad \square$$

Given an invertible sheaf $\mathcal{L}$, for any $f \in \Gamma(X, \mathcal{L})$, let

$$X_f = \{x \in X \mid f(x) \neq 0 \text{ in } \mathcal{L}(x)\}.$$

By Proposition 5.14 (3), X_f is open. If $\mathcal{L}'$ is another invertible sheaf on X, and $f' \in \Gamma(X, \mathcal{L}')$, then

$$X_f \cap X_{f'} = X_{f\otimes f'}.$$

Linear algebra constructions on locally free sheaves (of finite rank) preserve local freeness. For example, if $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of finite rank, then, the following properties hold:

(1) $\mathcal{F} \otimes \mathcal{G}$ is locally free and

$$\mathrm{rk}(\mathcal{F} \otimes \mathcal{G}) = \mathrm{rk}(\mathcal{F})\mathrm{rk}(\mathcal{G}).$$

(2) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally free and

$$\mathrm{rk}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \mathrm{rk}(\mathcal{F})\mathrm{rk}(\mathcal{G}).$$

(3) $\mathcal{F}^D$, the dual sheaf of $\mathcal{F}$, is locally free and of the same rank as $\mathcal{F}$.

(4) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is isomorphic to $\mathcal{F}^D \otimes_{\mathcal{O}_X} \mathcal{G}$.

(5) $\bigwedge^p \mathcal{F}$, the pth exterior power (or wedge) of $\mathcal{F}$, is locally free and

$$\mathrm{rk}\left(\bigwedge^p \mathcal{F}\right) = \binom{n}{p},$$

where $n = \mathrm{rk}(\mathcal{F})$.

(6) $\bigotimes^p \mathcal{F} = \mathcal{F}^{\otimes p}$, the pth tensor power of $\mathcal{F}$, is locally free and

$$\mathrm{rk}(\mathcal{F}^{\otimes p}) = (\mathrm{rk}(\mathcal{F}))^p.$$

(7) $S^p(\mathcal{F})$, the pth symmetric power of $\mathcal{F}$, is locally free of rank

$$\binom{n+p-1}{p},$$

the number of monomials of degree p in n variables, where $n = \mathrm{rk}(\mathcal{F})$. Here, $S^p(\mathcal{F})$ is $\mathcal{F}^{\otimes p}/\mathfrak{S}_p$, where $\mathfrak{S}_p$ is the symmetric group on p elements operating on the factors.

Indeed, the reader can check easily (using finite presentation) that all these operations commute with taking stalks and fibres.

Let $s_1, \ldots, s_p \in \Gamma(U, \mathcal{F})$ be some sections over U, for some open subset, U, of X. Then,

$$(s_1 \wedge \cdots \wedge s_p)(x) = s_1(x) \wedge \cdots \wedge s_p(x)$$

and

$$(s_1 \wedge \cdots \wedge s_p)_x = (s_1)_x \wedge \cdots \wedge (s_p)_x.$$

Consequently, as $s_1, \ldots, s_p$ are linearly independent at x iff $s_1(x) \wedge \cdots \wedge s_p(x) \neq 0$, the set

$$\{x \in X \mid s_1, \ldots, s_p \text{ are linearly independent at } x\}$$

is open in X.

Proposition 5.15 *Let U be open in X and $\mathcal{F}$ be a locally free sheaf of rank n. A necessary and sufficient condition that there is an isomorphism*

$$\mathcal{O}_X^n \restriction U \cong \mathcal{F} \restriction U$$

is that there exist some sections $s_1, \ldots, s_n \in \Gamma(U, \mathcal{F})$ everywhere linearly independent on U.

Proof. If $\mathcal{O}_X^n \restriction U \cong \mathcal{F} \restriction U$, take $s_1, \ldots, s_n$ to be the (images of the) canonical sections $e_1, \ldots, e_n$ of $\mathcal{O}_X^n$.

Conversely, assume that we have $s_1, \ldots, s_n$ linearly independent on all of U. Each s_i gives a map $\mathcal{O}_X \restriction U \longrightarrow \mathcal{F} \restriction U$. Thus, we get a map $\mathcal{O}_X^n \restriction U \longrightarrow \mathcal{F} \restriction U$. We want to check that it is an isomorphism. Since this is a local property on U, we may assume that $\mathcal{F} = \mathcal{O}_X^n \restriction U$. At $x \in U$, the elements $s_1(x), \ldots, s_n(x)$ are linearly independent, and since $\mathrm{rk}(\mathcal{F}) = n$, the elements $s_1(x), \ldots, s_n(x)$ form a basis at x. Consequently, the map $s_j(x) \mapsto e_j(x)$, for $j = 1, \ldots, n$, is an isomorphism at x. So, near x, it remains an isomorphism; and this provides a local inverse to our map. $\square$

Proposition 5.16 *Let $\mathcal{F}$ be a locally free sheaf of rank r and $\mathcal{G}$ be a locally free sheaf of rank $n \geq r$. Then, for every $x \in X$, a necessary and sufficient condition that a map $u \colon \mathcal{F} \to \mathcal{G}$ be injective on some open U containing x, and that $\mathcal{G} \restriction U$ be the direct sum of $\mathcal{F} \restriction U$ (really $u(\mathcal{F} \restriction U)$) and another submodule $\mathcal{H} \subseteq \mathcal{G} \restriction U$, is that*

$$u(x) \colon \mathcal{F}(x) \to \mathcal{G}(x)$$

be an injection.

Proof. Let U be an open subset containing x and choose U so that

$$(\mathcal{F} \restriction U) \coprod \mathcal{H} \cong \mathcal{G} \restriction U.$$

Taking stalks at x, we get

$$\mathcal{F}_x \coprod \mathcal{H}_x \cong \mathcal{G}_x.$$

Now, tensor with $\kappa(x)$;

$$\mathcal{F}(x) \coprod \mathcal{H}(x) \cong \mathcal{G}(x).$$

Thus,

$$u(x) \colon \mathcal{F}(x) \to \mathcal{G}(x)$$

is injective.

Conversely, assume that $u(x)\colon \mathcal{F}(x) \to \mathcal{G}(x)$ is injective. Since $\mathcal{F}$ is locally free and our question is local, we may assume that $\mathcal{F} = \mathcal{O}_X^r$. We have canonical sections $e_1, \ldots, e_r$ of $\mathcal{O}_X^r$, so let $s_j = u(e_j)$. At x, as $u(x)$ is injective, $s_1(x), \ldots, s_r(x)$ are linearly independent. Thus, there exist $b_{r+1}, \ldots, b_n$ filling out a basis for $\mathcal{G}(x)$. However, $\mathcal{G}$ is locally free, so that $b_{r+1}, \ldots, b_n$ come from sections $s_{r+1}, \ldots, s_n$ in $\Gamma(V, \mathcal{G})$ for some open subset V. Then,

$$s_1(x), \ldots, s_r(x), s_{r+1}(x), \ldots, s_n(x)$$

are linearly independent, and so, $s_1, \ldots, s_n$ are linearly independent on a small open U around x. We can define a map $\mathcal{O}_X^{n-r} \restriction U \longrightarrow \mathcal{G} \restriction U$ via

$$f_j \mapsto s_j,$$

for $j = r+1, \ldots, n$, where the f_js are canonical sections of $\mathcal{O}_X^{n-r} \restriction U$. The image is isomorphic to some locally free sheaf $\mathcal{H} \subseteq \mathcal{G} \restriction U$. Thus, we get

$$\mathcal{F}(x) \coprod \mathcal{H}(x) = \mathcal{G}(x),$$

and on U,

$$\mathcal{F} \coprod \mathcal{H} = \mathcal{G}. \qquad \square$$

In geometry, in order to move linear algebra techniques into position to aid in arguments, we need to have the concept of a space, V, over our given X, whose fibres are just vector spaces. Moreover, it is necessary that such a space V not have any local complication in order that well-known properties of linear algebra should carry over with minimal pain and maximal effet. For ringed spaces, we have effectively already done this, in our notion of locally free sheaf. Here is the connection:

Let $\mathbb{A}^n = \mathrm{Spec}\, \mathbb{Z}[T_1, \ldots, T_n]$, and if $W = \mathrm{Spec}\, A$ of X is an affine scheme, observe that

$$W \coprod_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, \mathbb{A}^n = \mathrm{Spec}\, A \coprod_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, \mathbb{Z}[T_1, \ldots, T_n] \cong \mathrm{Spec}\, A[T_1, \ldots, T_n].$$

Write

$$\mathbb{A}_W^n = \mathrm{Spec}\, A[T_1, \ldots, T_n],$$

and more generally, if U is any scheme, we let

$$\mathbb{A}_U^n = U \coprod_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, \mathbb{Z}[T_1, \ldots, T_n] = U \coprod_{\mathrm{Spec}\, \mathbb{Z}} \mathbb{A}^n.$$

Definition 5.3 A *rank n vector bundle over the scheme, X,* is a scheme, V, together with a surjective morphism $p\colon V \to X$ (called *projection*) so that the following properties hold:

(1) There is some open covering $\{U_\alpha \to X\}$ of X and isomorphisms
$$f_\alpha\colon p^{-1}(U_\alpha) \to \mathbb{A}^n_{U_\alpha} = U_\alpha \prod_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^n.$$
(This property is called *local triviality*.)

(2) For every α, the following diagram commutes:
$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "f_\alpha"] \arrow[dr, "p"'] & & U_\alpha \prod_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^n \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

(3) For all α, β, set
$$g_\alpha^\beta = f_\beta \circ f_\alpha^{-1} \restriction \mathbb{A}_{U_\alpha \cap U_\beta};$$
we require that g_α^β be induced by a linear automorphism of $\mathcal{O}_{U_\alpha\cap U_\beta}[T_1, \ldots, T_n]$.

If V and W are vector bundles over X, by taking common refinements, we may assume that they have the same trivializing cover. A morphism $\varphi\colon V \to W$ is an X-morphism (i.e., a morphism $\varphi\colon V \to W$ so that $p_U = p_W \circ \varphi$) *linear* on fibres, i.e.,
$$\varphi \restriction U_\alpha \colon U_\alpha \prod \mathbb{A}^r \to U_\alpha \prod \mathbb{A}^s$$
is the identity on the first factor and a function with values in $\mathrm{Hom}_{\,\mathrm{lin}}(\mathbb{A}^r, \mathbb{A}^s)$. The notion of isomorphism is clear. Note, for each $x \in U_\alpha$, the stalk φ_x is a homomorphism
$$\mathcal{O}_{X,x}[T_1, \ldots, T_s] \longrightarrow \mathcal{O}_{X,x}[\widetilde{T}_1, \ldots, \widetilde{T}_r]$$
which is the identity on the coefficients and takes each T_j to a linear form in $\widetilde{T}_1, \ldots, \widetilde{T}_r$. As x varies, the coefficients will vary.

Proposition 5.17 *Let X be a scheme, $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module (of finite rank) and V be a vector bundle on X. Then, $\mathcal{O}_X(V)$, the sheaf of local sections of V over X, is locally free of rank equal to* $\mathrm{rk}(V)$. *There exists a vector bundle, $\mathbb{V}(\mathcal{F})$, of the same rank as $\mathcal{F}$ so that the functors*
$$V \mapsto \mathcal{O}_X(V) \quad \textit{and} \quad \mathcal{F} \mapsto \mathbb{V}(\mathcal{F})$$
establish an equivalence of the categories, $\mathrm{Vect}(X)$, *of vector bundles over X and the category, $\mathcal{LF}(\mathcal{O}_X)$, of locally free $\mathcal{O}_X$-modules (of finite rank).*

Proof. Check that $U \mapsto \Gamma(U, V)$ where

$$\Gamma(U, V) = \{s\colon U \to V \mid p \circ s = \mathrm{id}\} = \mathrm{Hom}_{\mathcal{SCH}}(U, \mathbb{A}^r)$$

is, in fact, a sheaf (DX). By definition, this is the sheaf $\mathcal{O}_X(V)$. For every $x \in X$, there is some open subset, U, with $x \in U$, so that $p^{-1}(U) \cong U \prod_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^r$. Also, $\Gamma(U, V)$ is the set of scheme maps $s\colon U \to \mathbb{A}^r$, but we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{SCH}}(U, \mathbb{A}^r) &= (\mathrm{Hom}_{\mathcal{SCH}}(U, \mathbb{A}))^r \\ &= \mathrm{Hom}_{\mathrm{Alg}}(\Gamma(\mathbb{A}^1), \Gamma(U, \mathcal{O}_X \restriction U))^r \\ &= \Gamma(U, \mathcal{O}_X \restriction U)^r \\ &= \Gamma(U, \mathcal{O}_X^r \restriction U). \end{aligned}$$

Thus,

$$\mathcal{O}_X(V) \restriction U \cong \mathcal{O}_X^r \restriction U,$$

and $\mathcal{O}_X(V)$ is locally free.

Now consider $\mathcal{F}$. Form the dual sheaf, $\mathcal{F}^D$. We know that $\mathcal{F}^D$ is locally free of the same rank as $\mathcal{F}$. We can form $\bigotimes^n(\mathcal{F}^D)$, a new locally free $\mathcal{O}_X$-module, and we form the tensor algebra

$$\mathcal{T}(\mathcal{F}^D) = \coprod_{n \geq 0} \bigotimes^n (\mathcal{F}^D).$$

This is a noncommutative $\mathcal{O}_X$-algebra, and we take the quotient of $\mathcal{T}(\mathcal{F}^D)$ by the two-sided ideal generated by the elements of the form

$$a \otimes b - b \otimes a.$$

This gives us the *symmetric algebra of* $\mathcal{F}^D$:

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) = \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}^D) = \mathcal{T}(\mathcal{F}^D)/(a \otimes b - b \otimes a).$$

It is easily checked that

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) \restriction U = \mathcal{S}_{\mathcal{O}_X \restriction U}(\mathcal{F}^D \restriction U).$$

As $\mathcal{F}^D = \mathcal{O}_X^r$ locally, for a small open U, we get

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) \restriction U = \mathcal{S}_{\mathcal{O}_X \restriction U}(\mathcal{O}_X^r \restriction U) = \mathcal{S}_{\mathcal{O}_U}(\mathcal{O}_U^r).$$

Now,

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) = \coprod_{n \geq 0} \mathcal{S}^n_{\mathcal{O}_X}(\mathcal{F}^D) = \coprod_{n \geq 0} \bigotimes^n (\mathcal{F}^D/\text{symmetrized}),$$

and so, $\mathcal{S}^n(\mathcal{O}_U^r)$ has as a basis the "monomials" in r variables, with $\mathcal{S}^0(\mathcal{F}^D) = \mathcal{O}_X$. Therefore,

$$\mathcal{S}(\mathcal{O}_X^r) \cong \mathcal{O}_X[T_1, \ldots, T_r] \cong \mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \ldots, T_r].$$

Locally on X, i.e., on small enough affines, we have

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) \restriction U = \mathcal{O}_U[T_1, \ldots, T_r],$$

and so, we form $\mathrm{Spec}(\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D) \restriction U) \cong U \prod \mathbb{A}^r$, and glue via the data for $\mathcal{F}^D$. We get a scheme $\mathcal{S}pec(\mathcal{S}(\mathcal{F}^D))$, and we set

$$\mathbb{V}(\mathcal{F}) = \mathcal{S}pec(\mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D)).$$

By our remarks above, this is a vector bundle on X. We still have to check that $\mathbb{V}$ does the right thing on morphisms of schemes. Observe that if $\mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves, we get a map $\mathcal{G}^D \longrightarrow \mathcal{F}^D$, and hence, $\mathcal{S}_{\mathcal{O}_X}(\mathcal{G}^D) \longrightarrow \mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D)$. Then, we obtain a map of schemes:

$$\mathcal{S}pec\, \mathcal{S}_{\mathcal{O}_X}(\mathcal{G}^D) \longrightarrow \mathcal{S}pec\, \mathcal{S}_{\mathcal{O}_X}(\mathcal{F}^D).$$

Thus, we defined a map $\mathbb{V}(\mathcal{F}) \longrightarrow \mathbb{V}(\mathcal{G})$. The reader should check that this gives an equivalence of categories. $\square$

We can now carry over results about locally free sheaves to vector bundles. First, note that for a locally free sheaf $\mathcal{L}$, we can show using a finite presentation that

$$(\mathcal{L}_x)^D = (\mathcal{L}^D)_x,$$

and thus,

$$(\mathcal{L}_x^{\otimes n})^D = (\mathcal{L}^{D\otimes n})_x = ((\mathcal{L}^{\otimes n})^D)_x.$$

We also have

$$\mathcal{T}(\mathcal{L}_x) = \mathcal{T}(\mathcal{L})_x,$$

and passing to Sym and fibres (i.e., mod $\mathfrak{m}_x$), we get

$$\mathbb{V}(\mathcal{L})(x) = \mathbb{V}(\mathcal{L}(x)) = \mathbb{A}^{r(x)}_{\kappa(x)}.$$

We can also do linear algebra on vector bundles. The following properties hold.

(0) $\Gamma(U, \mathcal{O}_X(V)) = \Gamma(U, V)$ and $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathbb{V}(\mathcal{F}))$.

(1) $\mathbb{V}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathrm{Vect}}(\mathbb{V}(\mathcal{F}), \mathbb{V}(\mathcal{G}))$, where $\mathcal{H}om_{\mathrm{Vect}}(V, W)$ is the vector bundle whose fibres are $\mathrm{Hom}_{\kappa(x)}(V(x), W(x))$–(the reader should have no difficulty making sense of this).

(2) $\mathbb{V}(\mathcal{L}^D) = \mathbb{V}(\mathcal{L})^D$

(3) $\mathbb{V}(\mathcal{F} \coprod \mathcal{G}) = \mathbb{V}(\mathcal{F}) \prod \mathbb{V}(\mathcal{G})$

(4) $\mathbb{V}(\mathcal{F} \otimes \mathcal{G}) = \mathbb{V}(\mathcal{F}) \otimes \mathbb{V}(\mathcal{G})$, where the tensor product means the vector bundle whose fibres are the tensor products of the respective fibres.

(5) $\bigwedge^p \mathbb{V}(\mathcal{F}) = \mathbb{V}(\bigwedge^p \mathcal{F})$

(8) $S^p(\mathbb{V}(\mathcal{F})) = \mathbb{V}(S^p(\mathcal{F}))$

(7) A necessary and sufficient condition that a morphism $\varphi\colon V \to W$ of vector bundles be an injection with image a direct summand near x is that

$$\varphi(x)\colon V(x) \to W(x)$$

be injective.

(8) Given $s_1, \ldots, s_p \in \Gamma(X, V)$, where V is a vector bundle over X, we have

$$s_1(x) \wedge \cdots \wedge s_p(x) = (s_1 \wedge \cdots \wedge s_p)(x);$$

which implies that $s_1, \ldots, s_p$ are linearly independent at x iff $s_1, \ldots, s_p$ are linearly independent near x, and also implies that for every $s \in \Gamma(V, X)$, the set $\{x \mid s(x) \neq 0\}$ is open in X.

Proposition 5.18 *Let $\varphi\colon V \to W$ be a morphism of vector bundles. Then, the following properties hold:*

(1) $\operatorname{Im}\varphi$ *is a sub-bundle of W and locally a direct summand of W iff* $\operatorname{rk}(\varphi)$ *is locally constant.*

(2) $\operatorname{Ker}\varphi$ *is a sub-bundle of V and locally a direct summand iff* $\operatorname{rk}(\varphi)$ *is locally constant.*

(3) Under this rank condition, $W/\operatorname{Im}\varphi$ is a vector bundle.

Proof. (1) The implication $\Rightarrow$ is clear. Conversely, assume that $\operatorname{rk}(\varphi)$ is locally constant. Our contention is a local question, and thus, we may assume that V is the trivial bundle $X \prod \mathbb{A}^r$ and that $\operatorname{rk}(\varphi)$ is constant. Pick $x \in V$. Consider $\varphi(V)(x) \subseteq W(x)$. Let $\mathcal{H}(x)$ be a complement in $V(x)$ of $\operatorname{Ker}\varphi(x)$. Make the trivial bundle

$$\mathcal{H} = X \prod \mathcal{H}(x).$$

We have the injection of vector bundles

$$i\colon \mathcal{H} \to V,$$

and it has constant rank. Composing with φ, we get

$$\widetilde{\varphi}\colon \mathcal{H} \xrightarrow{i} V \xrightarrow{\varphi} W,$$

and the following properties hold:

(a) $\widetilde{\varphi}(\mathcal{H}) \subseteq \varphi(V)$.

(b) $\widetilde{\varphi}(x)$ is an injection at x.

By (7) above, $\widetilde{\varphi} \restriction U$ is an injection of constant rank, and the image is locally a direct summand, for some open U with $x \in U$. Pick $y \in U$. Then, we have

$$\mathrm{rk}(\widetilde{\varphi}(y)) = \mathrm{rk}(\widetilde{\varphi}(x)) = \mathrm{rk}(\varphi(x)) = \mathrm{rk}(\varphi(y)), \tag{$\dagger$}$$

where the last equation follows from the hypothesis of constant rank. However, (a) shows that

$$\widetilde{\varphi}(\mathcal{H}) \restriction U = \varphi(V) \restriction U,$$

and thus, $\varphi(V)$ is a sub-bundle and a direct summand on U.

(2) The implication $\Rightarrow$ is also clear. Conversely, assume that $\mathrm{rk}(\varphi)$ is locally constant. Again, our contention is a local question, and thus, we may assume that $\mathrm{rk}(\varphi)$ is constant. Then, $\mathrm{rk}(\varphi^D)$ is constant, where $\varphi^D\colon W^D \to V^D$. By (1), $\mathrm{Im}\,\varphi^D$ is a vector bundle, locally a direct summand, which implies that $\mathrm{Coker}\,(\varphi^D)$ is a vector bundle. As $V^D \longrightarrow \mathrm{Coker}\,(\varphi^D)$ is surjective, by dualizing again, we get the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}\,\varphi \arrow[r] \arrow[d] & V \arrow[d] \\
(\mathrm{Coker}\,(\varphi^D))^D \arrow[r] & V^{DD}
\end{tikzcd}$$

with the bottom horizontal arrow an injection. The righthand side vertical arrow is a natural isomorphism, and thus, the lefthand side vertical arrow is also a natural isomorphism, and (by reasons of rank) $\mathrm{Ker}\,\varphi$ is a vector bundle with the desired properties. For (3), we merely note that $W/\mathrm{Im}\,\varphi \approx \mathrm{Coker}\ \varphi$. $\square$

Corollary 5.19 *Let $\varphi\colon V \to W$ be a morphism of vector bundles. For any $x \in X$, there is some open subset U with $x \in U$ so that for every $y \in U$, $\mathrm{rk}(\varphi(x)) \leq \mathrm{rk}(\varphi(y))$.*

Proof. Use ($\dagger$) from the proof of Proposition 5.18. Now, $\widetilde{\varphi}(\mathcal{H}) \subseteq \varphi(V)$ implies that

$$\mathrm{rk}(\widetilde{\varphi}(y)) \leq \mathrm{rk}(\varphi(y)).$$

But, by the first parts of ($\dagger$), we get

$$\mathrm{rk}(\widetilde{\varphi}(y)) = \mathrm{rk}(\varphi(x)),$$

and the corollary is proved. $\square$

Corollary 5.20 *Suppose $\varphi\colon V \to W$ is a morphism of vector bundles. Then, the set*

$$\{x \mid \varphi(V)(x) \text{ is not a sub-bundle}\}$$

is closed in X and not equal to X.

Since vector bundles and locally free sheaves correspond to one another and automorphisms of one become automorphisms of the other (property (1) before Proposition 5.18), the classification of locally free sheaves as elements of a Čech cohomology group proved in Corollary 5.12, carries over and we get:

Theorem 5.21 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $r \in \mathbb{N}$, $r \geq 1$. Then, vector bundles (over X) of rank r are classified by $\check{H}^1(X, \mathbb{GL}(X, r))$.*

We will use the notation $\mathrm{Vect}_r(X)$ for the collection of rank r vector bundles over X, even though, as a set, it is just $\mathcal{LF}_r(\mathcal{O}_X)$.

Example 5.1 Consider $\mathbb{A}^{n+1} = \mathbb{A}^{n+1}_{\mathbb{C}}$, let 0 be the the origin in $\mathbb{A}^{n+1}$, and let $V = B_0(\mathbb{A}^{n+1})$ be the blowup at 0. We know that V is the subvariety $\mathbb{A}^{n+1} \prod \mathbb{P}^n$ given by the equations

$$x_i y_j = x_j y_i,$$

where $(x_0, \ldots, x_n)$ are the coordinates in $\mathbb{A}^{n+1}$ and $(y_0 : \cdots : y_n)$ are homogeneous coordinates in $\mathbb{P}^n$. Consider the second projection

$$pr_2 \colon V \to \mathbb{P}^n.$$

We already know that the fibres are lines. We claim that V is a line bundle on $\mathbb{P}^n$; for this, we only have to check local triviality. As usual, let

$$U_j = \{y = (y_0 : \cdots : y_n) \in \mathbb{P}^n \mid y_j \neq 0\}.$$

Look at $V \restriction U_j$, given by

$$V \restriction U_j = \{(x, y) \mid x_i y_j = x_j y_i \text{ and } y_j \neq 0\}.$$

Since $y_j \neq 0$, we have

$$(x, y) \in V \restriction U_j \quad \text{iff} \quad x_i = \left(\frac{y_i}{y_j}\right) x_j,$$

for all $i \neq j$. We can define the map $\sigma \colon U_j \to V$ by

$$\sigma(y_0 \colon \ldots : y_n) = \left\langle \frac{y_0}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, 1, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_n}{y_j}; y_0 \colon \ldots : y_n \right\rangle,$$

where $y_i \neq 0$. Since the ith component of the "x-part" of the image of σ is 1, the map σ is everywhere a nonzero section on U_j. Then, the map

$$\theta_j \colon U_j \prod \mathbb{A}^1 \to V \restriction U_j$$

defined by

$$\theta_j(y, \lambda) = \left\langle \lambda\frac{y_0}{y_j}, \ldots, \lambda\frac{y_{j-1}}{y_j}, \lambda, \lambda\frac{y_{j+1}}{y_j}, \ldots, \lambda\frac{y_n}{y_j}; y_0 \colon \ldots : y_n \right\rangle,$$

is a vector bundle isomorphism, showing that $V \restriction U_j$ is isomorphic to a trivial bundle. As an exercise, the reader should compute explicitly the transition functions of this bundle.

In case X is a scheme over a field and W is a vector bundle on X whose rank is large with respect to the dimension of X, then W may be "simplified" to a bundle of lower rank. The precise statement is the following theorem due to Atiyah and Serre:

Theorem 5.22 *Let X be a scheme over an algebraically closed field for which $\dim(X) = d$ makes sense, and assume that X may be covered by countably many open subschemes, each of which is quasi-compact. If W is a vector bundle on X of finite rank r and W is generated by its sections (which means that the map $\Gamma(X, W) \longrightarrow W(x)$, via $\sigma \mapsto \sigma(x)$, is surjective for all $x \in X$), then there is a trivial sub-bundle $\mathbb{I}^{r-d}$ of W of rank $r - d$, and an exact sequence of bundle maps*

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow W \longrightarrow W'' \longrightarrow 0,$$

where W'' is a vector bundle of rank at most d (of course, if $r \geq d$, then W'' has rank d).

Proof. To begin, replace X by one of its quasi-compact open subschemes. For every closed point $x \in X$, by hypothesis,

$$\Gamma(X, W) \longrightarrow W(x) \longrightarrow 0$$

is exact (via the map $\sigma \mapsto \sigma(x)$). As $W(x)$ is finite dimensional, some finite dimensional subspace of $\Gamma(X, W)$, call it Γ_x, maps onto $W(x)$. However, by previous work, we find that there is some open U containing x so that the map $U \prod \Gamma_x \longrightarrow W \restriction U$ is surjective. We can cover X by finitely many such open subsets and we get that there is a finitely generated subspace of $\Gamma(W, X)$ which generates W. Consequently, we may restrict attention to a finite dimensional subspace of $\Gamma(X, W)$, call it W_0, generating W. Let $\mathbb{P}_0 = \mathbb{P}(W_0)$, and let

$$\mathrm{Ker}\,(x) = \mathrm{Ker}\,(W_0 \longrightarrow W(x)).$$

Consider the Zariski closure, Z_0, of

$$\bigcup_{x \in X} \mathbb{P}(\mathrm{Ker}\,(x)).$$

Since $\dim(\mathbb{P}(\mathrm{Ker}\,(x)) = \dim(\mathbb{P}_0) - \mathrm{rk}(W)$, we get

$$\dim(Z_0) = \dim(X) + \dim(\mathbb{P}_0) - \mathrm{rk}(W),$$

and thus,

$$\mathrm{codim}(Z_0, \mathbb{P}_0) = \mathrm{rk}(W) - \dim(X) = r - d.$$

In the general case, by hypothesis, X can be written as the union of a countable ascending chain of opens, X_α, and for each X_α we can choose a projective space, $\mathbb{P}_\alpha$, as above, and observe that

$$\mathrm{codim}(Z_\alpha, \mathbb{P}_\alpha) = \mathrm{rk}(W) - \dim(X_\alpha) = r - d$$

remains constant. Therefore, in the limit,

$$\mathrm{codim}(Z, \mathbb{P}) = \mathrm{rk}(W) - \dim(X) = r - d.$$

By considering the limiting $\mathbb{P}$ as the projectivization of the limiting spaces of sections, there is a projective subspace S of dimension $r-d-1$ so that

$$S \cap Z = \emptyset.$$

Then, S corresponds to a vector subspace $\widetilde{\Gamma}$ of $\Gamma(X, W)$, and $\dim(\widetilde{\Gamma}) = r - d$. Consider

$$\mathbb{I}^{r-d} = X \prod \widetilde{\Gamma}.$$

We have the map

$$\varphi\colon \mathbb{I}^{r-d} \longrightarrow X \prod \Gamma(X, W) \xrightarrow{\theta} W,$$

where $\theta(x, \sigma) = \sigma(x)$. However, $\operatorname{Ker} \varphi = (0)$, since at x,

$$\operatorname{Ker}(x) \cap \widetilde{\Gamma} = (0),$$

by the choice of S. Thus, we have the exact sequence

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow W \longrightarrow W'' \longrightarrow 0,$$

where W'' is a vector bundle of rank at most d. $\square$

Let us draw some consequences of Theorem 5.22. Let $\bigwedge^{\bullet} W$ denote the highest wedge power of W. Then,

$$\overset{\bullet}{\bigwedge} W = \overset{\bullet}{\bigwedge} W'' \otimes \overset{\bullet}{\bigwedge} \mathbb{I}^{r-d} \cong \overset{\bullet}{\bigwedge} W''.$$

Now, when we come to Chern classes, we will see that the Chern class of a vector bundle is equal to the Chern class of its highest wedge. Hence $c_1(W) = c_1(W'')$, where $c_1(W)$ denotes the first Chern class of W, etc.

Observe that in the complex case, we have an inclusion

$$\operatorname{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

If X is a curve and $r \geq 1$, then the highest wedge, $\bigwedge^{\bullet} W''$, of W'' is isomorphic to W'', as W'' is already a line bundle. Thus, on a curve, the first Chern class of a vector bundle will be easy to compute from the Atiyah-Serre theorem, provided we can make the latter effective.

5.2 Divisors

In studying algebraic varieties, or more generally schemes, the intuitive geometric idea of studying subvarieties of codimension one is directly appealing. Unfortunately, to carry out this idea we will need some restrictive assumptions on X. Moreover, if $\dim X$ is big then each of the studied objects also has big dimension and this suggests an inductivee program. The assumptions we need to carry out the germ of such a program are the following:

(1) X is noetherian.

(2) X is reduced and irreducible.

(3) X is separated.

(4) For any $x \in X$, the local ring $\mathcal{O}_{X,x}$ has dimension 1. That is, if $\overline{\{x\}}$ is a codimension 1 subscheme of X, then we require $\mathcal{O}_{X,x}$ to be a regular local ring. This condition is abbreviated by saying: X is *regular in codimension* 1.

Condition (1) is not too restrictive an assumption. It is merely a finiteness hypothesis. For condition (2), the assumption that X is irreducible is again not too restrictive, because in view of (1), X is a finite union of such. However, the assumption of reducedness is definitely restrictive, because in "deforming" a variety, X, we will have to make use of nilpotent elements in the structure sheaf. As for (3), this is again not very restrictive since most varieties are glued together correctly out of their affine opens. Condition (4) is once again rather restrictive. However, it is possible to arrange for (4) by a canonical procedure. To see this, note that if X is normal then (4) is true. So, it suffices to pass to the normalization of X in order to achieve (4) whenever normalization makes sense. Note as well that (4) holds iff $\mathcal{O}_{X,x}$ is a DVR for every x such that $\overline{\{x\}}$ is a codimension 1 subscheme of X. When conditions (1)–(4) are met, we say that X is a *W-scheme*.

Example 5.2

1. The surface in $\mathbb{A}^3$ consisting of the cylinder based on the cuspidal cubic $y^2 = x^3$ is not a W-scheme, because the entire line above $x = y = 0$ is singular on the surface.

2. A regular surface patch is a W-scheme.

3. A cone of equation $z^2 = xy$ is a W-scheme, because the singular point: $(0, 0, 0)$ is of codimension 2

Definition 5.4 A closed irreducible subscheme of X of codimension 1 (with the *reduced* induced structure) is a *prime divisor*, and the *Weil divisor group* of X, denoted by $\mathrm{WDiv}(X)$, is the free abelian group on the prime divisors of X. Here, X is a W-scheme. The elements of $\mathrm{WDiv}(X)$ are called *Weil divisors* or *W-divisors*.

Let X be a W-scheme, $U = \mathrm{Spec}\, A$ an open affine in X (so that A is a noetherian domain), and let $\mathcal{M}\mathrm{er}(U) = \mathrm{Frac}(A)$. We know that $\mathcal{M}\mathrm{er}(U)$ is independent of U, and thus, $\mathcal{M}\mathrm{er}(U) = \mathcal{M}\mathrm{er}(X)$. We denote by $\mathcal{M}\mathrm{er}(X)^*$ the set of nonidentically zero elements in $\mathcal{M}\mathrm{er}(X)$. Pick a prime divisor $Y \subseteq X$, and let y be its unique generic point (recall: Y is irreducible). Then, $\mathcal{O}_{Y,y}$ is a regular local ring and thus, a DVR.

Given y, it determines Y; this is the valuative criterion for separation (DX, see Section 3.3). Thus, we can write $\mathcal{O}_{X,Y}$ instead of $\mathcal{O}_{X,y}$. Now,

$$\mathrm{Frac}(\mathcal{O}_{X,Y}) = \mathcal{M}\mathrm{er}(X);$$

So, if $F \in \mathcal{M}\mathrm{er}(X)^*$ is chosen, the number

$$\mathrm{ord}_Y(F) = \mathrm{ord}_y(F)$$

is defined. Hence, if we can prove that $\mathrm{ord}_Y(F) = 0$ for all but finitely many Y's, we get a Weil divisor

$$\sum_{Y \text{ prime}} \mathrm{ord}_Y(F) Y \in \mathrm{WDiv}(X). \tag{$*$}$$

Proposition 5.23 *We have* $\mathrm{ord}_Y(F) = 0$ *for all but finitely many* Y*'s.*

Proof. Let $F \in \mathcal{M}\mathrm{er}(X)^*$. Then, there is some affine open $U \subseteq X$ where $U = \mathrm{Spec}\, A$, so that $F \restriction U$ is a morphism $U \longrightarrow \mathbb{A}^1$. Consider $Z = X - U$. It is closed and not equal to X, and thus, since X is noetherian, Z is a finite union of irreducible components. Only at most finitely many prime divisors, Y, can appear among these irreducible components. If we exclude these, we may assume that F is holomorphic. Now,

$$\mathrm{ord}_Y(F) \geq 0 \quad \text{for all } Y,$$

and

$$\begin{aligned} \mathrm{ord}_Y(F) > 0 \quad &\Longleftrightarrow \quad F \in \mathfrak{m}_y \\ &\Longleftrightarrow \quad \text{the ideal } A \cdot F \subseteq \mathfrak{m}_y \\ &\Longleftrightarrow \quad y \in V(F) \\ &\Longleftrightarrow \quad Y \subseteq V(F). \end{aligned}$$

But $V(F) \neq X$ as $F \not\equiv 0$ and $V(F)$ is closed. As X is noetherian, $V(F)$ is a finite union of irreducible components. Thus, only finitely many Y's can appear when F is holomorphic. Since finitely many Y's can appear in the complement, Z, of the holomorphic locus, only finitely many Y's appear in the entire sum $(*)$, above. $\square$

We set

$$(F) = \sum_{Y \text{ prime}} \mathrm{ord}_Y(F) Y \in \mathrm{WDiv}(X),$$

and call it the *W-divisor of F*. Any such divisor is also called a *principal W-divisor*. The group of principal W-divisors is denoted by $\mathrm{PDiv}(X)$.

It is easy to show that

$$\begin{aligned} (FG) &= (F) + (G) \\ \left(\frac{F}{G}\right) &= (F) - (G) \\ (C) &= 0 \end{aligned}$$

if C is a constant. Thus, we have a homomorphism $\mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathrm{WDiv}(X)$, and we set

$$\mathrm{WCl}(X) = \mathrm{WDiv}(X)/\mathrm{PDiv}(X),$$

and call it the *Weil class group of* X.

Remarks:

(1) What is the kernel of the homomorphism $\mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathrm{WDiv}(X)$? To answer this, let us introduce the notion of "codimension one skeleton of X." This is just the union of the codimension one subschemes of X, each considered as an atomic object. Then $\mathcal{O}_X^* \restriction$ (codimension one skeleton of X) is the kernel of the map $\mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathrm{WDiv}(X)$. For, an element F is in this kernel iff F is in $\mathcal{O}_{X,Y}^*$ for each codimension one Y in X.

(2) The group $\mathrm{WCl}(X)$ is a fundamental invariant of the scheme X. It can be very subtle to compute, and at times, the computed answer is surprising.

(3) Suppose $f\colon X \to Y$ is a morphism of W-schemes. If D is a prime divisor on Y, then $f^*(D)$ has codimension one in X. Hence, as X is a W-scheme, $f^*(D)$ is a linear combination of prime divisors of X. Consequently, the map f^* takes Weil divisors on Y to Weil divisors on X. Since f^* also takes $\mathcal{M}\mathrm{er}(Y)$ to $\mathcal{M}\mathrm{er}(X)$, we find that f^* induces a map of $\mathrm{WCl}(Y)$ to $\mathrm{WCl}(X)$.

Example 5.3 Let A be a noetherian domain. Then, A is a UFD iff A is normal and $\mathrm{WCl}(\mathrm{Spec}\, A) = (0)$. This is a standard argument from commutative algebra because a noetherian domain has factorization and the factorization is unique iff every minimal prime is principal. The reader should have no difficulty in completing the details based on these ideas. What this example shows is that the subtle invariant, $\mathrm{WCl}(\mathrm{Spec}\, A)$, plays the role of ideal class group in number theory (which vanishes there iff one has unique factorization). The reader with experience in number theory will recognize that computation of divisor class groups is difficult, hence he should expect no less of Weil class groups.

Two facts are mainly used to determine Weil class groups.

Proposition 5.24 *Let X be a W-scheme and Z be a closed subscheme (with the reduced induced structure) with $Z \neq X$, and let $U = X - Z$. Then, the map* $\mathrm{WCl}(X) \xrightarrow{\theta} \mathrm{WCl}(U)$ *via*

$$\sum_{Q \text{ prime}} n_Q Q \mapsto \sum_{Q \text{ prime}, Q\cap U \neq \emptyset} n_Q (Q \cap U)$$

is well-defined and surjective. If $\mathrm{codim}(Z, X) \geq 2$*, it is an isomorphism. If* $\mathrm{codim}(Z, X) = 1$ *and Z is irreducible, then there exists an exact sequence*

$$\mathbb{Z} \xrightarrow{\theta} \mathrm{WCl}(X) \xrightarrow{res} \mathrm{WCl}(U) \longrightarrow 0, \tag{FI}$$

where $\theta(1) = [Z]$. Moreover, if we write $Z = \bigcup_{i=1}^{q} Y_i$, where each Y_i is an irreducible component of Z (with reduced induced structure), then there exists an exact sequence

$$\coprod_{i=1}^{q} \mathbb{Z} \xrightarrow{\theta} \mathrm{WCl}(X) \xrightarrow{res} \mathrm{WCl}(U) \longrightarrow 0, \tag{FI$'$}$$

where $\theta(e_i) = [Y_i]$, and e_i is the canonical generator $(\underbrace{0, \ldots, 1, \ldots, 0}_{i})$ of $\coprod_{i=1}^{q} \mathbb{Z}$.

Proof. Note that if $Q \cap U \neq \emptyset$ then $Q \cap U$ is of codimension 1 in U and irreducible. Thus, the righthand side of the map makes sense at the divisor level. Let $F \in \mathcal{M}\mathrm{er}(X) = \mathcal{M}\mathrm{er}(U)$. Since

$$(F) = \sum_{Y \text{ prime}} \mathrm{ord}_Y(F) Y,$$

we get

$$(F \restriction U) = \sum_{Y \cap U \neq \emptyset} \mathrm{ord}_Y(F)(Y \cap U) = res(F).$$

Thus, our map descends and we get res on class groups. Given $W \subseteq U$, a prime divisor of U, let $\overline{W}$ be the closure of W in X with the reduced induced structure. Then

$$\overline{W} \cap U = W,$$

and $\overline{W}$ is a prime divisor of X. This shows that our map is onto. Nothing on either side of our map involves Z's of codimension ≥ 2, which implies that the map res is an isomorphism in the case that $\mathrm{codim}(Z, X) \geq 2$.

For the rest of the proof, assume at first that Z has codimension 1 and is irreducible. The kernel of the map is generated by the classes of prime divisors that miss U. The support of such a divisor is

$$\left(\bigcup_{n_Q \neq 0,\, Q \subseteq Z,\, Q \text{ prime}} Q \right) \subseteq Z.$$

Now, Z is irreducible so each such Q is Z itself, and thus, any divisor in the kernel is of the form dZ for some $d \in \mathbb{Z}$. Consequently, we obtain the sequence (FI), as claimed, but note that the map θ may not be injective.

If Z is not irreducible, then we can write $Z = \bigcup_{i=1}^{q} Y_i$, where the Y_i's are irreducible. Note that $Y_i \cap Y_j$ for $i \neq j$ has codimension at least two in X, consequently by the first part of the proof, we may assume that Z is the disjoint union of the Y_i's. In this case, we will use induction on q and the argument is the same as the case $q = 2$, which runs as follows:

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 & & 0 & \\
\mathbb{Z} \arrow[r, "\theta_1"] \arrow[d, equal] & \mathrm{WCl}(X - Y_2) \arrow[r] \arrow[u] & \mathrm{WCl}(X - Y_1 - Y_2) \arrow[r] \arrow[u] & 0 \\
\mathbb{Z} \arrow[r, "\theta_1"] & \mathrm{WCl}(X) \arrow[r] \arrow[u] \arrow[ur] & \mathrm{WCl}(X - Y_1) \arrow[r] \arrow[u] & 0 \\
 & \mathbb{Z} \arrow[u, "\theta_2"] \arrow[r, equal] & \mathbb{Z} \arrow[u, "\theta_2"] &
\end{tikzcd}$$

The reader should chase the diagram to find that the kernel of the diagonal arrow is exactly the group generated by $\theta(e_1)$ and $\theta(e_2)$ (i.e., by $[Y_1]$ and $[Y_2]$ in $\mathrm{WCl}(X)$). $\square$

Note that if $A = k[T_1, \ldots, T_n]$, then

$$\mathrm{WCl}(\mathrm{Spec}\, A) = \mathrm{WCl}(\mathbb{A}^n) = (0).$$

Moreover, the same is true if $A = D[T_1, \ldots, T_n]$, where D is a noetherian UFD. Hence,

$$\mathrm{WCl}(\mathbb{A}^n_D) = (0),$$

where D is a noetherian UFD.

Proposition 5.25 *We have* $\mathrm{WCl}(\mathbb{P}^n) = \mathbb{Z}$.

Proof. Let Z and $\widetilde{Z}$ be two hypersurfaces in $\mathbb{P}^n$ such that $\deg(Z) = \deg(\widetilde{Z})$, but Z and $\widetilde{Z}$ are not necessarily assumed irreducible. Then, there are some homogeneous forms f and $\widetilde{f}$ of the same degree d such that $Z = V(f)$ and $\widetilde{Z} = V(\widetilde{f})$. Let $F = f/\widetilde{f}$, then, F is a meromorphic function on $\mathbb{P}^n$, and

$$(F) = Z_{\mathrm{div}} - \widetilde{Z}_{\mathrm{div}}, \tag{$\dagger$}$$

where Z_{div} stands for Z as a divisor. This means that if we write

$$f = \prod_{j=1}^{t} f_j^{n_j},$$

where the f_j's are irreducible, and similarly for $\widetilde{f}$, then,

$$Z_{\mathrm{div}} = \sum_{j=1}^{t} n_j Z_j$$

(and similarly for $\widetilde{Z}_{\mathrm{div}}$), where $Z_j = V(f_j)$ is a prime divisor. But then, by (†), we have

$$Z_{\mathrm{div}} \sim \widetilde{Z}_{\mathrm{div}}$$

in $\mathrm{WDiv}(\mathbb{P}^n)$. We define a map from $\mathrm{WDiv}(\mathbb{P}^n)$ to $\mathbb{Z}$ *via*

$$\deg\left(\sum_{j=1}^{t} n_j Z_j\right) = \sum_{j=1}^{t} n_j \deg(Z_j).$$

Note that $\mathrm{Ker}\,(\deg)$ consists of the principal divisors (by the above). Consequently, we get an injection from $\mathrm{WCl}(\mathbb{P}^n)$ to $\mathbb{Z}$. This map is onto because hyperplanes go to 1. $\square$

Corollary 5.26 *Let Z be an irreducible hypersurface of degree d in $\mathbb{P}^n$, and write $U = \mathbb{P}^n - Z$. Then, U is an affine variety and $\mathrm{WCl}(U) = \mathbb{Z}/d\mathbb{Z}$. In fact,*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{WCl}(\mathbb{P}^n) \longrightarrow \mathrm{WCl}(U) \longrightarrow 0$$

is exact, where $\mathbb{Z} \longrightarrow \mathrm{WCl}(\mathbb{P}^n)$ is induced by $1 \mapsto [Z]$. If Z is not necessarily irreducible and consists of the union of irreducible hypersurfaces $Y_1, \ldots, Y_q$ of dimensions $d_1, \ldots, d_q$, then $\mathrm{WCl}(\mathbb{P}^n - Z) = \mathbb{Z}/d_1\mathbb{Z} \otimes \cdots \otimes \mathbb{Z}/d_q\mathbb{Z}$.

Proof. By (FI) in Proposition 5.24, we know that

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{WCl}(\mathbb{P}^n) = \mathbb{Z} \longrightarrow \mathrm{WCl}(U) \longrightarrow 0$$

is exact, where $\mathbb{Z} \longrightarrow \mathrm{WCl}(\mathbb{P}^n)$ is induced by $1 \mapsto [Z]$. Since $[Z] = d$, we find that $\mathrm{WCl}(U) = \mathbb{Z}/d\mathbb{Z}$. For the second part of the corollary, we use the exact sequence (FI$'$) and note that the group generated by $[Y_1], \ldots, [Y_q]$ is merely the subgroup $r\mathbb{Z}$, where $r = \gcd(d_1, \ldots, d_q)$. $\square$

The second fundamental fact is invariance under homotopy.

Proposition 5.27 *Let X be a W-scheme. Then, $X \prod \mathbb{A}^1$ is again a W-scheme, and the projection $pr_1 \colon X \prod \mathbb{A}^1 \to X$ induces an isomorphism*

$$\mathrm{WCl}(X) \longrightarrow \mathrm{WCl}\left(X \prod \mathbb{A}^1\right). \tag{FII}$$

Proof. First, we have to prove that $X \prod \mathbb{A}^1$ is a W-scheme, and for this, the only problem is nonsingularity in codimension one. Let $x \in X \prod \mathbb{A}^1$, of codimension 1; there are two possible cases:

The point x can be a *vertical* point, which means that $\overline{x} = pr_1(x)$ has codimension 1 in X. Then, $pr_1^{-1}(\overline{x})$ is the fibre through x and te point x is generic for the fibre. In this case,

$$\mathcal{O}_{X,x} = (\mathcal{O}_{X,\overline{x}}[T])_{\mathfrak{m}}$$

where the localization is made at a maximal ideal, because $\overline{x}$ has codimension one in X. But then, $\mathcal{O}_{X,x}$ is a regular local ring.

The point x can be a *horizontal* point, which means that $\overline{x} = pr_1(x)$ is generic in X. In this case,

$$\mathcal{O}_{X,\overline{x}} = \mathcal{M}\mathrm{er}(X),$$

which implies that

$$\mathcal{O}_{X,x} = (\mathcal{M}\mathrm{er}(X)[T])_{\mathfrak{m}}\,,$$

where again the localization is made at a maximal ideal, as x has codimension one. (The closure of x is a scheme whose projection down to X is dense in X, see Figure 5.1.)

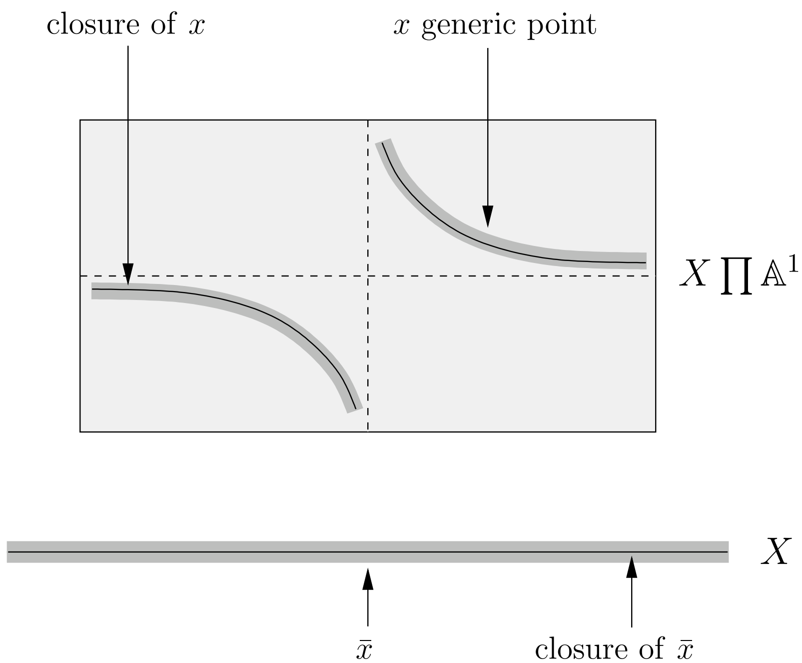

Figure 5.1: Case of a horizontal point

Now, $\mathcal{M}\mathrm{er}(X)[T]$ is a PID and so its localization, $\mathcal{O}_{X,x}$, is a DVR. Therefore, $X \prod \mathbb{A}^1$ is indeed a W-scheme.

Given a divisor D in X, the divisor $pr_1^*(D)$ is a sum of vertical prime divisors with appropriate multiplicities. So, we must show that each horizontal prime divisor is linearly equivalent to a sum of such vertical divisors. Let $\xi \in X$ be a generic point, and let

$$\Xi = pr_1^{-1}(\xi).$$

If we take D horizontal and look at $D \cap \Xi$, we get a divisor on

$$\Xi = \mathrm{Spec}(\mathcal{M}\mathrm{er}(X)[T]).$$

Now, $\mathcal{M}\text{er}(X)[T])$ is a PID and so its Weil class group is (0). Therefore, $D \cap \Xi$ is a principal divisor, (F), for some $F \in \mathcal{M}\text{er}(X)[T]$. However,

$$\mathcal{M}\text{er}(X)[T] \subseteq \mathcal{M}\text{er}(X \prod \mathbb{A}^1),$$

thus, $F \in \mathcal{M}\text{er}(X \prod \mathbb{A}^1)$. Consider the divisor, (F), where F is now considered as an element of $\mathcal{M}\text{er}(X \prod \mathbb{A}^1)$. Any horizontal component of (F) projects down to a dense subset of X. However, we chose ξ generic in X so each projection contains ξ. Therefore, any horizontal component of (F) meets Ξ. But $(F) \cap \Xi$ is just D. Hence,

$$(F) = D + \sum_i V_i,$$

where the V_i are vertical fibres. Thus, D is equivalent to a vertical divisor, as claimed. $\square$

Example 5.4 Consider the cone C whose equation is

$$xy = z^2$$

in $\mathbb{A}^3$. This cone has a singularity at the origin $0 = (0, 0, 0)$, and no other singularity. Let R be the locus of

$$y = z = 0$$

in the cone C and give this locus the reduced-induced structure. Consider $U = C - R$. Because R is given the reduced-induced structure, the locus R is isomorphic to Spec $k[x]$ (a simple algebraic way of seeing this is that when we set $y = 0$, our ring maps onto $k[x][z]/(z^2 = 0)$, and when taking the reduced-induced structure, we just get $k[x]$ by setting $z = 0$). Consequently, U is obtained by inverting y:

$$U = C - R = \mathrm{Spec}((k[x, y, z]/(xy - z^2))_y).$$

As y is invertible, we have

$$x = \frac{z^2}{y} \quad \text{in } \mathcal{O}_U$$

and thus,

$$U = C - R = \mathrm{Spec}(k[y, z, y^{-1}]) = \mathrm{Spec}((k[y, z])_y).$$

However, $k[y, z]$ is a UFD, which implies that $(k[y, z])_y$ is also a UFD. Example 5.3 shows that $\mathrm{WCl}(C - R) = (0)$, and then, (FI) implies that

$$\mathbb{Z} \longrightarrow \mathrm{WCl}(C) \longrightarrow 0$$

is exact, where the first map is induced by $1 \mapsto [R]$. Let's look at y as a function on C. What is the principal divisor (y)?

When $y = 0$, on C, we get R or nR for some n. Thus,

$$(y) = nR.$$

Look at a generic point ζ on R. We must have $\zeta = (\xi, 0, 0)$, where ξ is transcendental. In the local ring at ζ, note that x is invertible (there, it is equal to $\xi \neq 0$). Thus, in $\mathcal{O}_{C,\zeta}$, the equation $xy = z^2$ implies

$$y = \frac{z^2}{x},$$

and, as x is a unit in $\mathcal{O}_{C,\zeta}$,

$$\mathrm{ord}_R(y) = \mathrm{ord}_\zeta(y) = 2.$$

In other words, we conclude that

$$\mathrm{WCl}(C) = \begin{cases} (0) & \text{if } [R] = 0. \\ \mathbb{Z}/2\mathbb{Z} & \text{if } [R] \neq 0. \end{cases}$$

We claim that $[R] \neq 0$. Were $[R] = 0$, then R would be (f) for some function f. Let $\mathfrak{p}$ be the prime ideal defining R. We must show that $\mathfrak{p}$ is not principal. Now, $y = z = 0$ on R; so $y, z \in \mathfrak{p}$. Compute at the origin, the singularity. Write $\mathfrak{m}$ for the maximal ideal of the origin on C. Then, $\mathfrak{m} = (x, y, z)$. The Zariski-cotangent space at 0 to C is $\mathfrak{m}/\mathfrak{m}^2$ and $\overline{x}, \overline{y}, \overline{z} \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent, since 0 is singular, and $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3$. Assume that $\mathfrak{p}$ is principal. Since $0 \in R$, we have $\mathfrak{p} \subseteq \mathfrak{m}$ and principality implies that there is some f so that $y = \lambda f$ and $z = \mu f$. But then,

$$\overline{y} = \lambda \overline{f}, \quad \overline{z} = \mu \overline{f},$$

and so, $\overline{y}, \overline{z}$ are linearly dependent, a contradiction. We find, finally,

$$\mathrm{WCl}(C) = \mathbb{Z}/2\mathbb{Z}.$$

Example 5.5 Consider the nonsingular quadric Q in $\mathbb{P}^3$ over an algebraically closed field. In the four coordinates of $\mathbb{P}^3$, our quadric Q is given in matrix form as

$$X^\top A X = 0,$$

where A is the symmetric matrix of its coefficients and X represents the vector of coordinates. If we change coordinates via a $\mathrm{PGL}(3)$, say $X = BX'$, then Q becomes the quadric

$$X'^\top C X' = 0,$$

where $C = B^\top AB$. However, symmetric matrices may be diagonalized in such a manner, and as A is nonsingular, we find that after diagonalization, Q is given by

$$Q(x, y, w, z) = x^2 + y^2 + w^2 + z^2$$

(we have used the algebraic closedness of k to incorporate the nonzero coefficients in our variables so as to make all the coefficients 1). The well-known transformation

$$X = x + iy, \quad Y = x - iy, \quad Z = -(w + iz), \quad W = w - iz,$$

makes our diagonal quadric the quadric

$$XY = ZW.$$

We have already seen that this quadric is isomorphic to

$$\mathbb{P}^1 \prod \mathbb{P}^1.$$

So, on Q, we have two rulings R_1 and R_2. Hence, by (FI),

$$\mathbb{Z} \longrightarrow \mathrm{WCl}(Q) \longrightarrow \mathrm{WCl}(Q - R_j) \longrightarrow 0, \qquad (E_j)$$

where the first map is induced by $1 \mapsto [R_j]$. Note that

$$Q - R_j \cong \mathbb{P}^1 \prod \mathbb{A}^1.$$

By (FII), we have

$$\mathrm{WCl}(Q - R_j) \cong \mathbb{Z}.$$

Look at exact sequence E_1:

$$\mathbb{Z} \longrightarrow \mathrm{WCl}(Q) \longrightarrow \mathrm{WCl}(Q - R_1) = \mathbb{Z} \longrightarrow 0, \qquad (E_1)$$

where the first map is induced by $1 \mapsto [R_1]$. But (1) is the generator of $\mathrm{WCl}(\mathbb{P}^1)$ and the map from $\mathrm{WCl}(\mathbb{P}^1)$ to $\mathrm{WCl}(\mathbb{P}^1 \prod \mathbb{A}^1)$ is the pullback map, and the image of 1 under the pullback is just R_2. This shows that the map $1 \mapsto [R_2]$ is a splitting of the exact sequence E_1, that is,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{WCl}(Q) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is split-exact. Consequently,

$$\mathrm{WCl}(Q) = \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore, every element of $\mathrm{WCl}(Q)$ is of the form (α, β), and the pair (α, β) is called the *type* of the Weil divisor.

We can use our determination of $\mathrm{WCl}(Q)$ to make some tiny progress on an old problem. The problem is the following. We have seen that the twisted cubic curve in $\mathbb{P}^3$ is not the scheme theoretic intersection of two surfaces (3 is a prime and has no factor). But, is the twisted cubic the *set* theoretic intersection of two surfaces?

Let Z be a surface in $\mathbb{P}^3$. We say that Z is *well-positioned for* Q iff $Q \not\subseteq Z$. Then, $Z \cap Q$ is a divisor of Q. The surface Z is given by some form f. Look at $f \restriction Q$. We know that $f \not\equiv 0$ on Q, by hypothesis. If R is a prime divisor of Q and ρ is a generic point of R, then for any affine open U with $R \cap U \neq \emptyset$, we find $\rho \in R \cap U$. The function $f \restriction U$ is well-defined, and $\mathrm{ord}_\rho(f \restriction U)$ makes sense. It is easily checked that $\mathrm{ord}_\rho(f \restriction U)$ is independent of the affine open, U, provided $R \cap U \neq \emptyset$. We define $\mathrm{ord}_R(Z \cdot Q)$ by

$$\mathrm{ord}_R(Z \cdot Q) = \mathrm{ord}_\rho(f \restriction U)$$

and

$$(Z \cdot Q) = \sum_R \mathrm{ord}_R(Z \cdot Q) R,$$

for all prime divisors R on Q. If, on the other hand, Z is not well positioned for Q, then, as a divisor,

$$Z = rQ + \sum_{Q_\alpha \neq Q} n_\alpha Q_\alpha,$$

where the Q_α are well positioned for Q. If H is any hyperplane well positioned for Q, let

$$F = \frac{f_H^{2r}}{f_Q^r},$$

where f_H and f_Q are forms defining H and Q. Then,

$$(F) = 2rH - rQ,$$

and thus,

$$Z + (F) = 2rH + \sum_{Q_\alpha \neq Q} n_\alpha Q_\alpha.$$

This shows that $Z \sim \widetilde{Z}$, with $\widetilde{Z}$ well positioned for Q. We write $Z \cdot Q = \widetilde{Z} \cdot Q$, and leave it to the reader to check that $\widetilde{Z} \cdot Q$ is independent of $\widetilde{Z}$, as long as $\widetilde{Z}$ is well positioned for Q. Thus, $Z \cdot Q$ is well-defined for all Z. We call $Z \cdot Q$ *the intersection cycle* of Z and Q. Of course, if $Z \sim Z'$, we get $Z \cdot Q = Z' \cdot Q$. Thus, we obtain a map

$$\mathrm{WCl}(\mathbb{P}^3) \longrightarrow \mathrm{WCl}(Q)$$

given by

$$Z \mapsto Z \cdot Q.$$

Now, we have $Z \sim dH$ for some d and some good hyperplane; for instance, the hyperplane $X = 0$ (Q is not contained in this hyperplane). Since Q is given by $XY = ZW$, we see that $Q \cap H$ consists of the two lines defined by

$$X = Z = 0 \quad \text{and} \quad X = W = 0.$$

The function $f = \frac{X}{Y}$ has ord $= 1$ on both of these lines, and these are the generating lines for $\mathrm{WCl}(Q)$. Thus,

$$H \cdot Q = (1, 1),$$

which implies that

$$Z \cdot Q = (d, d)$$

if $\deg(Z) = d$.

Now, look at the twisted cubic $C \subseteq \mathbb{P}^3$. Parametrically, it is defined by

$$X = t^3, Y = u^3, Z = t^2u, W = tu^2.$$

Since $XY = ZW$, the cubic C lies on Q. To find out what C is in $\mathrm{WCl}(Q)$, look at the cone, Γ, given by

$$W^2 = YZ.$$

The intersection $\Gamma \cap Q$ is given by

$$XY = ZW, \quad W^2 = YZ,$$

which implies that

$$XW^2 = XYZ = WZ^2.$$

Either $W = 0$, or $W \neq 0$ and $XW = Z^2$. If $W = 0$, then $XY = 0$ and $YZ = 0$. This implies that $Y = 0$, since otherwise, $X = Z = W = 0$, a point, which is a contradiction. Thus, the intersection consists of the line $Y = W = 0$ and of the curve defined by

$$XY = ZW, W^2 = YZ, XW = Z^2,$$

which is the twisted cubic. As $\Gamma \cap Q$ is the union of a line l and the twisted cubic, and as l is one of the rulings, we find that $l = (0, 1)$ in $\mathrm{WCl}(Q)$, and

$$C + l = \Gamma \cdot Q = (2, 2).$$

We conclude that $C = (2, 1)$ in $\mathrm{WCl}(Q)$. This implies that there cannot be any surface Σ in $\mathbb{P}^3$ so that, even as sets,

$$\Sigma \cap Q = C.$$

For if we had $\Sigma \cap Q = C$, then we would have $\Sigma \cdot Q = \rho C$, for some ρ. But Σ has degree d, and so, $\Sigma \cdot Q = (d, d)$ in $\mathrm{WCl}(Q)$. On the other hand, we saw that $\rho C = (2\rho, \rho)$ in $\mathrm{WCl}(Q)$, and now, $(2\rho, \rho) = (d, d)$, which is impossible.

The previous discussion is much too restrictive for it limits the construction of the invariant $\mathrm{WCl}(X)$ to those X which are W-schemes. P. Cartier (1957) had the idea of admitting just divisors "given locally by one equation." This idea would finesse the restrictions forced on us by the previous discussion. While the idea is quite simple, perforce, a real execution of this must be more abstract.

Let X be any scheme. Cover X by affine opens, X_α, and consider $A_\alpha = \Gamma(X_\alpha, \mathcal{O}_X)$. Let S_α be the set of all nonzero divisors of A_α, a multiplicative set. The rings $S_\alpha^{-1}A_\alpha$ glue together on overlaps and give an $\mathcal{O}_X$-algebra, $\mathcal{K}_X$, the *total fraction sheaf of* $\mathcal{O}_X$. We have an embedding $\mathcal{O}_X \longrightarrow \mathcal{K}_X$. Let $\mathcal{K}_X^*$ be the sheaf of invertible elements of $\mathcal{K}_X$, which means that

$$\Gamma(X_\alpha, \mathcal{K}_X^*) = (S_\alpha^{-1}A_\alpha)^* = \left\{\frac{g}{h},\ g \in S_\alpha,\ h \in S_\alpha\right\}.$$

Recall that $\mathcal{O}_X^*$ is the sheaf given by

$$\Gamma(X_\alpha, \mathcal{O}_X^*) = A_\alpha^*,$$

and that we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^*.$$

Definition 5.5 The quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ is a sheaf of abelian groups, denoted by $\mathcal{D}_X$, and called the *sheaf of germs of Cartier divisors*. A *Cartier divisor on* X is a global section of $\mathcal{D}_X$, i.e., an element of $\Gamma(X, \mathcal{D}_X)$.

Since

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{D}_X \longrightarrow 0 \tag{$\dagger$}$$

is exact, every $\sigma \in \Gamma(X, \mathcal{D}_X)$ yields an open covering by affine subschemes, X_α, and elements $f_\alpha \in \Gamma(X_\alpha, \mathcal{K}_X^*)$, so that $f_\alpha \mapsto \sigma \restriction X_\alpha$ under the map $\mathcal{K}_X^* \longrightarrow \mathcal{D}_X$. Hence, there are elements θ_α^β in $\Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X^*)$, so that

$$f_\beta = f_\alpha \cdot \theta_\alpha^\beta. \tag{$\dagger\dagger$}$$

Thus, every Cartier divisor, $\sigma \in \Gamma(X, \mathcal{D}_X)$, yields a family (X_α, f_α) satisfying condition ($\dagger\dagger$). Conversely, a family (X_α, f_α) satisfying condition ($\dagger\dagger$) determines a Cartier divisor.

Observe that for every $g_\alpha \in \Gamma(X_\alpha, \mathcal{O}_X^*)$, the family $(X_\alpha, f_\alpha g_\alpha)$ defines the same σ as (X_α, f_α).

If X is an integral scheme, which means that each A_α is an integral domain, then $\mathcal{K}_X$ is equal to the constant sheaf, $\mathcal{M}\mathrm{er}(X)$.

The sheaf $\mathcal{K}_X$ is not a constant sheaf in general. Merely take $X = X_1 \coprod X_2$, for integral schemes X_1, X_2 of different dimensions.

A *principal Cartier divisor* is a Cartier divisor arising from $\Gamma(X, \mathcal{K}_X^*)$, i.e., it is given by a family (X_α, f), where $f \in \Gamma(X, \mathcal{K}_X^*)$, that is, f does not depend on α; to repeat: A divisor σ is a principal Cartier divisor iff it belongs to the image of the natural map

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{D}_X).$$

Let us call such an f a generalized meromorphic function. For a Cartier divisor, (X_α, f_α), if the f_α's actually come from $\Gamma(X_\alpha, \mathcal{O}_X)$, we will call the divisor an *integral divisor* or an *effective divisor*. Note, these f_α's, while in $\Gamma(X_\alpha, \mathcal{O}_X)$ need not be units of $\Gamma(X_\alpha, \mathcal{O}_X)$–that would make our Cartier divisor trivial. We write $\mathrm{CDiv}(X)$ in place of $\Gamma(X, \mathcal{D}_X)$ and define the *(Cartier) class group* $\mathrm{Cl}(X)$ as the quotient

$$\mathrm{Cl}(X) = \mathrm{CDiv}(X)/\mathrm{Im}\,(\Gamma(X, \mathcal{K}_X^*)).$$

If we apply cohomology to (†), we get the injection

$$\delta\colon \mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X).$$

Note that line bundles are defined on all schemes (even ringed spaces), Cartier divisors are defined on schemes, and Weil divisors are defined on W-schemes.

Here is the main proposition relating Weil divisors and Cartier divisors.

Proposition 5.28 *If X is a Noetherian, normal, integral scheme, then X is a W-scheme, and there are natural injections*

$$\mathrm{CDiv}(X) \longrightarrow \mathrm{WDiv}(X) \quad \textit{and} \quad \mathrm{Cl}(X) \longrightarrow \mathrm{WCl}(X).$$

Furthermore, if $\mathcal{O}_{X,x}$ is a UFD for every $x \in X$, then the inclusions are isomorphisms.

Proof. Since X is Noetherian and integral, to prove that X is a W-scheme we need only show that X is regular in codimension one. But X is normal, so $\mathcal{O}_{X,x}$ is a normal one–dimensional local ring if X is a point of codimension one. Such a local ring is a DVR, therefore, is regular. We construct the map from $\mathrm{CDiv}(X)$ to $\mathrm{WDiv}(X)$ as follows: Pick $\sigma \in \Gamma(X, \mathcal{D}_X) = \mathrm{CDiv}(X)$, then σ corresponds to a family $\{X_\alpha, f_\alpha\}$, where the X_α are an open covering and the f_α are in $\mathcal{K}_X^*$. Let Q be a prime divisor on X and write ζ for his generic point. If $Q \cap X_\alpha \neq \emptyset$, then ζ belongs to X_α and thus, $\mathrm{ord}_\zeta(f_\alpha)$ makes sense, by regularity in codimension one. Of course, we set

$$\mathrm{ord}_Q(f_\alpha) = \mathrm{ord}_\zeta(f_\alpha).$$

Should $Q \cap X_\alpha \cap X_\beta$ be nonempty, then as

$$f_\beta = f_\alpha \theta_\alpha^\beta$$

with $\theta_\alpha^\beta \in \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X^*)$, we find that

$$\mathrm{ord}_\zeta(f_\beta) = \mathrm{ord}_\zeta(f_\alpha)$$

(of course, ζ belongs to $X_\alpha \cap X_\beta$). This show that $\mathrm{ord}_Q(f_\alpha)$ is independent of α as long as $Q \cap X_\alpha \cap X_\beta \neq \emptyset$. Hence, we can define $\mathrm{ord}_Q(\sigma)$ to be $\mathrm{ord}_Q(f_\alpha)$ for any α with $Q \cap X_\alpha \neq \emptyset$, and $\mathrm{ord}_Q(\sigma) = 0$ if $Q \cap X_\alpha = \emptyset$. Our map is now defined by

$$\sigma \mapsto D(\sigma) = \sum_Q \mathrm{ord}_Q(\sigma) Q,$$

and this is clearly a homomorphism.

We have to check that it is injective. If $D(\sigma)=0$, then $\mathrm{ord}_Q(f_\alpha)=0$ for all Q and all α. This means that f_α is a unit in the one–dimensional local ring $\mathcal{O}_{X,Q}$. However, X is normal, so that

$$\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}) = \bigcap_{Q\cap X_\alpha\neq\emptyset} \mathcal{O}_{X,Q}.$$

It follows immediately that f_α is a unit in $\Gamma(X_\alpha, \mathcal{O}_X^*)$, that is, that σ comes from $H^0(X, \mathcal{O}_X^*)$. Hence, $\sigma = 0$.

To see that $\mathrm{Cl}(X)$ maps to $\mathrm{WCl}(X)$, observe that $\mathcal{K}_X^*$, for a W-scheme, is just the constant sheaf $\mathcal{M}\mathrm{er}(X)$. If $\sigma = \{(X_\alpha, f)\}_\alpha$ is a principal Cartier divisor, then f is a meromorphic function; so, $D(\sigma)$ is just the principal Weil divisor (f). There results the map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{WCl}(X).$$

Given $\sigma \in \mathrm{Cl}(X)$, write, as usual, $\sigma = (X_\alpha, f_\alpha)$. Suppose that $D(\sigma) = (f)$, for some fixed $f \in \mathcal{M}\mathrm{er}(X)$. Then, on X_α,

$$\mathrm{ord}_Q(f_\alpha) = \mathrm{ord}_Q(f)$$

for all Q with $Q\cap X_\alpha \neq \emptyset$. By normality again, the element f_α/f is a unit of $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$. Consequently, σ equals (X_α, f) in $\mathrm{CDiv}(X)$; that is, σ is a principal Cartier divisor, which proves our map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{WCl}(X)$$

is an indeed injective.

Assume now that X is locally factorial. Given any $x \in X$, any prime divisor Q gives us a height–one prime in $\mathcal{O}_{X,x}$, or $\mathcal{O}_{X,x}$ itself, corresponding to the case $x \in Q$, or $x \notin Q$. But, as X is locally factorial, the ideal of $\mathcal{O}_{X,x}$ is just $q_x\mathcal{O}_{X,x}$, where q_x is either an irreducible element ($X \in Q$) or the unit element 1 ($X \notin Q$). Given D in $\mathrm{WDiv}(X)$, define an element f_x in $\mathcal{M}\mathrm{er}(X)$ by

$$f_x = \prod_Q q_x^{\mathrm{ord}_Q(D)} \in \mathrm{Frac}(\mathcal{O}_{X,x}) = \mathcal{M}\mathrm{er}(X).$$

If we prove that there is some open subset, U_x, with $x \in U_x$, so that on U_x, the (locally) principal divisor (f_x) is equal to D, then these U_x will cover X, and they will define a Cartier divisor, $\sigma = (U_x, f_x)$, and we have $D(\sigma) = D$. Consequently, all is reduced to the assertion that there is an open subset, U_x, and that on U_x,

$$(f_x)\cap U_x = D \cap U_x.$$

In the definition of f_x, only finitely many Q appear, and each such Q is given by some equations on an affine open, U, containing x (we can use the same affine open merely by shrinking the possibly different affine opens, there being only finitely many Q). We also know that the element, q_x, is defined on some small open about x, and we may take the above affine open to be this open, and write q_x simply as q. As Q is given by finitely many equations, and as all these are multiples of q_x on $\mathcal{O}_{X,x}$, we may shrink U even further an find that the equations for Q on this open U are $d^{(j)}q$, for $j = 1, \ldots, t$. If we set

$$f_U = \prod_Q q_V^{\mathrm{ord}_Q(D)},$$

then at our $x \in U$, the elements f_U and f_x are the same. This means that in a possibly smaller affine open, they agree and hence, D and f_U define the same divisor on U. $\square$

Remark: It is known (Auslander-Buchsbaum [4], and Serre [52], Chapter IV, Section D.1, Corollary 4 of Theorem 9) that all regular local rings are UFD's. So, if X is regular, then our UFD condition follows. In Bourbaki's terminology, when $\mathcal{O}_{X,x}$ is always a UFD, X is called a locally factorial scheme. We know that if X is an ordinary variety and X is nonsingular, then X is regular (Chapter 2, Theorem 2.31 (Zariski)).

Corollary 5.29 *We have* $\mathrm{Cl}(\mathbb{A}^n) = (0)$*,* $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$*,* $\mathrm{Cl}((xy = z^2)) = (0)$*, and* $\mathrm{Cl}(Q) = \mathbb{Z} \oplus \mathbb{Z}$*, if Q is a nonsingular quadric in $\mathbb{P}^3$.*

Remark: In the case of the cone $(xy = z^2)$, we showed that R, a conic generator, is not principal. Hence, R is not in the image of the map $\mathrm{Cl}(X) \longrightarrow \mathrm{WCl}(X)$, but $\mathcal{O}_{X,0}$ is not a UFD.

5.3 Divisors and Line Bundles

When is the map

$$\mathrm{Cl}(X) \xrightarrow{\delta} \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*),$$

we get from the cohomology sequence, surjective? First, we examine the connection between Cartier divisors and invertible sheaves.

Let D be a Cartier divisor on X, say given by (U_α, f_α). Look at the module $\mathcal{L} \subseteq \mathcal{K}_X$ defined as follows: On U_α, take the submodule

$$\frac{1}{f_\alpha}(\mathcal{O}_X \restriction U_\alpha) \hookrightarrow \mathcal{K}_X \restriction U_\alpha.$$

On $U_\alpha \cap U_\beta$, we appear to have two submodules

$$\frac{1}{f_\alpha}(\mathcal{O}_X \restriction U_\alpha \cap U_\beta) \quad \text{and} \quad \frac{1}{f_\beta}(\mathcal{O}_X \restriction U_\alpha \cap U_\beta).$$

However, $f_\beta/f_\alpha \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$, and thus, the submodules are identical. We get a submodule of $\mathcal{K}_X$, denoted by $\mathcal{O}_X(D)$. The map

$$\varphi_\alpha \colon \mathcal{O}_X(D) \upharpoonright U_\alpha \longrightarrow \mathcal{O}_X \upharpoonright U_\alpha$$

defined via multiplication by f_α is an isomorphism of modules; therefore, $\mathcal{O}_X(D)$ is invertible and has an embedding

$$\iota_D \colon \mathcal{O}_X(D) \to \mathcal{K}_X.$$

There results a map

$$D \in \mathrm{CDiv}(X) \mapsto (\mathcal{O}_X(D), \iota_D),$$

where $\mathcal{O}_X(D) \in \mathrm{Pic}(X)$ and ι_D is an embedding of $\mathcal{O}_X(D)$ into $\mathcal{K}_X$.

Now, suppose we have an invertible sheaf which is a submodule, $\mathcal{L}$, of $\mathcal{K}_X$. This means that there are embeddings $\iota_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{K}_X$ and $\iota_{\mathcal{L}^{-1}} \colon \mathcal{L}^{-1} \to \mathcal{K}_X$. Then, there exists: A cover $\{U_\alpha\}$ of X and, some isomorphisms

$$\psi_\alpha \colon \mathcal{L} \upharpoonright U_\alpha \longrightarrow \mathcal{O}_X \upharpoonright U_\alpha.$$

Since $1 \in \mathcal{O}_X \upharpoonright U_\alpha$, we find an element

$$\xi_\alpha = \psi_\alpha^{-1}(1) \in \mathcal{L} \upharpoonright U_\alpha \hookrightarrow \mathcal{K}_X \upharpoonright U_\alpha.$$

Thus, we have $\iota_{\mathcal{L}}(\xi_\alpha) \in \mathcal{K}_X \upharpoonright U_\alpha$. For simplicity of notation, we also denote $\iota_{\mathcal{L}}(\xi_\alpha)$ by ξ_α. Since $\mathcal{L}^{-1}$ is also embedded in $\mathcal{K}_X$, the ξ_α's are non-zero divisors. Therefore, we get a Cartier divisor $D = (U_\alpha, \xi_\alpha^{-1})$, and $\mathcal{O}_X(D) = \mathcal{L}$. Of course, for any D and $\widetilde{D}$,

$$\mathcal{O}_X(D \pm \widetilde{D}) = \mathcal{O}_X(D) \otimes \mathcal{O}_X(\widetilde{D})^{\pm 1}.$$

Say $\mathcal{L} = \mathcal{O}_X(D)$ is isomorphic (abstractly, not as submodules of $\mathcal{K}_X$) to $\mathcal{O}_X(\widetilde{D})$. To ask what this means is the same as setting $E = D - \widetilde{D}$ and asking: What does it mean that $\mathcal{O}_X(E) \cong \mathcal{O}_X$?

We have an isomorphism $\varphi \colon \mathcal{O}_X(E) \to \mathcal{O}_X$ and an embedding $\iota \colon \mathcal{O}_X(E) \to \mathcal{K}_X$. Thus, we get

$$F = (\iota \circ \varphi^{-1})(1) \in \Gamma(X, \mathcal{K}_X^*).$$

We also know that

$$\mathcal{O}_X(E) \upharpoonright U_\alpha = \frac{1}{e_\alpha} \mathcal{O}_X \upharpoonright U_\alpha \hookrightarrow \mathcal{K}_X.$$

Therefore,

$$\frac{1}{e_\alpha} = F \upharpoonright U_\alpha,$$

where we have absorbed a unit of $\mathcal{O}_X \restriction U_\alpha$ in the element e_α, which is all right as we consider abstract isomorphism. We find that e_α lifts to $1/F$ on U_α, but $1/F$ is a global function, which implies that e_α comes from $\Gamma(X, \mathcal{K}_X^*)$, and

$$E = \left(\frac{1}{F}\right).$$

Thus,

$$E \sim (0),$$

where, of course, we write $E \sim E'$ when and only when $E - E' \sim (0)$, and the latter simply means that $E - E'$ is principal. Running the argument backwards, we get the following proposition:

Proposition 5.30 *There is a bijection*

$$D \longleftrightarrow \mathcal{O}_X(D)$$

between Cartier divisors on X and invertible submodules of $\mathcal{K}_X$ such that linear equivalence of Cartier divisors corresponds to abstract isomorphism of invertible sheaves. We have an inclusion

$$\mathrm{Cl}(X) \hookrightarrow \mathrm{Pic}(X),$$

and the following diagram commutes:

$$\begin{tikzcd} & H^1(X, \mathcal{O}_X^*) \arrow[dd] \\ \mathrm{Cl}(X) \arrow[ur, "\delta"] \arrow[dr] & \\ & \mathrm{Pic}(X) \end{tikzcd}$$

Proof. We only need to prove the last part of the proposition. Let D be a Cartier divisor, and assume that D is given by (U_α, f_α). Recall that we have isomorphisms

$$\mathcal{O}_X(D) \restriction U_\alpha \longrightarrow \mathcal{O}_X \restriction U_\alpha$$

given by multiplication by f_α. This implies that the transition functions $g_\alpha^\beta \colon \mathcal{O}_X \restriction U_\alpha \to \mathcal{O}_X \restriction U_\beta$ satisfy

$$g_\alpha^\beta = \frac{f_\beta}{f_\alpha}.$$

From this, it is easy to see that

$$\delta(D) = [(U_\alpha \cap U_\beta, f_\beta/f_\alpha)],$$

where the brackets mean "class of." □

We now make a momentary digression on the cycle map and the moving lemma. Let $Y \hookrightarrow X$ be a closed immersion, and let $D \in \mathrm{CDiv}(X)$. Then, D is given by (U_α, f_α). We will say that D *is well positioned for* Y if

$$f_\alpha \upharpoonright Y_\alpha \in (S_{Y_\alpha}^{-1} B_\alpha)^*,$$

where $Y_\alpha = U_\alpha \cap Y$, the ring $B_\alpha = \Gamma(Y_\alpha, \mathcal{O}_Y)$, and S_α is its collection of nonzero divisors. Then, $(Y_\alpha, f_\alpha \upharpoonright Y)$ gives a Cartier divisor on Y denoted by $D \cdot Y$. The divisor $D \cdot Y$ is the *intersection cycle* of D and Y. Assuming that $f \in \Gamma(X, \mathcal{K}_X^*)$, we require that (f) be well-positioned for Y. This means

$$f \upharpoonright Y \in \Gamma(Y, \mathcal{K}_Y^*)$$

and then,

$$(f) \cdot Y = (f \upharpoonright Y).$$

We get a map from the subset consisting of well positioned C-divisors on X to C-divisors on Y, and well positioned principal C-divisors on X map to principal C-divisors on Y.

Let us now assume that

$$Y \hookrightarrow X \overset{\theta}{\hookrightarrow} \mathbb{P}^N$$

and let D be a Cartier divisor on X. Eventually, we will show that

(1) There is some embedding θ and some effective Cartier divisors τ, μ well positioned vis-a-vis Y so that

$$D \sim r\tau - s\mu,$$

where $r, s \in \mathbb{Z}$ are large enough. Then, we can set

$$D \cdot Y = r(\tau \cdot Y) - s(\mu \cdot Y)$$

and

(2) The class of $D \cdot Y$, as just defined, is independent of the embedding θ and of τ, μ, r, s. Then, we get the *moving lemma*, due to Chow:

Moving Lemma If our varieties X and Y lie over a field, and if

$$Y \hookrightarrow X \hookrightarrow \mathbb{P}^N, \text{(closed immersion)}$$

then we can move any $D \in \mathrm{CDiv}(X)$ to another Cartier divisor, D', so that $D' \cdot Y$ makes sense and $D \sim D'$. This yields a homomorphism

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y).$$

Note that we can use cohomology to give a crisper "proof" of the existence of this map (intersection cycle class map), namely consider the diagram

$$\begin{tikzcd} \mathrm{Cl}(Y) \arrow[r] & H^1(Y, \mathcal{O}_Y^*) \\ \mathrm{Cl}(X) \arrow[r] & H^1(X, \mathcal{O}_X^*). \arrow[u] \end{tikzcd}$$

If we could show that the composed map $\mathrm{Cl}(X) \longrightarrow H^1(Y, \mathcal{O}_Y^*)$ factors through the top horizontal inclusion, we would have the required cycle class map.

If $X = \mathbb{P}^3$ and $Y = Q$ (a nonsingular quadric), we know that the moving lemma holds. Therefore, we can define $Y \cdot Y$. We know that $Y \sim 2H$, which implies that $Y \cdot Y = 2(Y \cdot H)$, that is, $Y \cdot Y$ is of type $(2, 2)$.

We now go back to the question: When the map

$$\mathrm{Cl}(Y) \longrightarrow H^1(X, \mathcal{O}_X^*)$$

is onto. Consider a vector bundle, V, of rank r on X. Assume that there is a section $\sigma \in \Gamma(X, V)$. We know that there is an open cover, (U_α), and isomorphisms, φ_α

$$\varphi_\alpha \colon V \restriction U_\alpha \to U_\alpha \prod \mathbb{A}^r.$$

The transition functions g_α^β lie in $\mathrm{GL}(r, U_\alpha \cap U_\beta)$. Since $\sigma \colon X \to V$, we have
$\sigma_\alpha = \sigma \restriction U_\alpha \colon U_\alpha \to V \restriction U_\alpha$;

$$\varphi_\alpha \circ \sigma_\alpha \colon U_\alpha \to U_\alpha \prod \mathbb{A}^r,$$

and $pr_1 \circ \varphi_\alpha \circ \sigma_\alpha = \mathrm{id}$. Let

$$pr_2 \circ \varphi_\alpha \circ \sigma_\alpha = (f_1^{(\alpha)}, \ldots, f_r^{(\alpha)}),$$

where $f_j^{(\alpha)} \in \mathrm{Hom}\,(U_\alpha, \mathbb{A}^1)$, a holomorphic function on U_α. Note that the assertion: σ is a section, is equivalent to the transition equations:

$$g_\alpha^\beta(f_1^{(\alpha)}, \ldots, f_r^{(\alpha)}) = (f_1^{(\beta)}, \ldots, f_r^{(\beta)}). \tag{$*$}$$

Let

$$Z_\alpha(\sigma) = Z_\alpha = \{x \in U_\alpha \mid f_j^{(\alpha)}(x) = 0,\ j = 1, \ldots, r\}.$$

The closed subscheme, $Z_\alpha(\sigma)$, of U_α patches with the corresponding closed subscheme, $Z_\beta(\sigma)$, on $U_\alpha \cap U_\beta$, by $(*)$. Thus, we get a closed subscheme, $Z(\sigma)$, corresponding to the section $\sigma \in \Gamma(X, V)$. Since $\mathrm{rk}(V) = r$, the scheme $Z(\sigma)$ is defined locally by r equations. Consequently, the subscheme, $Z(\sigma)$, should have its proper codimension, i.e.,

$$\min\{r, \dim(X)\},$$

where $\mathrm{rk}(V) = r$, at least for generic σ. Apply these considerations to the case: $r = 1$, line bundles.

Let L be a line bundle on X, and *assume that L has lots of sections*. Then, every section, σ, of L yields on each U_α a holomorphic function, h_α, and if g_α^β is the transition function for L on $U_\alpha \cap U_\beta$, we get

$$h_\beta = g_\alpha^\beta h_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

We say that σ is a *good section* of L if the h_α's are non-zero divisors in $\Gamma(U_\alpha, \mathcal{O}_{U_\alpha})$. Under these conditions, we can make the submodule, $\frac{1}{h_\alpha}\mathcal{O}_{U_\alpha}$, of $\mathcal{K}_\alpha = \mathcal{K}_X(U_\alpha)$, and we have

$$\frac{h_\beta}{h_\alpha} = g_\alpha^\beta \in \mathcal{K}_X^*(U_\alpha \cap U_\beta).$$

So, if σ is a good section, then the following facts hold:

(1) The pairs (U_α, h_α) define a Cartier divisor, D. This divisor is *effective*.

(2) We have $L \cong \mathcal{O}_X(D)$, because the transition functions for L are the g_α^β's and the transition functions for $\mathcal{O}_X(D)$ are the h_β/h_α, and they agree.

Write $D \geq 0$ when D is an effective Cartier divisor. This defines a partial order on $\mathrm{CDiv}(X)$. If X is a scheme over a field k, we get a cone called the *effective cone*.

Assume that $L = \mathcal{O}_X(D)$ and that σ is some given good section of L. What about $Z(\sigma)$, i.e., what's the relation between $Z(\sigma)$ and D? The section σ yields (U_α, h_α), where the h_α's define the effective Cartier divisor $Z(\sigma) \geq 0$, and

$$h_\beta = g_\alpha^\beta h_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

However,

$$g_\alpha^\beta = \frac{f_\beta}{f_\alpha},$$

where (U_α, f_α) defines D. We get

$$h_\beta = \frac{f_\beta}{f_\alpha} h_\alpha \quad \text{on } U_\alpha \cap U_\beta,$$

and so,

$$\frac{h_\beta}{f_\beta} = \frac{h_\alpha}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta.$$

Therefore, these quotients patch, and we get a generalized meromorphic function, F, in $\Gamma(X, \mathcal{K}_X^*)$, *via*

$$F \upharpoonright U_\alpha = \frac{h_\alpha}{f_\alpha}.$$

Now, from

$$f_\alpha F = h_\alpha,$$

we see that

$$(F) + D = Z(\sigma), \quad \text{with} \quad Z(\sigma) \geq 0.$$

Conversely, if $D \sim E$ where $E \geq 0$, then $E = (U_\alpha, h_\alpha)$, where $h_\alpha \in \mathcal{O}_X(U_\alpha) \cap \mathcal{K}_X^*(U_\alpha)$. However, $D = (U_\alpha, f_\alpha)$, with $f_\alpha \in \mathcal{K}_X^*(U_\alpha)$, and as $D \sim E$, there is some $F \in \Gamma(X, \mathcal{K}_X^*)$ with

$$h_\alpha = f_\alpha F \quad \text{on } U_\alpha.$$

Then,

$$g_\alpha^\beta h_\alpha = g_\alpha^\beta f_\alpha F = f_\beta F = h_\beta,$$

which implies that the section, σ, given on U_α by the functions h_α, is a good section. We have now proved most of the following proposition:

Proposition 5.31 *Let X be a scheme, L a line bundle on X, and D a Cartier divisor on X. If σ is a good section of L, then $Z(\sigma)$ is an effective Cartier divisor on X, and $L \cong \mathcal{O}_X(Z(\sigma))$. Next, for $\mathcal{O}_X(D)$, there is a bijection between*

(1) The collection of good sections, σ, of $\mathcal{O}_X(D)$ and

(2) The set of all "meromorphic" functions, F, on X, which satisfy the inequality

$$(F) + D \geq 0.$$

Moreover, there is a bijection among the following three sets:

(A) Good sections, σ, of $\mathcal{O}_X(D)$ modulo the action of global invertible holomorphic functions on X (where invertible holomorphic functions act on sections by multiplication).

(B) Effective Cartier divisors, E, with $E \sim D$.

(C) Global "meromorphic" functions, F, with

$$(F) + D \geq 0,$$

modulo the action of $\Gamma(X, \mathcal{O}_X^*)$, $(F \mapsto \lambda F)$.

Proof. Only (A), (B), (C), have not been proved yet. We need only check (A) and (B), and for this, we need the following fact: If σ and τ are good sections of $\mathcal{O}_X(D)$, and $Z(\sigma) = Z(\tau)$ as Cartier divisors, then $\sigma = \lambda\tau$ for some $\lambda \in \Gamma(X, \mathcal{O}_X^*)$. Since $Z(\sigma) = Z(\tau)$, there are elements $q_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X^*)$ so that

$$h_\alpha = q_\alpha k_\alpha,$$

where $Z(\sigma)$ and $Z(\tau)$ are defined by h_α, resp. k_α on U_α. We know that $\{h_\alpha\}$ and $\{k_\alpha\}$ are sections, and so

$$h_\beta = g_\alpha^\beta h_\alpha,$$

which implies that

$$q_\beta k_\beta = g_\alpha^\beta q_\alpha k_\alpha = q_\alpha g_\alpha^\beta k_\alpha = q_\alpha k_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

Therefore,

$$(q_\beta - q_\alpha)k_\beta = 0, \quad \text{on } U_\alpha \cap U_\beta.$$

Since k_β is not a zero-divisor, we get

$$q_\beta = q_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

The functions, q_α, therefore patch on the overlaps and thus define a global invertible holomorphic function. $\square$

If X lies over a field,k, then elements $\lambda \in k^*$ lie in $\Gamma(X, \mathcal{O}_X^*)$. So,

$$\{\sigma \in \Gamma(X, \mathcal{O}_X(D)) \mid \sigma \text{ is good}\}$$

is a k-vector space and

$$\{\sigma \in \Gamma(X, \mathcal{O}_X(D)) \mid \sigma \text{ is good}\}/k^* = \mathbb{P}(\{\sigma \in \Gamma(X, \mathcal{O}_X(D)) \mid \sigma \text{ is good}\})$$

maps onto

$$\{E \mid E \geq 0,\ E \sim D\}.$$

This surjection is an isomorphism if $\Gamma(X, \mathcal{O}_X^*) = k^*$. If X is proper, we will show that $\Gamma(X, \mathcal{O}_X) = k$. From now on, let us denote the set $\{\sigma \in \Gamma(X, \mathcal{O}_X(D)) \mid \sigma \text{ is good}\}$ by $\Gamma_{\text{good}}(X, \mathcal{O}_X(D))$.

We define $|D|$ by

$$|D| = \{E \mid E \geq 0,\ E \sim D\},$$

and call $|D|$ the *complete linear system* determined by D. Of course, when X is proper $|D|$ is just $\mathbb{P}(\Gamma_{\text{good}}(X, \mathcal{O}_X(D))$; hence, in this case, $|D|$ has the structure of a projective space.

It turns out that for X a proper scheme over a field k, the dimension of $\Gamma(X, \mathcal{O}_X(D))$ is finite, and this number is a very important invariant of the divisor D. The *Riemann-Roch problem*, in its simplest form, is to compute the dimension of $\Gamma(X, \mathcal{O}_X(D))$ in terms of other, simpler invariants of X and D. More generally we can say that the *Riemann-Roch problem* is to compute

$$\dim(H^0(X, \mathcal{F})) = \dim(\Gamma(X, \mathcal{F})),$$

where $\mathcal{F}$ is a QC sheaf on a given scheme X. Of course, the computation should be made in terms of invariants of X and simpler invariants of $\mathcal{F}$; and further, as stated in our form, we have to assume that X is defined over a field. We shall return to these questions in Chapter 9.

Now, let D be an effective divisor of X and (U_α, f_α) its "equations." We know that $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X) \cap \Gamma(U_\alpha, \mathcal{K}_X^*)$, and the transition functions, g_α^β, for $\mathcal{O}_X(D)$ are given by $g_\alpha^\beta = f_\beta / f_\alpha$. Hence, $f_\beta = g_\alpha^\beta f_\alpha$, so that the f_α's define a good section σ. The subscheme, $Z(\sigma)$, is just defined locally by the ideal $f_\alpha \mathcal{O}_X$ on U_α. However,

$$f_\alpha \mathcal{O}_X \subseteq \mathcal{O}_X \subseteq \mathcal{K}_X,$$

and this locally free $\mathcal{O}_X$-module is just $\mathcal{O}_X(-D)$. Thus, we have the following proposition:

Proposition 5.32 *If D is an effective Cartier divisor on X, let Y be the locally principal subscheme of X given by the "equations" D and write $\mathfrak{J}_Y$ for the ideal sheaf of Y. Then,*

(1) $\mathfrak{J}_Y = \mathcal{O}_X(-D)$,

(2) The sequence
$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$
is exact.

We have seen that divisors relate to line bundles and now must inquire into the difference between the two concepts. Generally speaking, $\mathrm{Cl}(X)$ is a proper subgroup of $\mathrm{Pic}(X)$, but there are conditions on X which will ensure the equality of the two groups. We will give just two such criteria and not pretend to any real generality.

Proposition 5.33 *Let X be a scheme and suppose that there is a line bundle, L, having the following property:*

(Amp) For every line bundle $\mathcal{L}$ on X, there exists an integer $M(\mathcal{L}) \geq 1$, so that if $q \geq M(\mathcal{L})$ then $\mathcal{L} \otimes L^{\otimes q}$ has a good section.

Then, $\mathrm{Cl}(X) = \mathrm{Pic}(X)$.

Proof. Apply (Amp) to L itself. If $q \geq N = M(L) - 1$, then $L^{\otimes q}$ has a good section. A previous argument implies that
$$L^{\otimes q} \cong \mathcal{O}_X(D_q),$$
for some Cartier divisor D_q. Pick any line bundle $\mathcal{L}$, and let

$$M = \max\{M(\mathcal{L}), N\}.$$

Assume that $q \geq M$. Then, $\mathcal{L} \otimes L^{\otimes q}$ has a good section, which implies that

$$\mathcal{L} \otimes L^{\otimes q} \cong \mathcal{O}_X(E_q),$$

where $E_q \in \mathrm{CDiv}(X)$. Since $q \geq N$, we also have

$$L^{\otimes q} \cong \mathcal{O}_X(D_q),$$

and thus,

$$\mathcal{L} \cong \mathcal{O}_X(E_q) \otimes \mathcal{O}_X(D_q)^{-1} \cong \mathcal{O}_X(E_q - D_q).$$

$\square$

Remark: Since it is not clear if any X we know satisfies (Amp) or even how to satisfy (Amp), we need a more tractable criterion.

Proposition 5.34 *If X is an integral scheme (i.e., reduced and irreducible), then* $\mathrm{Cl}(X) = \mathrm{Pic}(X)$*.*

Proof. The sheaf $\mathcal{K}_X = \mathcal{M}\mathrm{er}(X)$ is constant, as X is integral. Therefore, the sheaf $\mathcal{K}_X$ is flasque. As a consequence,

$$H^r(X, \mathcal{K}_X) = (0) \quad \text{for all } r > 0,$$

see Appendix B. Similarly, the sheaf $\mathcal{K}_X^*$ is flasque and

$$H^r(X, \mathcal{K}_X^*) = (0) \quad \text{for all } r > 0.$$

However, we have the exact sequences

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{P}_X \longrightarrow 0 \tag{ML}$$

and

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{D}_X \longrightarrow 0, \tag{W}$$

where, $\mathcal{P}_X$ is, by definition, the quotient sheaf, in exact sequence (ML). Applying cohomology to the sequence (W), we get

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{O}_X^*) \arrow[r] & H^0(X, \mathcal{K}_X^*) \arrow[r] & \mathrm{CDiv}(X) \arrow[dll, rounded corners] \\ & \mathrm{Pic}(X) \arrow[r] & H^1(X, \mathcal{K}_X^*) = 0. & \end{tikzcd}$$

This shows that the map $\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$ is onto, and thus, an isomorphism. $\square$

Exact sequences (ML) and (W) have many interesting and important consequences. If we continue the cohomology sequences arising from (ML) and (W) in the case where $\mathcal{K}_X$ is flasque (e.g., if X is an integral scheme) then we obtain the isomorphisms

(a) $H^r(X, \mathcal{D}_X) \cong H^{r+1}(X, \mathcal{O}_X^*)$ for all $r \geq 1$, and

(b) $H^r(X, \mathcal{P}_X) \cong H^{r+1}(X, \mathcal{O}_X)$ for all $r \geq 1$.

Now, the cohomology of $\mathcal{O}_X$ is, in many cases, a tractable group to compute; so, isomorphism (b) gives us information about the cohomology of $\mathcal{P}_X$. The cohomology of $\mathcal{O}_X^*$ is, in general, more difficult to compute. But in the complex analytic case we can connect $\mathcal{O}_X$ and $\mathcal{O}_X^*$. In this case, we have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0.$$

Moreover, for any scheme, X we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathbb{GL}(n+1) \longrightarrow \mathbb{PGL}(n) \longrightarrow 0,$$

where, by definition, $\mathbb{PGL}(X, n)$ is the quotient sheaf, and the map $\mathcal{O}_X^* \longrightarrow \mathbb{GL}(X, n+1)$ is the diagonal map. Note that $\mathrm{Im}\,(\mathcal{O}_X^*)$ is contained in the center of $\mathbb{GL}(n+1)$. It is known that this condition guarantees the existence of a partial long cohomology exact sequence

$$\begin{array}{l}
0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathbb{GL}(n+1)) \longrightarrow \Gamma(X, \mathbb{PGL}(n)) \longrightarrow \\
\longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, \mathbb{GL}(n+1)) \longrightarrow H^1(X, \mathbb{PGL}(n)) \longrightarrow \\
\longrightarrow H^2(X, \mathcal{O}_X^*).
\end{array}$$

However, in this partial cohomology exact sequence, the terms $H^1(X, \mathbb{GL}(n+1))$ and $H^1(X, \mathbb{PGL}(n))$ are just sets with a distinguished element. The first term, $H^1(X, \mathbb{GL}(n+1))$, is already known to us as $\mathrm{Vect}(n+1, X)$ and we shall shortly have an interpretation of the second term, $H^1(X, \mathbb{PGL}(n))$. If, in addition, X is proper over an affine scheme, then the first part of the sequence consisting of the global sections is already exact by itself. As a consequence, we get the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Vect}(n+1, X) \longrightarrow H^1(X, \mathbb{PGL}(n)) \longrightarrow H^2(X, \mathcal{O}_X^*), \tag{$\dagger$}$$

where X is proper over an affine base.

We would like to apply the exponential sequence; for this, assume now that X is a variety over $\mathbb{C}$. We have the corresponding analytic space, X^{an}, as previously explained, and the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{\exp} \mathcal{O}_{X^{\mathrm{an}}}^* \longrightarrow 0.$$

Taking the cohomology sequence, we get

$$\begin{aligned}
0 &\longrightarrow \mathbb{Z} \longrightarrow \Gamma(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \xrightarrow{\exp} \Gamma(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \longrightarrow \\
&\longrightarrow H^1(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{an}}) \longrightarrow \\
&\longrightarrow H^2(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \longrightarrow \\
&\longrightarrow H^3(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \cdots
\end{aligned}$$

If X is proper over $\mathbb{C}$ then, as above, $\Gamma(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = \mathbb{C}$, and the existence of the logarithm shows that the first part of the sequence consisting of the global sections is already exact. If we assume more, we can say more. For example, if $X \subseteq \mathbb{P}^r$ as a closed subvariety then the analytic cohomology and the algebraic cohomology will agree by the results of GAGA due to Serre [48]. That is,

$$H^k(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = H^k(X, \mathcal{O}_X),$$

and

$$H^k(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) = H^k(X, \mathcal{O}^*_X) \quad \text{for all } k \geq 0.$$

Then, we have

$$\begin{aligned}
0 &\longrightarrow H^1(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \\
&\longrightarrow H^2(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}^*_X) \longrightarrow \\
&\longrightarrow H^3(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \cdots
\end{aligned}$$

Now, X^{an} is a complex analytic space and $H^k(X^{\mathrm{an}}, \mathbb{Z})$ is the Betti (= ordinary) cohomology. The group $H^1(X^{\mathrm{an}}, \mathbb{Z})$ is a discrete subgroup of $H^1(X, \mathcal{O}_X)$ and we shall show that $H^1(X, \mathcal{O}_X)/H^1(X^{\mathrm{an}}, \mathbb{Z})$ is a complex torus. Consequently, $\mathrm{Pic}(X)$ has as a subgroup a complex torus with a quotient a subgroup of the discrete group $H^2(X^{\mathrm{an}}, \mathbb{Z})$. When we have studied Chern classes, it will turn out that the map $\mathrm{Pic}(X) \longrightarrow H^2(X^{\mathrm{an}}, \mathbb{Z})$ is exactly the map from a line bundle to its first Chern class. For now, let $X = \mathbb{P}^r$. Then, $X^{\mathrm{an}} = \mathbb{P}^r_{\mathbb{C}}$ and we know that

$$H^t(\mathbb{P}^r_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} (0) & \text{if } r \text{ is odd.} \\ \mathbb{Z} & \text{if } r \text{ is even.} \\ (0) & \text{if } t > 2r. \end{cases}$$

We will see in the next section that

$$H^p(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^r}) = (0) \quad \text{for all } p > 0$$

and
$$H^0(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^r}) = \mathbb{C}.$$
In particular, we find that
$$\mathrm{Pic}(\mathbb{P}^r_{\mathbb{C}}) = \mathbb{Z}$$
(as we already knew) and
$$H^2(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^r}) = (0),$$
We shall prove this latter fact in a moment in discussing the interpretation of $H^1(X, \mathbb{PGL}(n))$.

But first, we use the former fact: $\mathrm{Pic}(\mathbb{P}^r_k) = \mathbb{Z}$, where k is a field. It shows that a line bundle over $\mathbb{P}^n_k$ is of the form
$$\mathcal{O}_{\mathbb{P}^n}(H)^{\otimes q},$$
where H is some hyperplane and $q \in \mathbb{Z}$, i.e., of the form $\mathcal{O}_{\mathbb{P}^n}(qH)$. The usual notation for $\mathcal{O}_{\mathbb{P}^n}(qH)$ is $\mathcal{O}_{\mathbb{P}^n}(q)$. Let us assume momentarily that $q > 0$, and look at $\mathcal{O}_{\mathbb{P}^n}(q)$. We know that an equation for a Cartier divisor corresponding to qH is
$$Z_0^q = 0,$$
where $(Z_0\colon \cdots : Z_n)$ are the homogeneous coordinates on $\mathbb{P}^n$. Cover $\mathbb{P}^n$ by the usual affine opens
$$U_j = \{(Z_0\colon \cdots : Z_n) \mid Z_j \neq 0\}, \quad \text{where } j = 0, \ldots, n.$$
On U_0, our divisor is given by the function 1. On U_j, for $j > 0$, our divisor is given by the function
$$\left(\frac{Z_0}{Z_j}\right)^q.$$
We also have
$$g_{ij} = \frac{(Z_0/Z_j)^q}{(Z_0/Z_i)^q} = \left(\frac{Z_i}{Z_j}\right)^q.$$
Note: this also holds for $q < 0$. Thus, these g_{ij}'s are the transition functions for $\mathcal{O}_{\mathbb{P}^n}(q)$. A section, $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$, is a family of holomorphic functions, σ_j, on U_j with
$$\sigma_j = g_{ij}\sigma_i \quad \text{on } U_i \cap U_j.$$
We know
$$\sigma_j = h_j\left(\frac{Z_0}{Z_j}, \ldots, \frac{Z_n}{Z_j}\right), \quad \text{where } h_j \text{ is a polynomial.}$$
For patching as a section, we need to have
$$h_j\left(\frac{Z_0}{Z_j}, \ldots, \frac{Z_n}{Z_j}\right) = \frac{Z_i^q}{Z_j^q} h_i\left(\frac{Z_0}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right) \quad \text{on } U_i \cap U_j \text{ for all } i, j.$$

This means:

$$Z_j^q h_j\left(\frac{Z_0}{Z_j},\ldots,\frac{Z_n}{Z_j}\right)=Z_i^q h_i\left(\frac{Z_0}{Z_i},\ldots,\frac{Z_n}{Z_i}\right)\quad\text{for all } i,j. \tag{$*$}$$

Go back to the case $q>0$. Equation $(*)$ show that each h_j is the dehomogenization of a singe form h of degree q in the variables $Z_0,\ldots,Z_n$. If $q<0$ the equation $(*)$ is impossible as the reader should easily check. Now, it turns out that we even have

$$H^2(\mathbb{P}^n_A,\mathcal{O}_{\mathbb{P}^n_A})=(0)$$

when A is a commutative ring and $\mathbb{P}^n_A$ means the projective space over $\mathrm{Spec}\,A$–a concept to be introduced in Chapter 7–but that we use here for the convenience of the reader in stating the next proposition (set $A=k$, a field in the next proposition to see a statement that we have actually proved).

Proposition 5.35 *For $\mathbb{P}^n_k$, with k a field, the Cartier divisor classes are in bijection with $\mathbb{Z}$ via the map*

$$q\mapsto\mathcal{O}_{\mathbb{P}^n}(qH).$$

Moreover, the space of global sections, $\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(q))$, is given by

$$\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(q))=\begin{cases}\textit{vector space of all forms of degree } q, & \textit{if } q\geq 0,\\ (0), & \textit{if } q<0.\end{cases}$$

Later, we shall generalize Proposition 5.35 to take care of the case where k is replaced by a commutative ring, A. For now, we use it to prove the following proposition:

Proposition 5.36 *(Fundamental theorem of projective geometry)*

$$\mathrm{Aut}_k(\mathbb{P}^n_k)=\mathrm{PGL}(n,k).$$

Proof. Let $\sigma\in\mathrm{Aut}_k(\mathbb{P}^n_k)$. Then, the map $\sigma^*\colon L\mapsto\sigma^*L$ is a k-automorphism of $\mathrm{Pic}(\mathbb{P}^n_k)$. As $\mathrm{Pic}(\mathbb{P}^n_k)=\mathbb{Z}$, we get

$$\sigma^*=\pm 1.$$

Also,

$$\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))=\mathcal{O}_{\mathbb{P}^n}(\pm 1)$$

since $\mathcal{O}_{\mathbb{P}^n}(1)$ is a generator. We can find $f\in\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$, with $f\neq 0$, and then

$$\sigma^*f\in\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(\pm 1))\quad\text{and}\quad\sigma^*f\neq 0.$$

But $\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(-1))=(0)$, so $\sigma^*=\mathrm{id}$. Therefore, σ takes hyperplanes to hyperplanes. It follows that (DX) σ is linear, and we are done. $\square$

Proposition 5.36 helps us interpret the group $H^1(X,\mathbb{PGL}(n))$. Recall that in the case of $\mathbb{GL}(n)$, the cohomology group $H^1(X,\mathbb{GL}(n))$ was identified with the isomorphism classes of rank n vector bundles on X (see Theorem 5.11). The reader should look at the proof of this theorem and see that the sole place where $\mathbb{GL}(n)$ entered the argument was in the description of the automorphisms of the fibres of the total space of the vector bundle. Exactly the same argument applied to a projective fibre bundle (space Y over X, locally trivial on X, modeled as the product $U \prod F$ (U open in X, and F a fixed space–the fibre) whose fibre, F, is $\mathbb{P}^n_k$), shows that the isomorphism classes of these are in bijective correspondence with the elements of $H^1(X, \mathbb{PGL}(n))$. Of course, we have used the fundamental fact that the automomorphisms of the fibre, $\mathbb{P}^n$, comprise the group $\mathrm{PGL}(n, k)$. Now, the exact sequence (†) (after Proposition 5.34) shows that a necessary and sufficient condition that a projective fibre bundle with fibre $\mathbb{P}^n$ arises by "projectivizing" a vector bundle of rank $n+1$ is that the image in $H^2(X, \mathcal{O}_X^*)$ of the cohomology class, α, representing our projective fibre bundle vanishes. In this way, some elements of $H^2(X, \mathcal{O}_X^*)$ arise as "obstructions" to viewing a given projective fibre bundle as a projectivization of a vector bundle of one–higher rank.

For projective space, $\mathbb{P}^n$, we can make an interpretation of the line bundles $\mathcal{O}_X(q)$ in a geometric fashion. We shall first do this for the case $q = \pm 1$.

We know that $B_0(\mathbb{A}^{n+1})$ is a line bundle on $\mathbb{P}^n$ (over a field $k \subseteq \Omega$). We also showed that there are no nonzero sections (see Chapter 2, Proposition 2.54). So,

$$B_0(\mathbb{A}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(q) \quad \text{with } q < 0.$$

Let U_j be the standard open, as usual, and let $\alpha = (\alpha_0 \colon \cdots \colon \alpha_n) \in U_j$. The fibre in $B_0(\mathbb{A}^{n+1})$ over α is the line L_α given by

$$Z_k = \alpha_k t, \quad \text{where } t \in \Omega, \text{ and } Z_0, \ldots, Z_n \text{ are coordinates on } \mathbb{A}^{n+1}.$$

The local never-zero trivializing section, σ_j, is given

$$\sigma_j(\alpha) = \left\langle \frac{\alpha_0}{\alpha_j}, \ldots, \frac{\alpha_n}{\alpha_j} \right\rangle,$$

point on L_α corresponding to $t = 1/\alpha_j$. Thus, σ_j gives the isomorphism

$$U_j \prod \mathbb{A}^1 \longrightarrow B_0(\mathbb{A}^{n+1}) \restriction U_j,$$

via

$$\langle \alpha, t \rangle \mapsto \left(\left\langle \frac{t\alpha_0}{\alpha_j}, \ldots, \frac{t\alpha_{j-1}}{\alpha_j}, t, \frac{t\alpha_{j+1}}{\alpha_j}, \ldots, \frac{t\alpha_n}{\alpha_j} \right\rangle ; (\alpha_0 \colon \cdots \colon \alpha_n) \right),$$

the last tuple on the right-hand side representing the point, α, of U_j. The inverse

$$\varphi_j \colon B_0(\mathbb{A}^{n+1}) \restriction U_j \longrightarrow U_j \prod \mathbb{A}^1$$

is given by

$$((t\beta_0, \ldots, t\beta_n); (\beta_0 \colon \cdots \colon \beta_n)) \mapsto ((\beta_0 \colon \cdots \colon \beta_n), t\beta_j).$$

Then,

$$g^i_j\left(\frac{\beta_k}{\beta_j}\right)=\frac{\beta_k}{\beta_i}$$

implies that g^i_j is multiplication by $\frac{\beta_j}{\beta_i}=\left(\frac{\beta_i}{\beta_j}\right)^{-1}$, which implies that $q=-1$. We see that

$$B_0(\mathbb{A}^{n+1})\cong\mathcal{O}_{\mathbb{P}^n}(-1).$$

Now, for a geometric view of $\mathcal{O}_{\mathbb{P}^n}(1)$. Let $P\in\mathbb{P}^{n+1}$, and choose coordinates so that

$$P=(0\colon\ \cdots:0\colon 1).$$

View $\mathbb{P}^n\hookrightarrow\mathbb{P}^{n+1}$ as the hyperplane $Z_{n+1}=0$, and let $B=\mathbb{P}^{n+1}-\{P\}$. Project B onto $\mathbb{P}^n$. Given any Q in the hyperplane, the fibre of the projection is the line l_{PQ}. By our choice of coordinates, the projection is given by

$$\pi(\alpha_0\colon\ \cdots:\alpha_{n+1})=(\alpha_0\colon\ \cdots:\alpha_n),$$

and the equation of l_{PQ} is

$$(u\colon t)\mapsto(u\alpha_0\colon\ \cdots:u\alpha_n\colon t),\quad\text{where } Q=(\alpha_0\colon\ \cdots:\alpha_n).$$

We have

$$\mathbb{A}^1=l_{PQ}-\{P\}\subseteq B=\mathbb{P}^{n+1}-\{P\},$$

and the equation of this affine line is

$$\tau\mapsto(\alpha_0\colon\ \cdots:\alpha_n\colon\tau)\quad(\text{where } \tau=t/u).$$

So, B is a line family, and it is locally trivial. For if $Q\in U_j$, define

$$\sigma_j(Q)=\sigma_j(\alpha_0\colon\ \cdots:\alpha_n)=(\alpha_0\colon\ \cdots:\alpha_n\colon\alpha_j);$$

we get an everywhere nonzero function from U_j to B. Thus, B is a line bundle. The section, σ_j, gives the trivialization

$$U_j\coprod\mathbb{A}^1\longrightarrow B\restriction U_j,$$

where

$$(Q,t)\mapsto(\alpha_0\colon\ \cdots:\alpha_n\colon t\alpha_j).$$

The inverse isomorphism is

$$\varphi_j\colon B\restriction U_j\longrightarrow U_j\coprod\mathbb{A}^1,$$

given by

$$\varphi_j(\beta_0\colon\ \cdots:\beta_{n+1})=\left\langle(\beta_0\colon\ \cdots:\beta_n),\frac{\beta_{n+1}}{\beta_j}\right\rangle.$$

For the transition functions, g^i_j, as $g_{ij}\sigma_j = \sigma_i$, we find that g^i_j is multiplication by $\frac{\beta_i}{\beta_j}$. Thus, $q = 1$, and $B = \mathcal{O}_{\mathbb{P}^n}(1)$. More geometrically, hyperplanes in $\mathbb{P}^{n+1}$ are given by equations of the form

$$\sum_{j=0}^{n+1} a_j Z_j = 0.$$

The hyperplanes, H, through P form a $\mathbb{P}^n$; namely, in the above, $a_{n+1} = 0$. So, the correspondence is $H \longleftrightarrow (a_0\colon \cdots \colon a_n)$. The rest of the hyperplanes fill out a copy of $\mathbb{A}^{n+1}$; namely, these are the hyperplanes given by equations of the form

$$\sum_{j=0}^{n} a_j Z_j + Z_{n+1} = 0, \tag{$*$}$$

because they correspond to homogeneous coordinates $(a_0\colon \cdots \colon a_n\colon a_{n+1})$ with $a_{n+1} \neq 0$. Let $\overrightarrow{a} = (a_0, \ldots, a_n)$ and $H_{\overrightarrow{a}}$ be the hyperplane given by $(*)$. Pick $Q = (\alpha_0\colon \cdots \colon \alpha_n) \in \mathbb{P}^n$ (recall that $P = (0\colon \cdots \colon 0\colon 1)$), then the intersection, $H_{\overrightarrow{a}} \cap l_{PQ}$, of $H_{\overrightarrow{a}}$ with the line l_{PQ} is the point

$$\left(\alpha_0\colon \cdots \colon \alpha_n\colon -\sum_{j=0}^{n} a_j\alpha_j\right),$$

since the points on the line l_{PQ} other than P have coordinates $(\alpha_0\colon \cdots \colon \alpha_n\colon t)$. Define

$$\sigma_{\overrightarrow{a}}(Q) = H_{\overrightarrow{a}} \cap l_{PQ}.$$

The map, $\sigma_{\overrightarrow{a}}$, gives a section of our bundle. Conversely, if σ is a section of our bundle, then $\sigma(\mathbb{P}^n)$ is contained in $\mathbb{P}^{n+1}$. Now, $\sigma(\mathbb{P}^n)$ is closed (by Theorem 2.36, since σ is a proper map) and irreducible. It is also of dimension n since σ is injective. Therefore, $\sigma(\mathbb{P}^n)$ is an irreducible hypersurface. Since σ is a section, $\sigma(\mathbb{P}^n) \cap l_{PQ}$ is a single point, and thus, $\deg(\sigma(\mathbb{P}^n)) = 1$. So, $\sigma(\mathbb{P}^n)$ must be a hyperplane in $\mathbb{P}^{n+1} - \{P\}$. Therefore, the maps:

$$\overrightarrow{a} \mapsto \sigma_{\overrightarrow{a}}$$

and

$$\sigma \mapsto \sigma(\mathbb{P}^n) = H$$

establish an isomorphism between $\mathbb{A}^{n+1}$, the hyperplanes not through P, i.e., the linear forms $\sum_{j=0}^{n} a_j Z_j$, and the sections of our bundle. There is only one line bundle on $\mathbb{P}^n$ whose sections are the linear forms $\sum_{j=0}^{n} a_j Z_j$, and we find once again the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

In case q is not ± 1, we can make similar geometric arguments to interpret the total spaces of these line bundles. Namely, consider the q-uple embedding $\mathbb{P}^n \longrightarrow \mathbb{P}^N$, where $N = \binom{n+q}{q} - 1$ (see Section 2.5), and apply the above to $\mathbb{P}^N$ to give an interpretation of $\mathcal{O}_{\mathbb{P}^N}(\pm 1)$. Then, the pull back of $\mathcal{O}_{\mathbb{P}^N}(\pm 1)$ to $\mathbb{P}^n$ gives the bundles $\mathcal{O}_{\mathbb{P}^n}(\pm q)$. Hence, as the reader should verify, we obtain a description of the total spaces of $\mathcal{O}_{\mathbb{P}^n}(\pm q)$.

5.4 Further Readings

Locally free sheaves, bundles, and divisors are key concepts in algebraic geometry. These concepts are covered quite extensively in Chapter II and III of Hartshorne [33], although bundles are relegated to the exercises. Locally free sheaves are discussed in EGA I ([22, 30]), and Cartier divisors are introduced in EGA IVd ([29], Section 21). A more informal discussion of all these concepts can also be found in Danilov's survey [11], and in Shafarevich [54].

Chapter 6

Tangent and Normal Bundles; Normal Sheaves and Canonical Sheaves

6.1 Flat Morphisms–Elementary Theory

Morphisms in algebraic geometry, say $f\colon X \to Y$, can be thought of as families of schemes, $f^{-1}(y)$, each over its corresponding residue field, $\kappa(y)$, with the consistency of the family being guaranteed by the fact that the fibres all come from one scheme X. However, this direct and easy notion of a family of schemes is usually not a correct mathematical embodiment of our intuitive notion of a "continuously varying" family over Y. For example, if X is the blowup of a point of Y, then even the fibre dimension jumps. And, when the fibre dimension does not jump, what guarantee have we that more subtle invariants of the fibre vary continuously–which is to say, for discrete invariants, remain locally constant? The algebraic notion of flatness (originally due to Serre [48]), seems to capture the desired continuity we see in our minds quite efficiently indeed. This will come out over an extended sequence of results, some of which will be proved in this section and others of which will be taken up later on.

Definition 6.1 Let $f\colon X \to Y$ be a morphism and $\mathcal{F}$ a QC $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *flat over* Y *at* $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module (recall that there is a ring map $f^a\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, and since $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module, it can also be viewed as a $\mathcal{O}_{Y,f(x)}$-module). We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at x for all $x \in X$. Finally, f is *flat* if $\mathcal{O}_X$ is flat over Y. This means that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$.

An easy case with which to begin is when $f\colon X \to Y$ is a finite flat surjective morphism. Let us call such an X a *finite flat cover* of Y. Let us also assume at first that X and Y are integral schemes (reduced and irreducible). Then, there is an inclusion of fields

$$\mathcal{M}\mathrm{er}(Y) \hookrightarrow \mathcal{M}\mathrm{er}(X),$$

and it is a finite extension. The degree, δ, of this extension is called the *degree* of the morphism. Even if X and Y are not integral but if Y is locally noetherian, then $\mathcal{O}_X$ is a finite rank locally free $\mathcal{O}_Y$-module. If Y is also connected, this rank is constant and is just the degree δ introduced above in the integral case. Let V be a vector bundle of rank n over X. We can view V as a locally free sheaf, thus coherent, and $f_*(V)$ is again a coherent $\mathcal{O}_Y$-module (see remark after Corollary 4.20). We have the following useful lemma:

Lemma 6.1 *Let $f\colon X \to Y$ be a proper morphism. If U is any nonempty open set in X, there exists a maximal open $W \subseteq Y$ so that $f^{-1}(W) \subseteq U$. The set W is nonempty iff there exists a closed point $y \in Y$ so that $C \cap f^{-1}(y) = \emptyset$, where $C = X - U$.*

Proof. As above let $C = X - U$, which is closed in X. Then, $D = f(C)$ is closed in Y (since f is proper). Let $W = Y - D$. Note that $W \neq \emptyset$ iff $f(C) \neq Y$. If $\xi \in f^{-1}(W)$, then $f(\xi) \in W$, which implies that $f(\xi) \notin D$. Thus, $\xi \notin C$, so that $\xi \in U$. That W is maximal should be clear. Now, the closed points of Y are dense in Y, and the condition $C \cap f^{-1}(y) = \emptyset$ is exactly the condition that $y \notin D$. Since D is closed in Y, it equals Y iff it is dense, that is if and only if D contains every closed point of Y. $\square$

Proposition 6.2 *If X and Y are schemes, with Y locally noetherian and connected, and if $f\colon X \to Y$ is a finite flat cover of degree δ, then the direct image, f_*V, of any rank n vector bundle, V, on X is again a vector bundle on Y, but of rank $n\delta$.*

Proof. The question being local on Y we may and do assume Y is a noetherian affine connected scheme. As in the remark after Corollary 4.20, the sheaf f_*V is coherent and of the form $\widetilde{M}$, where M is a B-module ($X = \mathrm{Spec}\, B$, $Y = \mathrm{Spec}\, A$, and B is a finite flat A-module), and by hypothesis is localy free as B-module. If $\mathfrak{p}$ is a prime of A, then $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module–and so, $M_{\mathfrak{p}}$ is a free finitely generated $A_{\mathfrak{p}}$-module. By the usual persistence of pointwise (stalk) properties to small open sets around the points, we see that M is a locally free A-module. This means exactly that f_*V is a vector bundle. The rank of f_*V is manifestly $n\delta$. $\square$

If $\mathcal{L} \in \mathrm{Pic}(X)$, then $f_*\mathcal{L} \in \mathrm{Vect}_\delta(Y)$. Hence, we can form $\bigwedge^\delta(f_*(\mathcal{L}))$, and this is an element of $\mathrm{Pic}(Y)$. We will denote $\bigwedge^\delta(f_*(\mathcal{L}))$ by $\mathcal{N}_{X/Y}(\mathcal{L})$, and call it the "norm" of $\mathcal{L}$. In the case that X and Y are integral schemes, what is $\mathcal{N}_{X/Y}$ on Cartier divisors?

We know that $\mathcal{M}\mathrm{er}(X)$ is a degree δ extension of $\mathcal{M}\mathrm{er}(Y)$. This view of $\mathcal{M}\mathrm{er}(X)$ is properly speaking a view of $f_*\mathcal{M}\mathrm{er}(X)$ as sheaf on Y. If U is a sufficiently small affine open of Y, we can arrange matters so that

(1) $f_*\mathcal{O}_X \restriction U$ is free of rank δ over $\mathcal{O}_Y \restriction U$ (as f is finite, flat, surjective.

(2) A basis used in (1) is a gain a basis of $f_*\mathcal{M}\mathrm{er}(X)$ over $\mathcal{M}\mathrm{er}(Y)$.

Over this open, each section $\sigma \in \Gamma(f^{-1}(U), \mathcal{O}_X)$ or $\sigma \in \Gamma(f^{-1}(U), \mathcal{M}\mathrm{er}(X))$ acts as a linear transformation on its respective module *via* the formula

$$T_\sigma(\tau) = \sigma \cdot \tau.$$

By applying this to our basis, we find a matrix for T_σ and the determinant of this matrix (which is the determinant of T_σ as linear transformation and is independent of the basis) is an element of the respective module $\mathcal{O}_Y$ or $\mathcal{M}\mathrm{er}(Y)$ and is the norm of σ in the sense of linear algebra. Let us denote this norm by $\mathcal{N}_{X/Y}(\sigma)$. We therefore obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, "\mathcal{N}_{X/Y}"'] & \mathcal{M}\mathrm{er}(X)^* \arrow[r] \arrow[d, "\mathcal{N}_{X/Y}"'] & \mathcal{D}_X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_Y^* \arrow[r] & \mathcal{M}\mathrm{er}(Y)^* \arrow[r] & \mathcal{D}_Y \arrow[r] & 0 \end{tikzcd} \tag{6.1}$$

where the righthand vertical arrow comes from the two lefthand side vertical ones, and we actually have maps of sheaves because on any overlaps of affines the norm computed in the two different bases is the same. Now, for a matrix, its determinant is just its highest wedge (as a linear transformation of one-dimensensional free modules). Since $\mathcal{N}_{X/Y}$ is a Čech-cochain map, diagram 6.1 extended by cohomology gives us the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathcal{O}_X^*) \arrow[r] \arrow[d, "\mathcal{N}_{X/Y}"'] & \mathcal{M}\mathrm{er}(X)^* \arrow[r] \arrow[d, "\mathcal{N}_{X/Y}"'] & \mathrm{CDiv}(X) \arrow[r] \arrow[d] & H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma(Y, \mathcal{O}_Y^*) \arrow[r] & \mathcal{M}\mathrm{er}(Y)^* \arrow[r] & \mathrm{CDiv}(Y) \arrow[r] & H^1(Y, \mathcal{O}_Y^*) \arrow[r] & 0. \end{tikzcd} \tag{6.2}$$

But then, the righthand vertical map is just given by the norm map of diagram 6.1 applied to representing cocycles and the latter is just the highest wedge as remarked above. Thus, in diagram 6.2, the righthand vertical map is what we call the norm of a line bundle. This shows that the norm map of lines bundles when viewed in the context of Cartier divisors is just the map induced by the obvious norm map $\mathcal{M}\mathrm{er}(X) \longrightarrow \mathcal{M}\mathrm{er}(Y)$.

Take $\mathcal{Q} \in \mathrm{Pic}(Y)$. Then, $f^*\mathcal{Q} \in \mathrm{Pic}(X)$. On Cartier divisors, let $\mathcal{Q} = (O_\alpha, g_\alpha)$, with $g_\alpha \in \mathcal{M}\mathrm{er}(Y)$; then,

$$f^*\mathcal{Q} = (f^{-1}O_\alpha, g_\alpha),$$

where $g_\alpha \in \mathcal{M}\mathrm{er}(X)$, as $\mathcal{M}\mathrm{er}(Y) \hookrightarrow \mathcal{M}\mathrm{er}(X)$. Form $\mathcal{N}_{X/Y}(f^*(\mathcal{Q})) \in \mathrm{Pic}(Y)$, then, we find

$$\mathcal{N}_{X/Y}(f^*\mathcal{Q}) = (O_\alpha, \mathcal{N}_{Y/X}g_\alpha) = (O_\alpha, g_\alpha^\delta),$$

because $g_\alpha \in \mathcal{M}\mathrm{er}(Y)$. Therefore,

$$\mathcal{N}_{X/Y}(f^*\mathcal{Q}) = \mathcal{Q}^{\otimes\delta},$$

or $\delta\mathcal{Q}$, a C-divisor.

For Weil divisors, when they make sense,

$$f^*(P) = \sum_{\substack{Q \subseteq X \text{ prime} \\ Q \longrightarrow P}} \mathrm{ord}_Q(f^{-1}(P))Q,$$

for any prime W-divisor, P, of Y. If Q is a prime divisor on X, then $\mathcal{N}_{X/Y} f_Q$ is in $\mathcal{M}\mathrm{er}(Y)^*$, and

$$\mathcal{N}_{X/Y}(Q) = \sum_{P \subseteq Y \text{ prime}} \mathrm{ord}_P(\mathcal{N}_{X/Y} f_Q) P.$$

Here, f_Q is a generator of the maximal ideal of $\mathcal{O}_{X,Q}$ defining Q as prime divisor. Hence, we obtain the following proposition:

Proposition 6.3 *If X, Y are integral schemes with Y locally noetherian and $f\colon X \to Y$ is a finite flat cover of degree δ, then there is a morphism*

$$\mathcal{N}_{X/Y}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

and the composition

$$\mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\mathcal{N}_{X/Y}} \mathrm{Pic}(Y)$$

is just multiplication by δ. We have the following formulae for Cartier divisors:

$$\mathcal{N}_{X/Y}(U_\alpha, g_\alpha) = (O_\alpha, \mathcal{N}_{X/Y} g_\alpha),$$

where $f^{-1}(O_\alpha) \subseteq U_\alpha$,

$$f^*(U_\alpha, g_\alpha) = (f^{-1}(U_\alpha), g_\alpha),$$

with $g_\alpha \in \mathcal{M}\mathrm{er}(Y) \subseteq \mathcal{M}\mathrm{er}(X)$. For Weil divisors:

$$f^*(P) = \sum_{\substack{Q \subseteq X \text{ prime} \\ Q \longrightarrow P}} \mathrm{ord}_Q(f^{-1}(P)) Q,$$

for any prime W-divisor, P, of Y and

$$\mathcal{N}_{X/Y}(Q) = \sum_{P \subseteq Y \text{ prime}} \mathrm{ord}_P(\mathcal{N}_{X/Y} f_Q) P,$$

where f_Q is the element of $\mathcal{O}_{X,Q}$ defining Q as prime divisor.

Assume that there is a notion of degree $\deg\colon \mathrm{Pic}(Y) \to \mathbb{Z}$ on Y. Then, we get a degree on $\mathrm{Pic}(X)$ *via*

$$\deg_X(\mathcal{L}) = \deg_Y(\mathcal{N}_{X/Y}\mathcal{L}).$$

We have

$$\deg_X f^*(\mathcal{L}) = \deg_Y(\mathcal{N}_{X/Y}(f^*(\mathcal{L}))) = \delta \deg_Y(\mathcal{L}).$$

We can apply this to any irreducible projective variety $X \subseteq \mathbb{P}^N$ of dimension n over a field. By Noether normalization, there is a finite covering map $f\colon X \to \mathbb{P}^n$.

** Steve, this needs fixing

Now, we can prove (using part (3) of the fibre dimension theorem, Theorem 2.9) that there is an open, U, of $\mathbb{P}^n$ where our map f is actually finite and flat from $f^{-1}(U)$ to U.

But $\mathcal{M}\mathrm{er}(X)$ and $\mathcal{M}\mathrm{er}(Y)$ are constant sheaves, so the norm continues to make sense even when restricted to $f^{-1}(U)$. Thus, if we take a Cartier divisor on X, with local equations, g_α, we get a divisor on $\mathbb{P}^n$ from the local equations $\mathcal{N}_{X/\mathbb{P}^n} g_\alpha$–because the zeros and poles of $\mathcal{N}_{X/\mathbb{P}^n} g_\alpha$ and $\mathcal{N}_{X/\mathbb{P}^n} g_\beta$ are the same on a sufficiently small open in $\mathbb{P}^n$. **

If we are over an algebraically closed field, then $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, we have a notion of degree, thus, the degree is defined on all line bundles on projective irreducible varieties over an algebraically closed field. Take $g \in \mathcal{M}\mathrm{er}(X)$. This yields the trivial bundle (g), and

$$\mathcal{N}_{X/Y}((g)) = (\mathcal{N}_{X/\mathbb{P}^n} g).$$

Therefore,

$$\deg_X((g)) = \deg_{\mathbb{P}^n}((\mathcal{N}_{X/\mathbb{P}^n} g)) = 0.$$

Corollary 6.4 *On a projective irreducible variety, X, over an algebraically closed field, for any $g \in \mathcal{M}\mathrm{er}(X)$, we have $\deg((g)) = 0$. So, the number of zeros of g is equal to the number of poles of g.*

Flat morphisms have good behavior with respect to cohomology. The situation is as follows: We have a finite-type separated morphism, $f \colon X \to Y$, between schemes, where Y is locally noetherian. Take any locally noetherian scheme, Y', and any morphism, $\theta \colon Y' \to Y$, and let $\mathcal{F}$ be a QC $\mathcal{O}_X$-module on X. We can form the fibred product $X' = X \prod_Y Y'$, and we obtain the following diagram:

$$\begin{tikzcd}
X \prod_Y Y' \arrow[r, "\theta'"] \arrow[d, "f'"'] & \mathcal{F} \text{ QC sheaf} \arrow[d, "\text{f.t. separated}"] \\
Y' \arrow[r, "\theta"'] & Y; \text{ loc. noeth.}
\end{tikzcd}$$

Then, we can form two QC sheaves on Y', and we claim that there is a canonical morphism, can, between these two sheaves:

$$\mathrm{can} \colon \theta^*(R^p f_*)(\mathcal{F}) \longrightarrow (R^p f'_*)(\theta'^*(\mathcal{F})).$$

To give a morphism can as above is equivalent to giving the corresponding morphism:

$$(R^p f_*)(\mathcal{F}) \longrightarrow \theta_*(R^p f'_*)(\theta'^*(\mathcal{F})).$$

Pick some open U in Y. We know that $(R^p f_*)(\mathcal{F})$ is the sheaf associated to the presheaf

$$U \mapsto H^p(f^{-1}(U), \mathcal{F}).$$

The righthand side is the sheaf

$$U \mapsto (R^p f'_*)(\theta'^*(\mathcal{F})(\theta^{-1}(U))).$$

But this is the sheaf associated to the presheaf

$$U \mapsto H^p(f'^{-1}(\theta^{-1}(U)), \theta'^*(\mathcal{F})).$$

However,

$$f'^{-1}(\theta^{-1}(U)) = (\theta \circ f')^{-1}(U) = (f \circ \theta')^{-1} = \theta'^{-1}(f^{-1}(U)).$$

Thus, the RHS is associated to the presheaf

$$U \mapsto H^p(\theta'^{-1}(f^{-1}(U)), \theta'^*(\mathcal{F})).$$

Let $Z = f^{-1}(U)$. Then, the LHS is associated to the presheaf

$$U \mapsto H^p(Z, \mathcal{F}),$$

and the RHS is associated to the presheaf

$$U \mapsto H^p(\theta'^{-1}(Z), \theta'^*(\mathcal{F})).$$

The contravariant nature of cohomology implies that there is a map

$$\theta'^*\colon H^p(Z, \mathcal{F}) \longrightarrow H^p(\theta'^{-1}(Z), \theta'^*(\mathcal{F})).$$

Therefore, we get a map, can, from the LHS presheaf to the RHS presheaf, and, by the universal property of sheafification, we get the desired map of sheaves.

Proposition 6.5 *Under our circumstances, if θ is flat, the canonical homomorphism*

$$\mathrm{can}\colon \theta^*(R^p f_*)(\mathcal{F}) \longrightarrow (R^p f'_*)(\theta'^*(\mathcal{F}))$$

is an isomorphism.

Proof. The whole statement is local on Y and Y'. Thus, we may assume that $Y = \mathrm{Spec}\, A$, $Y' = \mathrm{Spec}\, A'$, and that A and A' are noetherian rings. In this case, we know that

$$(R^p f_*)(\mathcal{F}) = \widetilde{H^p(X, \mathcal{F})}$$

and

$$(R^p f'_*)(\mathcal{F}') = \widetilde{H^p(X', \mathcal{F}')},$$

where $\mathcal{F}' = \theta'^* \mathcal{F}$. Further,

$$\theta^*(R^p f_*)(\mathcal{F}) = \widetilde{H^p(X, \mathcal{F})} \otimes_A A'.$$

We have the map

$$\mathrm{can}\colon H^p(X, \mathcal{F}) \otimes_A A' \longrightarrow H^p(X', \mathcal{F}');$$

we must show that this is an isomorphism. Since f is finite type, X and X' are noetherian, separated, and thus, affine open covers of each are both nerve-finite. Hence, cohomology of our QC sheaves, $\mathcal{F}$ and $\mathcal{F}'$, may be computed by the Čech method. Cover X by affines U_α. Then, X' is covered by affines $U'_\alpha = \theta'^{-1}(U_\alpha)$ (since θ' is an affine morphism). The Čech complex for $X', \mathcal{F}'$ is

$$\begin{aligned} C^\bullet(U'_\alpha, \mathcal{F}') &= \{\prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}'(U'_{\alpha_0}\cap\cdots\cap U'_{\alpha_p})\}_{p\geq 0} \\ &= \{\prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}'(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p})'\}_{p\geq 0} \\ &= \{\prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p})\otimes_A A'\}_{p\geq 0}. \end{aligned}$$

Thus, there is an isomorphism

$$\check{C}^\bullet(\{U_\alpha\to X\}, \mathcal{F})\otimes_A A' \longrightarrow \check{C}^\bullet(\{U'_\alpha\to X'\}, \mathcal{F}').$$

But computing cohomology commutes with $-\otimes_A A'$, since A' is flat over A. $\square$

Given a morphism $f\colon X\to Y$, for every $y\in Y$ we have

$$X_y = f^{-1}(y),$$

the fibre, a scheme over $\mathrm{Spec}(\kappa(y))$. Hence, a morphism "really is" an algebraic family of schemes parametrized by the base, each scheme of the family being a fibre, and defined over varying base fields: $\kappa(y)$. We get a behavior closer to our intuition if f is l.f.t., even better if f is f.t. Look at a QC sheaf, $\mathcal{F}$, on X and write $\mathcal{F}_y$ for the pullback of $\mathcal{F}$ to X_y. Inside Y is the scheme, Y', which is just $\overline{\{y\}}$ with reduced induced structure. On it, there is the constant sheaf $\kappa(y)$.

Proposition 6.6 *If $f\colon X\to Y$ is a f.t. separated morphism, if Y is locally noetherian, and if $y\in Y$, then, we have the isomorphisms*

$$H^p(X, \mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y)) \cong H^p(X_y, \mathcal{F}_y)\quad \textit{for all } p\geq 0.$$

Proof. Consider the diagram of schemes and morphisms

$$\begin{tikzcd} X_y \arrow[r, hook] \arrow[d, "f_y"'] & X \prod_Y Y' \arrow[r, hook] \arrow[d, "pr_2"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\,\kappa(y) \arrow[r, hook] & Y' \arrow[r, hook] & Y. \end{tikzcd}$$

In this diagram, $X' = X \prod_Y Y'$ and Y' are closed subschemes of X and Y respectively and $\mathrm{Spec}\,\kappa(y)$ is dense in Y'. A pictorial sketch of the situation is shown in Figure 6.1.

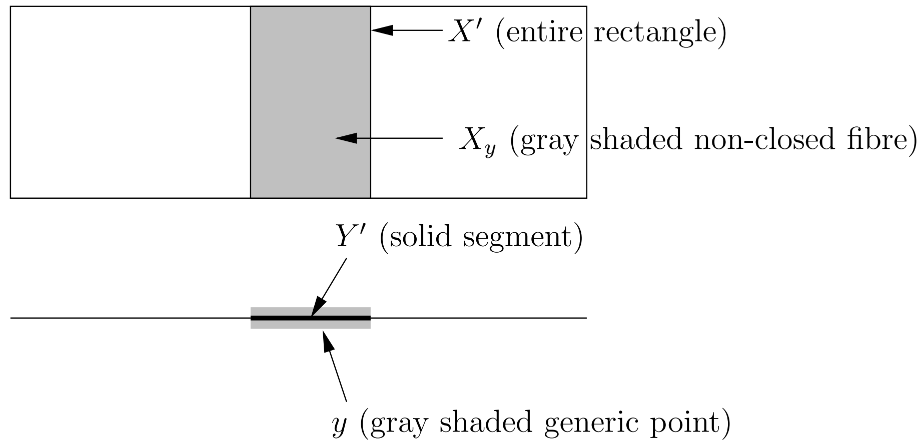

Figure 6.1: Illustration of the proof of Proposition 6.6.

Here, the shaded line on Y represents the (possibly) nonclosed point y, and the shaded region of X represents the (possibly) nonclosed fibre X_y.

Write $\mathcal{I}$ for the quasi–coherent ideal of $\mathcal{O}_Y$ defining the scheme Y' (so, $\mathcal{O}_{Y'} = \mathcal{O}/\mathcal{I}$), then $\mathcal{O}_{X'} = \mathcal{O}_X/f^*\mathcal{I}\cdot\mathcal{O}_X$. Both $\mathcal{F}$ and $\kappa(y)$ are $\mathcal{O}_Y$-modules and $\mathcal{F}' = \mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y)$ is a sheaf concentrated on Y'. Let us assume the following lemma which we shall prove after Proposition 6.6:

Lemma 6.7 *Let $i\colon X'\to X$ be a closed immersion and $\mathcal{F}$ a sheaf on X whose support is contained in X'. Then,*

$$H^p(X,\mathcal{F})\cong H^p(X',\mathcal{F}).$$

By this lemma, applied to the sheaf $\mathcal{F}'$ and the scheme X', we see that

$$H^p(X,\mathcal{F})\cong H^p(X',\mathcal{F}') = H^p(X',\mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y)).$$

However, $\mathcal{F}_y = \mathcal{F}'_y$. Therefore, we may and do assume that X, Y and $\mathcal{F}$ are replaced by X', Y', and $\mathcal{F}'$. But now, y is generic in Y and $\mathrm{Spec}\,\kappa(y)\longrightarrow Y$ is a flat morphism. Hence, we may apply Proposition 6.5 which says that

$$H^p(X,\mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y))\otimes_{\mathcal{O}_Y}\kappa(y)\cong H^p(X_y,\mathcal{F}_y).$$

Now, $H^p(X,\mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y))$ is already a $\kappa(y)$-module, and thus,

$$H^p(X,\mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y))\otimes_{\mathcal{O}_Y}\kappa(y) = H^p(X,\mathcal{F}\otimes_{\mathcal{O}_Y}\kappa(y)).\qquad\square$$

Proof of Lemma 6.7. Look at $i_*\mathcal{G}$ (where $\mathcal{G}$ is a sheaf on X'). Since X' is closed, $i_*\mathcal{G}$ is the extension by 0 outside X'. Resolve $\mathcal{G}$ on X' by flasque sheaves:

$$0\longrightarrow\mathcal{G}\longrightarrow G^\bullet(\mathcal{G}).$$

Applying i_* to this resolution, we get an acyclic resolution. But flasque sheaves are preserved under i_*, and thus, we get a resolution

$$0 \longrightarrow i_*\mathcal{G} \longrightarrow i_*G^\bullet(\mathcal{G}),$$

which gives the same cohomology. $\square$

Remarks: Each of the following statements is easy to prove:

(1) Every open immersion is a flat morphism.

(2) Any composition of flat morphisms is flat.

(3) Any base extension of a flat morphism is flat.

Under descent, flatness may not be preserved.

(4) If $\mathcal{F}$ is a QC f.g. $\mathcal{O}_X$-module and X is locally noetherian, then $\mathcal{F}$ is flat over X iff $\mathcal{F}$ is locally free. Also observe that the hypotheses imply that $\mathcal{F}$ is coherent.

6.2 Relative Differentials; Smooth Morphisms

A natural desire is to immitate as well as possible the elementary formalism of differentials and differential forms (familiar from analysis) in our present, rather abstract, context. This turns out to be quite possible to do and gives rise to the notion of relative differentials corresponding to a morphism of schemes. However, the notion of tangent and normal bundle does not make very much sense if our schemes are not regular (with regular local rings). The notion of dual of a sheaf, is perfectly general. We start in the affine, i.e., algebraic, case.

Let A be a ring and B an A-algebra. For any B-module M, we have a functor

$$M \mapsto \mathrm{Der}_A(B, M).$$

This functor is representable, and the representing object, $\Omega^1_{B/A}$, (a B-module) comes with a map $d \in \mathrm{Der}_A(B, \Omega^1_{B/A})$. There is a functorial isomorphism

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \cong \mathrm{Der}_A(B, M)$$

via

$$\varphi \mapsto \varphi \circ d.$$

Recall the construction of $\Omega^1_{B/A}$: We have the map

$$B \otimes_A B \xrightarrow{m} B,$$

where m is multiplication. If $I = \mathrm{Ker}\,(m)$, it turns out that I/I^2, as a B-module, is $\Omega^1_{B/A}$. Also, the derivation, d, is given by

$$db = 1 \otimes b - b \otimes 1 \pmod{I^2}.$$

The B-module, $\Omega^1_{B/A}$, is called the *module of relative differentials of B over A*. We have the following facts:

If A' is an A-algebra and $B' = B \otimes_A A'$, then

$$\Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes_B B'. \tag{$D1$}$$

Apply (D1) to $A' = S^{-1}A$, where $S \subseteq A$ is a multiplicative set. We get

$$\Omega^1_{S^{-1}B/S^{-1}A} \cong S^{-1}\Omega^1_{B/A}. \tag{$D2$}$$

If $S \subseteq B$ is a multiplicative set, then

$$\Omega^1_{S^{-1}B/A} \cong S^{-1}\Omega^1_{B/A}. \tag{$D3$}$$

Fact (D3) implies that we can patch modules of relative differentials on affines and make $\Omega^1_{X/Y}$, in the case that $f\colon X \to Y$ is a morphism of schemes. We can also do this directly as follows: Assume that we have a morphism $f\colon X \to Y$. Consider the immersion

$$\Delta_{X/Y}\colon X \to X \prod_Y X.$$

The image is locally closed; so, in some open, U, of the product it is given by a QC ideal, $\mathfrak{I}$, of $\mathcal{O}_U = \mathcal{O}_{X \prod_Y X} \upharpoonright U$. Look at $\mathfrak{I}/\mathfrak{I}^2$, and pull it back by Δ, to get the $\mathcal{O}_X$-module

$$\Delta^*_{X/Y}(\mathfrak{I}/\mathfrak{I}^2).$$

This is also $\Omega^1_{X/Y}$, for schemes X, Y (DX).

Property (D1) becomes the following property in terms of schemes: Assume that there are morphisms $f\colon X \to Y$ and $g\colon Y' \to Y$. The product diagram

$$\begin{tikzcd} X \prod_Y Y' \arrow[r, equal] & X' \arrow[r, "pr_1"] \arrow[d] & X \arrow[d, "f"] \\ & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

gives

$$\Omega^1_{X'/Y'} = pr_1^*\Omega^1_{X/Y}. \tag{$D1'$}$$

We call $\Omega^1_{X/Y}$ the *sheaf of relative differentials of* X *over* Y (or *relative* 1*-forms of* X *over* Y). It mainly depends on the structure of f, not on Y. Again, the differential, $d\colon \mathcal{O}_X \to \Omega^1_{X/Y}$, is given by

$$db = 1 \otimes b - b \otimes 1 \,(\mathrm{mod}\, \Delta^*_{X/Y}\mathfrak{I}^2).$$

Remark: For one case $f\colon X \to Y$, we can easily compute $\Omega^1_{X/Y}$. This is the case where X is the total space of a vector bundle of rank r over Y. Then, everywhere locally on Y,

$$X \cong Y \prod \mathbb{A}^r.$$

Thus, locally, $X = \mathrm{Spec}\, B$, where $B = A[T_1, \ldots, T_r]$, and $Y = \mathrm{Spec}\, A$. We find that $\Omega^1_{X/Y}$ is $\Omega^1_{B/A}$ locally, and $\Omega^1_{B/A}$ is the free B-module on the generators $dT_1, \ldots, dT_r$. Thus, $\Omega^1_{X/Y}$ is a locally free sheaf of rank r on X.

Now, look at rings A, B, C and maps $A \longrightarrow B \longrightarrow C$. Then, we get the exact sequence

$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0. \tag{$D4$}$$

For schemes, given morphisms

$$Z \xrightarrow{\theta} Y \longrightarrow X,$$

we get the exact sequence

$$\theta^*(\Omega^1_{Y/X}) \longrightarrow \Omega^1_{Z/X} \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0. \tag{$D4'$}$$

An important special case is the case where $B \longrightarrow C$ is surjective, i.e., $C = B/\mathfrak{B}$ for some ideal $\mathfrak{B} \subseteq B$. Now, when $B \longrightarrow C$ is surjective, we have $\Omega^1_{C/B} = (0)$, because

$$\mathrm{Hom}_C(\Omega^1_{C/B}, M) \cong \mathrm{Der}_B(C, M) = (0), \quad \text{for all } M.$$

In this special case, we have the map

$$\delta\colon \mathfrak{B}/\mathfrak{B}^2 \to \Omega^1_{B/A} \otimes_B B/\mathfrak{B},$$

given by

$$\delta(\overline{b}) = db \otimes_B 1.$$

We get the exact sequence

$$\mathfrak{B}/\mathfrak{B}^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B B/\mathfrak{B} \longrightarrow \Omega^1_{(B/\mathfrak{B})/A} \longrightarrow 0. \tag{$D5$}$$

For schemes:

$$Z \xrightarrow{i} Y \longrightarrow X$$

where i is a closed immersion, we have the exact sequence

$$\mathfrak{I}/\mathfrak{I}^2 \longrightarrow \Omega^1_{Y/X} \restriction Z \longrightarrow \Omega^1_{Z/X} \longrightarrow 0,$$

where $\mathfrak{I}$ is the ideal sheaf in $\mathcal{O}_Y$ defining Z.

($D6$) Let B be an A-algebra which is a localization of a finitely generated A-algebra, B_0. Then, $\Omega^1_{B/A}$ is a finitely generated B-module.
Proof. We have $B = S^{-1}B_0$, and hence,

$$\Omega^1_{B/A} = \Omega^1_{S^{-1}B_0/A} = S^{-1}\Omega_{B_0/A}.$$

Thus, we are reduced to the case where $B = B_0$. In this case, we have

$$B = B_0 = A[T_1, \ldots, T_r]/\mathfrak{B}.$$

Then, $\Omega^1_{B/A}$ is a homomorphic image of the free B-module on $dT_1, \ldots, dT_r$, which implies that it is finitely generated. In fact, we find that

$$\Omega^1_{B/A} = \left(\coprod_{i=1}^{r} B dT_i\right) / (df \ \mid \ f \in \mathcal{B}). \qquad \square$$

For the rest of this chapter, we assume that all schemes are locally noetherian.

Definition 6.2 The morphism $f\colon X \to Y$ is a *smooth morphism* (or X *is smooth over* Y) iff the following conditions hold:

(1) f is flat.

(2) f is a finite-type morphism.

(3) $\Omega^1_{X/Y}$ is a locally free $\mathcal{O}_X$-module (so, under our hypotheses, it has finite rank).

The following theorem whose proof will be relegated to the exercises gives equivalent conditions for smoothness of a morphism.

Theorem 6.8 *Let X, Y be locally noetherian schemes and $f\colon X \to Y$ a finite-type morphism. Then, the following statements are equivalent:*

(1) X is smooth over Y and $\Omega^1_{X/Y}$ has rank r (constant on the connected components of X).

(2) (Jacobian criterion) For any $x \in X$, there exist affine open subschemes Spec B *of X and* Spec A *of Y, with $x \in$* Spec B *and $y = f(x) \in$* Spec A*, so that*

$$B \cong A[T_1, \ldots, T_n]/(f_1, \ldots, f_{n-r}),$$

and

$$J = \left(\frac{\partial f_i}{\partial T_j}\right)$$

has maximal rank, $n - r$; i.e., some $(n-r) \times (n-r)$ minor of J has an invertible determinant (in B).

(3) *(Infinitesimal lifting criterion) Given any infinitesimal extension of Artinian local rings*

$$0 \longrightarrow I \longrightarrow \widetilde{C} \longrightarrow C \longrightarrow 0$$

with $I^2 = (0)$*, and given the commutative diagram (of solid arrows)*

$$\begin{tikzcd} \operatorname{Spec} C \arrow[r, "\theta"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} \widetilde{C} \arrow[r, "\widetilde{\theta}"'] \arrow[ur, dotted, "\Theta"] & Y \end{tikzcd}$$

there exists an extension $\Theta\colon \operatorname{Spec} \widetilde{C} \to X$ *of* θ*, shown as the dotted arrow, making the diagram commute.*

Remarks:

(1) Obviously, the equivalence (1) $\Longleftrightarrow$ (2) above, is our version of a fact familiar from elementary differential geometry (see also the material of Chapter 2, Sections 2.2 and 2.3).

(2) We can define $f\colon X \to Y$ to be *étale* iff it has (1) and (2) of Definition 6.2, and

(3) $\Omega^1_{X/Y} = (0)$.

There is a corresponding theorem to our Theorem 6.8 for étale morphisms. It says

Theorem 6.9 *Let* X, Y *be locally noetherian schemes and* $f\colon X \to Y$ *a finite-type morphism. Then, the following statements are equivalent:*

(1) X *is étale over* Y*.*

(2) *(Jacobian criterion) For any* $x \in X$*, there exist affine open subschemes* $\operatorname{Spec} B$ *of* X *and* $\operatorname{Spec} A$ *of* Y*, with* $x \in \operatorname{Spec} B$ *and* $y = f(x) \in \operatorname{Spec} A$*, so that*

$$B \cong A[T_1, \ldots, T_n]/(f_1, \ldots, f_n),$$

and

$$J = \left(\frac{\partial f_i}{\partial T_j}\right)$$

has maximal rank, n; i.e., J *has an invertible determinant (in* B*).*

(3) *(Infinitesimal lifting criterion) Given any infinitesimal extension of Artinian local rings*

$$0 \longrightarrow I \longrightarrow \widetilde{C} \longrightarrow C \longrightarrow 0$$

with $I^2 = (0)$*, and given the commutative diagram (of solid arrows)*

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[r, "\theta"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} \widetilde{C} \arrow[r, "\widetilde{\theta}"'] \arrow[ur, dotted, "\Theta"] & Y
\end{tikzcd}$$

there exists a **unique** *extension* $\Theta\colon \operatorname{Spec} \widetilde{C} \to X$ *of* θ*, shown as the dotted arrow, making the diagram commute.*

(3) From the Jacobian criterion, X is smooth over Y of relative dimension r iff locally, $f\colon X \to Y$ factors as $f\colon X \longrightarrow \mathbb{A}^r_Y \longrightarrow Y$, where $X \longrightarrow \mathbb{A}^r_Y$ is étale and $\mathbb{A}^r_Y \longrightarrow Y$ is the structure morphism (cf. the exercises).

(4) The Jacobian criterion in the étale case says that an étale morphism is quasi-finite. Therefore, if $f\colon X \to Y$ is étale, the fibres are finite and there is no ramification. So, $f\colon X \to Y$ étale is the analog of a covering space from topology; but, in topology (even in the C^∞-category), a covering map is a local homeomorphism or diffeomorphism. This is *false* for étale morphisms in algebraic geometry. The problem is that there is no implicit function theorem! All these matters will be explicated in the exercises.

Example 6.1 Let $Y = \operatorname{Spec} k$ and $X = \operatorname{Spec} K$, where k is not algebraically closed and K/k is a finite separable extension. Then $\operatorname{Spec} K$ is étale over k. If $\overline{k}$ is, as usual, an algebraic closure of k (or even a separable closure), then we have

$$K \otimes_k \overline{k} = \underbrace{\overline{k} \prod \cdots \prod \overline{k}}_{[K\colon k]},$$

(in the category of rings) and

$$\operatorname{Spec}(K) \prod_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \cong \coprod_{[K\colon k]} \operatorname{Spec} \overline{k}.$$

Consequently, *over* $\overline{k}$, the scheme $\operatorname{Spec} K$ becomes localy isomorphic to $\operatorname{Spec} \overline{k}$.

The reader will have noticed the similarity of material in our present (abstract) case with material presented in the section of Chapter 2, dealing with the implicit function theorem and nonsingularity (Section 2.3). In fact, we have the following more explicit relationship between our present material and that presented in Chapter 2:

Proposition 6.10 *Let X be an algebraic variety over an algebraically closed field, k, and assume X is equidimensional with $\dim(X) = d$. Then, X is regular (i.e., every $\mathcal{O}_{X,x}$ is a regular local ring) iff X is smooth over $\mathrm{Spec}\, k$ and $\mathrm{rk}(\Omega^1_{X/k}) = d$. Hence, X is non-singular iff X is smooth over k and $\mathrm{rk}(\Omega^1_{X/k}) = \dim(X)$. (Of course, X is assumed equidimensional.)*

Proof. In going from smoothness etc. to regularity of $\mathcal{O}_{X,x}$, we may assume that x is a closed point of X. Because if x_0 is a closed point in $\overline{\{x\}}$, then $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,x_0}$, and localizations of regular local rings are again regular (Serre [49]). Let x be a closed point of X, then $d = \dim(\mathcal{O}_{X,x})$. Look at the maps

$$k \longrightarrow \mathcal{O}_{X,x} \longrightarrow k = \kappa(x).$$

(Recall that since k is algebraically closed, $\kappa(x) = k$). Apply (D5) with $\mathfrak{B} = \mathfrak{m}_x$. We get

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\delta} \Omega^1_{\mathcal{O}_{X,x}/k} \otimes \kappa(x) \longrightarrow \Omega^1_{\kappa(x)/k} \longrightarrow 0 \quad \text{is exact.} \tag{$*$}$$

However, $\Omega^1_{\kappa(x)/k} = (0)$, since $\kappa(x) = k$.

Claim. The map, δ, is an isomorphism. To see this, by local freeness of $\Omega^1_{\mathcal{O}_{X,x}/k}$, it suffices to show the dual map, δ^D, is an isomorphism. But the dual of $(*)$ is the exact sequence

$$0 \longrightarrow \mathrm{Hom}_k(\Omega^1_{X,x} \otimes k, k) \xrightarrow{\delta^D} \mathrm{Hom}\,(\mathfrak{m}_x/\mathfrak{m}_x^2, k).$$

By Proposition 2.16 part (1) and the fact that $\mathrm{rk}(\Omega^1_{X,x}) = d$, the map δ^D is an isomorphism. Then, the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ is just d; by definition, $\mathcal{O}_{X,x}$ is then regular.

Conversely, when $x \in X$ is closed and $\mathcal{O}_{X,x}$ is regular, from $\dim(\mathcal{O}_{X,x}) = d = \dim(X)$, we find that

$$\dim(\Omega^1_{X,x} \otimes k) = d.$$

If $x \in X$ is generic, then $\Omega^1_{X,x}$ is the localization of Ω^1_{X,x_0}–where, x_0 is closed. Hence,

$$\Omega^1_{X,x_0} \otimes_{\mathcal{O}_{X,x_0}} K = \Omega^1_{K/k},$$

where K is $\mathcal{O}_{X,x}$. But the transcendence degree of K/k is just the rank of $\Omega^1_{K/k}$ as a K-module, since K is separably generated over k (remember, k is algebraically closed, and thus, perfect). Our result now follows from

Lemma 6.11 *Let A be a noetherian local domain and M a finitely generated A-module. If*

$$\dim(M \otimes_A \kappa(A)) = \dim(M \otimes_A \mathrm{Frac}(A)), \tag{$**$}$$

then M is free of rank the common dimension.

Proof. Let d be the common dimension in $(**)$. Since M is f.g., by Nakayama's lemma, we have the exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow A^d \longrightarrow M \longrightarrow 0.$$

If we tensor with K, the sequence remains exact as K is flat over A, and we get

$$0 \longrightarrow \mathfrak{A} \otimes_A K \longrightarrow K^d \longrightarrow M \otimes_A K \longrightarrow 0.$$

But, $M \otimes_A K$ has rank d as K-space, which implies that $\mathfrak{A} \otimes_A K = (0)$ and thus, $\mathfrak{A}$ is entirely torsion. Yet $\mathfrak{A}$ is contained in the free module A^d, a torsion-free module, so that $\mathfrak{A} = (0)$ and M is free. $\square$

In Proposition 6.10 we made heavy use of the fact that the rank of $\Omega^1_{X/k}$ was precisely the dimension of X. There are times where we merely know that $\Omega^1_{X/k}$ is locally free but are ignorant of its exact rank. In these cases the following theorem is frequently of use:

Theorem 6.12 *Let X be an irreducible nonsingular variety over k, where k is algebraically closed, and let $n = \dim(X)$. Suppose Y is an irreducible closed subscheme of X over k. Then, Y is nonsingular iff*

(1) $\Omega^1_{Y/k}$ is locally free, and

(2) In the exact sequence of (D5), where $\mathfrak{I}$ is the ideal sheaf of Y in X, we have exactness on the left:

$$0 \longrightarrow \mathfrak{I}/\mathfrak{I}^2 \xrightarrow{\delta} \Omega^1_{Y/k} \restriction Y \longrightarrow \Omega^1_{Y/k} \longrightarrow 0.$$

When Y is nonsingular, then $\mathfrak{I}/\mathfrak{I}^2$ is a locally free, rank r $(= \mathrm{codim}(Y, X))$ $\mathcal{O}_Y$-module and $\mathfrak{I}$ is locally generated by r elements.

Proof. Assume that (1) and (2) hold. By Proposition 6.10, we must show that $\mathrm{rk}(\Omega^1_{Y/k}) = \dim(Y)$. Let $q = \mathrm{rk}(\Omega^1_{Y/k})$, then $\mathfrak{I}/\mathfrak{I}^2$ is a locally free sheaf of rank $n-q$. By Nakayama, $\mathfrak{I}$ is locally generated (as an ideal) by $n-q$ elements. Thus, we can conclude that $\dim(Y) \geq q$. Now, pick a closed point $y \in Y$. Look at $\mathfrak{m}_y/\mathfrak{m}_y^2$. By previous work,

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \cong \Omega^1_{\mathcal{O}_{Y,y}/k} \otimes k.$$

Since $\mathrm{rk}(\Omega^1_{Y/k}) = q$, we get

$$\dim(\mathfrak{m}_y/\mathfrak{m}_y^2) = q \geq \dim(\mathcal{O}_{Y,y}) = \dim(Y).$$

This implies that $\dim(Y) = q$, and therefore, Y is nonsingular. Note, in this case, we know that $\mathfrak{I}/\mathfrak{I}^2$ is locally free of rank $n-q = \mathrm{codim}(Y, X)$, and $\mathfrak{I}$ is locally generated by $r = n-q$ elements.

Now assume that Y/k is nonsingular. Proposition 6.10 implies that $\Omega^1_{Y/k}$ is locally free and has rank $q = \dim(Y)$. Since Y and X are nonsingular, by Theorem 2.23, Y is a local complete intersection. Thus, $\mathfrak{I}$ is locally generated by $n-q$ elements, and $\mathfrak{I}/\mathfrak{I}^2$ is locally free of rank $n-q$ (see Lemma 6.11). As the ranks are correct, (D5) implies (2). The rest of the theorem has been remarked above. $\square$

A frequent occurrence in algebraic geometry is a situation where one has some geometrical object in or on a scheme and one wishes to see which if any of its properties persist when we restrict to sufficiently general subschemes of X. For example, say we are given a connected subscheme, Z, of a scheme X and we ask if $Z \cap Y$ is again connected for general subschemes, Y, or X. Or, given a vector bundle on X having a specific property, P, does the bundle when restricted to general subschemes of X retain this property. Of course, stated with this vagueness, either the desired results are false or unprovable. However, with sufficient restrictions they form a very interesting class of questions in algebraic geometry. The earliest theorem proved in this vein is that of Bertini. Its setup is as follows: We have a closed subvariety, Y, of $\mathbb{P}^n = X$, we assume Y is nonsingular and irreducible and place ourselves over an algebraically closed field, k. Then the question becomes: Is $H \cap Y$ again nonsingular and irreducible for sufficiently general hyperplanes, H? The next theorem—the classical Bertini theorem—gives an answer with precision. In Bertini's honor, theorems of this sort are usually called Bertini theorems.

Theorem 6.13 *(Bertini) Let $Y \subseteq \mathbb{P}^n$ be a closed, irreducible, nonsingular variety over an algebraically closed field k. Write* $\mathrm{Hyp}(\mathbb{P}^n)$ *for the projective space classifying all hyperplanes in $\mathbb{P}^n$. There is a Zariski open (hence dense) subset $U \subseteq \mathrm{Hyp}(\mathbb{P}^n)$ so that for every $H \in U$, the intersection $H \cap Y$ is everywhere regular and connected if* $\dim(Y) \geq 2$. *That is, almost every hyperplane section of a nonsingular irreducible variety of dimension at least* 2 *is again irreducible and nonsingular (note, in the presence of regularity, irreducible is equivalent to connected).*

Proof. We would like to apply the irreducibility criterion: Theorem 2.11. Pick y closed in Y, and let $\mathrm{Bad}(y)$ be the set of all hyperplanes through y so that either

(1) $Y \subseteq H$, or

(2) $Y \not\subseteq H$ but y is a nonregular point of $H \cap Y$.

Pick some fixed hyperplane H_0 with $y_0 \notin H_0 \cap Y$ and $Y \not\subseteq H_0$. Recall that hyperplanes are zeros of sections, σ, of $\mathcal{O}_{\mathbb{P}^n}(1)$. Write H_σ for the hyperplane: $\sigma = 0$. We have $H_0 = H_{\sigma_0}$ for some σ_0. Look at σ/σ_0, a meromorphic function on $\mathbb{P}^n$. This function has a pole at H_0, i.e., it is holomorphic on $\mathbb{P}^n - H_0$. Hence

$$\frac{\sigma}{\sigma_0} \restriction (Y - (Y \cap H_0))$$

is holomorphic on $Y - (Y \cap H_0)$. Define the linear map

$$\Phi_y \colon \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \mathcal{O}_{Y,y}/\mathfrak{m}_y^2$$

by

$$\Phi_y \colon \sigma \mapsto \overline{\left(\frac{\sigma}{\sigma_0}\right)} \in \mathcal{O}_{Y,y}/\mathfrak{m}_y^2.$$

Now, $y \in Y \cap H$ means that σ vanishes at y (here $H = H_\sigma$), i.e., $\frac{\sigma}{\sigma_0} \in \mathfrak{m}_y$. When is $y \in Y \cap H$ a singular point of $Y \cap H$? This happens if there are no linear terms in σ/σ_0 at y, i.e., everywhere $\frac{\sigma}{\sigma_0} \in \mathfrak{m}_y^2$. Hence, y is nonsingular on $Y \cap H$ iff $\Phi_y(\sigma) = 0$. Also, $Y \subseteq H$ iff $\sigma/\sigma_0 = 0$ on Y. Thus, we get $H \in \mathrm{Bad}(y)$ iff the σ defining H lies in $\mathrm{Ker}\,(\Phi_y)$, and thus

$$\mathrm{Bad}(y) = \mathbb{P}(\mathrm{Ker}\,(\Phi_y)).$$

What's the dimension of $\mathrm{Bad}(y)$?

Note that Φ_y is surjective onto $\mathcal{O}_{Y,y}/\mathfrak{m}_y^2$. Indeed, we get the constants from $\Phi_y(\lambda\sigma_0)$. As $\mathfrak{m}_y$ is generated by the linear forms in $T_0, \ldots, T_n$, the coordinates of $\mathbb{P}^n$, and as σ ranges over all linear forms, this implies surjectivity. Therefore,

$$\dim(\mathrm{Ker}\,(\Phi_y)) = \dim(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))) - \dim(\mathcal{O}_{Y,y}/\mathfrak{m}_y^2).$$

However, $\dim(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))) = n + 1$, and we have the exact sequence

$$0 \longrightarrow \mathfrak{m}_y/\mathfrak{m}_y^2 \longrightarrow \mathcal{O}_{Y,y}/\mathfrak{m}_y^2 \longrightarrow \mathcal{O}_{Y,y}/\mathfrak{m}_y \; (= \kappa(y) = k) \longrightarrow 0,$$

since y is a closed point and k is algebraically closed. Now, y is nonsingular, so

$$\dim(\mathfrak{m}_y/\mathfrak{m}_y^2) = \dim(Y) = r \quad \text{(say)},$$

and thus,

$$\dim(\mathrm{Ker}\,(\Phi_y)) = n + 1 - (r + 1) = n - r.$$

We find,

$$\dim(\mathrm{Bad}(y)) = n - r - 1$$

and this is *independent* of y. In particular, note that for every $y \in Y$, there is some bad hyperplane containing y. Now, consider the variety $Y \prod \mathrm{Hyp}(\mathbb{P}^n)$, and in it, the Bertini set

$$\mathcal{B} \subseteq Y \prod \mathrm{Hyp}(\mathbb{P}^n)$$

defined by

$$(y, H) \in \mathcal{B} \quad \text{iff} \quad H \in \mathrm{Bad}(y).$$

Clearly, $\mathcal{B}$ is a closed subvariety (under the reduced induced structure) and our points (y, H) are the closed points. As usual, we consider the projections

$$pr_1 \colon Y \prod \mathrm{Hyp}(\mathbb{P}^n) \to Y \quad \text{and} \quad pr_2 \colon Y \prod \mathrm{Hyp}(\mathbb{P}^n) \to \mathrm{Hyp}(\mathbb{P}^n).$$

Since $Y \prod \mathrm{Hyp}(\mathbb{P}^n)$ is projective, the maps are proper. We know that the fibres of pr_1 are all projective spaces, of constant dimension $n - r - 1$, thus, irreducible; by the irreducibility criterion, $\mathcal{B}$ is irreducible and its dimension is $r + n - r - 1 = n - 1$. The image of $\mathcal{B}$ in $\mathrm{Hyp}(\mathbb{P}^n) \cong \mathbb{P}^n$ is then irreducible, of dimension at most $n-1$, and closed. If U is the complement of $\mathrm{Im}\,(\mathcal{B})$ under pr_2, then $H \in U$ means that $H \notin \mathrm{Bad}(y)$ for all $y \in Y$. This means, U is the desired open set of good hyperplanes. For connectivity of $Y \cap H$ when $\dim(Y) \geq 2$, we will wait until the next Chapter (see Section 7.4, Remark (1) after Theorem 7.29). $\square$

Remark: As hyperplanes through a point form a closed subvariety (strictly contained) of $\mathrm{Hyp}(\mathbb{P}^n)$, we can throw these out for finitely many points and still retain the Bertini open set of good hyperplanes.

Nomenclature:

1. $\Omega^1_{X/Y}$ is the *sheaf of relative differentials* (or *relative* 1*-forms of X over Y*).

2. $\Omega^1_{X/Y}$ is also called the *relative cotangent sheaf of X over Y*, and, if $\Omega^1_{X/Y}$ is a bundle, then it is called the *relative cotangent bundle of X over Y*.

3. The dual of $\Omega^1_{X/Y}$, that is, $(\Omega^1_{X/Y})^D$, is called the *relative tangent sheaf of X over Y*, denoted $\mathcal{T}_{X/Y}$; and when $\Omega^1_{X/Y}$ is a bundle, we call $\mathcal{T}_{X/Y}$ the *relative tangent bundle of X over Y*.

4. If $\mathfrak{I}$ is the ideal sheaf defining Z as an S-subscheme of X (here, X lies over S), then $\mathfrak{I}/\mathfrak{I}^2$ is called the *conormal sheaf of Z in X*.

5. If $\mathfrak{I}$ is a bundle, it is called the *conormal bundle of Z in X* and its dual is the *normal bundle of Z in X*.

If $i \colon Z \to X$ (closed immersion), then observe that

$$\mathfrak{I}/\mathfrak{I}^2 = \mathfrak{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{I} = \mathfrak{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \mathfrak{I} \restriction Z = i^*\mathfrak{I}.$$

Thus, $\mathfrak{I}/\mathfrak{I}^2$ is a sheaf on Z. Say $S = \mathrm{Spec}\, k$, where k is algebraically closed, and suppose X and Y are nonsingular, then we know that

$$0 \longrightarrow \mathfrak{I}/\mathfrak{I}^2 \longrightarrow \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \longrightarrow \Omega^1_{Z/k} \longrightarrow 0 \qquad (\dagger)$$

is exact, and all are locally free. Taking duals, we get

$$0 \longrightarrow \mathcal{T}_{Z/k} \longrightarrow \mathcal{T}_{X/k} \restriction Z \longrightarrow (\mathfrak{I}/\mathfrak{I}^2)^D \longrightarrow 0.$$

Therefore, $(\mathfrak{I}/\mathfrak{I}^2)^D$ is the normal bundle of Z in X, denoted by $\mathcal{N}_{Z\hookrightarrow X}$, explaining our terminology above.

Definition 6.3 The sheaf

$$\Omega^r_{X/Y} = \bigwedge^r \Omega^1_{X/Y}$$

is the *sheaf of relative r-forms of X/Y*. It is a bundle if X is smooth over Y. The highest wedge of $\Omega^1_{X/Y}$, denoted $\omega_{X/Y}$ ($\omega_{X/Y} = \bigwedge^{\bullet} \Omega^1_{X/Y}$) is called the *relative canonical bundle of X/Y*. It is a line bundle on X, and any Cartier divisor representing $\omega_{X/Y}$ is called a *relative canonical divisor of X over Y*. (When $Y = \operatorname{Spec} k$ and k is algebraically closed, occurrences of the word "relative" are omitted.)

Again, assume that $Z \hookrightarrow X$ is a closed immersion, and that X lies over $\operatorname{Spec} k$, where k is algebraically closed. Further assume that Z and X are nonsingular, irreducible over k. Let $\mathfrak{I}$ be the ideal sheaf defining Z. From (†), we get

$$0 \longrightarrow \mathfrak{I}/\mathfrak{I}^2 \longrightarrow \Omega^1_{X/k} \restriction Z \longrightarrow \Omega^1_{Z/k} \longrightarrow 0,$$

which implies that

$$\omega_X \restriction Z = \omega_Z \otimes \bigwedge^{\bullet} \mathfrak{I}/\mathfrak{I}^2.$$

We know that $\mathfrak{I}/\mathfrak{I}^2$ is a rank r ($= \operatorname{codim}(Z \hookrightarrow X)$) bundle on Z, so that

$$\bigwedge^{\bullet} \mathfrak{I}/\mathfrak{I}^2 = \bigwedge^{r} \mathfrak{I}/\mathfrak{I}^2.$$

Therefore, moving $\bigwedge^r \mathfrak{I}/\mathfrak{I}^2$ to the other side, we get

$$\omega_Z = (\omega_X \restriction Z) \otimes \left(\bigwedge^{r} \mathfrak{I}/\mathfrak{I}^2\right)^D.$$

As a result, we get

Proposition 6.14 *(Adjunction formula) If Z is a closed Y-subscheme of X where both X and Z are smooth over Y, then*

$$\omega_{Z/Y} = (\omega_{X/Y} \restriction Z) \otimes_{\mathcal{O}_X} \bigwedge^{r} \mathcal{N}_{Z\hookrightarrow X} = \omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \otimes_{\mathcal{O}_X} \bigwedge^{r} \mathcal{N}_{Z\hookrightarrow X}.$$

Here, $r = \operatorname{codim}(Z \hookrightarrow X)$.

Consider the special case where $r = 1$, so that Z has codimension 1 in X. Then, Z is a Cartier divisor in X and $\mathfrak{I}$ is equal to $\mathcal{O}_X(-Z)$. Therefore,

$$\mathfrak{I}/\mathfrak{I}^2 = \mathfrak{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{X/\mathfrak{I}} = \mathfrak{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \mathcal{O}_X(-Z) \restriction Z = \mathcal{O}_Z(-Z),$$

and

$$\mathcal{N}_{Z\hookrightarrow X} = (\mathfrak{I}/\mathfrak{I}^2)^D = \mathcal{O}_X(Z) \restriction Z = \mathcal{O}_Z(Z).$$

Thus, when $r = 1$, the adjunction formula reads:

$$\omega_Z = \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_X(Z).$$

We can express the adjunction formula in terms of the canonical divisors associated with ω_X and ω_Z, especially in the case $r = 1$. Since

$$\mathcal{N}_{Z\hookrightarrow X} = \mathcal{O}_X(Z) \restriction Z$$

and Z is the divisor associated with $\mathcal{O}_X(Z)$, we see that $\mathcal{O}_X(Z) \restriction Z$ is just $Z \cdot Z$, as divisor on Z. Then, $\omega_X \otimes \mathcal{O}_X(Z)$ corresponds to the Cartier divisor $K_X + Z$ (where K_X is the canonical divisor associated with X). From all this, the adjunction formula becomes the following classical formula involving intersection cycles:

$$K_Z = (K_X + Z) \cdot Z = K_Z \cdot Z + (Z^2).$$

As an application of the adjunction formula, we will determine ω_Z when Z is a nonsingular hypersurface of degree d in $\mathbb{P}^n$ over an algebraically closed field. We will see in the next Chapter that we have the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

This implies the important formula:

$$\omega_{\mathbb{P}^n} = \bigwedge^{\bullet} \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n}(-(n+1)).$$

If Z is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, then $\mathfrak{I}_Z = \mathcal{O}_{\mathbb{P}^n}(-d)$. Thus, $\mathcal{N}_{Z\hookrightarrow \mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(d)$; so,

$$\omega_Z = \omega_X \otimes \mathcal{O}_X(Z) \otimes \mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \otimes \mathcal{O}_{\mathbb{P}^n}(d) \restriction Z = \mathcal{O}_Z(d-(n+1)). \qquad (*)$$

An important special case occurs if Z is a nonsingular hypersurface of degree $n+1$ in $\mathbb{P}^n$. Then, ω_Z is the trivial bundle on Z. This includes, for example, curves of degree 3 in $\mathbb{P}^2$ (elliptic curves), surfaces of degree 4 in $\mathbb{P}^3$ ($K3$-surfaces), and quintic 3-folds in $\mathbb{P}^4$ (Calabi-Yau 3-folds). In general, such hypersurfaces are called *Calabi-Yau hypersurfaces*.

If $d < n+1$, then ω_Z has no global sections. These hypersurfaces form a special class of hypersurfaces whose detailed study is possible—among them are the Fano varieties (see Chapter 7). Of course, the generic hypersurfaces have large degree, and so there are plenty of global sections of Ω_Z. Now, as a general fact the important geometry associated with any sheaf is exactly the geometry contained in the collection of its global sections. (For example, the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ has as global sections all the hypersurfaces of degree d of $\mathbb{P}^n$.) We want to see the geometric content of the sheaf ω_Z and sheaves derived from it.

Definition 6.4 Given a nonsingular variety X over an algebraically closed field k, we define the *rth plurigenus of* X denoted $p_r(X)$, by

$$p_r(X) = \dim_k(H^0(X, \omega_X^{\otimes r})).$$

When $r = 1$, the plurigenus $p_1(X)$ is also denoted by $p_g(X)$, or p_g, and it is called the *geometric genus of* X. When X is a nonsingular curve, the genus p_g is also denoted simply by g.

The notion of plurigenus was studied extensively by Castelnuovo, Enriques, and Severi. When $r = 1$, the geometric genus, $p_1(X)$, is the number of linearly independent holomorphic d-forms on X, where $d = \dim(X)$. Let Z be a nonsingular hypersurface in $\mathbb{P}^n$ of degree d over an algebraicaly closed field. What is $p_g(Z)$? We have the defining exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

and tensoring with $\mathcal{O}_{\mathbb{P}^n}(d)$, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_Z(d) \longrightarrow 0.$$

If we now tensor with $\mathcal{O}_{\mathbb{P}^n}(-(n+1))$ and use the adjunction formula $(*)$, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-(n+1)) \longrightarrow \omega_Z \longrightarrow 0.$$

Applying cohomology, we get

$$0 \longrightarrow 0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^r}(d-(n+1))) \longrightarrow H^0(Z, \omega_Z) \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-(n+1))).$$

From the next Chapter (Theorem 7.35), if we assume $n \geq 2$,

$$H^1(\mathbb{P}^n, \mathcal{L}) = (0) \quad \text{for any line bundle } \mathcal{L}.$$

Hence, we get

$$H^0(Z, \omega_Z) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^r}(d-(n+1))).$$

If $d < n+1$, then $p_g(Z) = 0$.

If $d = n+1$, then $p_g(Z) = 1$.

If $d > n+1$, then $p_g(Z)$ is the number of monomials in $n+1$ variables of degree $d-(n+1)$. This means,

$$p_g(Z) = \binom{d-1}{d-(n+1)} = \binom{d-1}{n}.$$

In particular, when $n = 2$, i.e., for Z a nonsingular plane curve, we get the genus

$$g = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2} \quad \text{if } d > n+1.$$

Thus, when $d = 1, 2$, we have $g = 0$, and when $d = 3$, we get $g = 1$, the elliptic case. When $d = 4$, we get $g = 3$. Therefore, no nonsingular curves of genus 2 embed in the plane.

6.3 Further Readings

Flatness was invented by Serre in the early fifties. Flatness is discussed extensively in EGA IVb ([27], Chapter IV) and EGA IVc ([28], Chapter IV), and also in Bourbaki [7] (Algèbre Commutative, Chapter 1). It is also discussed in Matsumura [40] (Chapter 3, Section 7 and Chapter 8), Hartshorne [33] (Chapter 3, Sections 9 and 10), and Mumford [43] (Chapter 3). Smooth and étale morphisms are discussed in EGA IVd ([29], Chapter IV) and in the last two references. Derivations and relative differentials are discussed in EGA IVa ([26], Chapter 0, Section 20), Matsumura [40] (Chapter 9), Hartshorne [33] (Chapter 2, Section 8), Mumford [43] (Chapter 3), Shafarevich [54] (Chapter VI), and Bourbaki [5] (Algèbre, Chapter III).

Chapter 7

Projective Schemes and Morphisms

7.1 Projective Schemes

Experience shows that the most important case in algebraic geometry is the case of projective varieties and the generalizations of them which we shall make in this section, namely, projective schemes. We already have considerable experience in projective matters (Chapter 2) and what we do here will be an extension whose structure should not be too surprising.

Projective schemes arise from the consideration of graded rings and homogeneous ideals. So, it's best to begin with these.

Let B be a graded ring, i.e.,

$$B = \coprod_{n\in\mathbb{Z}} B_n, \quad \text{and} \quad B_m \cdot B_n \subseteq B_{m+n}.$$

For any element $b \in B$, if $b \in B_n$ for some integer n, then b is called *homogeneous*, or a *form*, and the integer n is the *degree of* b, denoted $\deg(b)$. Any $b \in B$ can be written as a finite sum $b = b_1 + \cdots + b_k$, where each b_i belongs to some B_{n_i} (with $n_i \neq n_j$ whenever $i \neq j$), and each b_i is called a *homogeneous component* of b.

If A is a ring, we shall assume that the graded ring, B, is an A-algebra, so that each B_n is an A-module, and we have maps

$$A \longrightarrow B_0 \hookrightarrow B.$$

Graded rings come in all types and some are more amenable to the notions of geometry we want to stress than others. For example, there is no guarantee that elements of degree one generate B in any of the senses one can imagine. In topology, one frequently meets with rings having generators in many higher degree components. However, for our purposes, emphasis will be placed on those graded rings in which all generators appear in degree one. Let us call such graded rings, *good graded rings (ggr)*, a nomenclature which is by no means standard. That is, B is a good graded ring if

(1) $B = \coprod_{n\geq 0} B_n$, i.e., B is non-negatively graded, and

(2) $B^+ = \coprod_{n\geq 1} B_n$ is generated, as an ideal, by B_1.

This is equivalent to saying that $\mathrm{Sym}_{B_0}(B_1) \longrightarrow B$ is surjective.

Recall that an ideal $\mathfrak{B}$ is a *homogeneous ideal* of B if

$$\mathfrak{B} = \coprod_{n\in\mathbb{N}} \mathfrak{B} \cap B_n.$$

This means that each homogeneous component of an element of $\mathfrak{B}$ is again in $\mathfrak{B}$. Ideals which are simultaneously homogeneous and prime will, of course, be called *homogeneous prime ideals*. Testing primeness in a homogeneous ideal can be done using forms. Each non-negatively graded ring, B, gives rise to a scheme as follows:

Let

$$X = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a homogeneous prime ideal of } B \text{ and } B^+ \not\subseteq \mathfrak{p}\}.$$

A homogeneous prime ideal, $\mathfrak{p}$, such that $B^+ \not\subseteq \mathfrak{p}$ is called *relevant*. We take as basis for open sets, the sets

$$X_{(f)} = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$$

where $f \in B_i$ for some $i \geq 1$—that is, f *is to be a form*. Thus, the closed sets are exactly the sets

$$V(\mathfrak{A}) = \{\mathfrak{p} \in X \mid \mathfrak{A} \subseteq \mathfrak{p}, \quad \text{where } \mathfrak{A} \text{ is homogeneous and relevant}\}$$

Clearly,

$$V((f)) = (X_{(f)})^c.$$

Having made the underlying topological space, X, of our proposed scheme $(X, \mathcal{O}_X)$, we now need the sheaf part. For any $\mathfrak{p} \in X$, let

$$B_{(\mathfrak{p})} = \left\{ \frac{\xi}{\eta} \mid \xi, \eta \in B, \quad \text{with } \xi \text{ and } \eta \text{ forms, } \deg(\xi) = \deg(\eta) \text{ and } \eta \notin \mathfrak{p} \right\} \subseteq B_{\mathfrak{p}},$$

and if f is a form in B, write

$$B_{(f)} = \left\{ \frac{\xi}{f^r} \mid \xi \in B, \quad \text{with } \xi \text{ a form and } \deg(\xi) = r\deg(f) \right\} \subseteq B_f.$$

The sheaf $\mathcal{O}_X$ can now be defined as follows:

For any open subset, U, of X, define

$$\Gamma(U, \mathcal{O}_X) = \left\{ F\colon U \to \bigcup_{\mathfrak{p}\in U} B_{(\mathfrak{p})} \;\middle|\; \begin{array}{l} (1)\ F(\mathfrak{p}) \in B_{(\mathfrak{p})} \\ (2)\ (\forall \mathfrak{p} \in U)(\exists \text{ forms } f, g \in B)(g \notin \mathfrak{p}, \deg(f) = \deg(g)) \\ (3)\ (\forall \mathfrak{q} \in X_{(g)} \cap U) \left(F(\mathfrak{q}) = \text{image} \left(\frac{f}{g}\right) \text{ in } B_{(\mathfrak{q})} \right). \end{array} \right\}$$

This is a sheaf of local rings, and for every $x = \mathfrak{p} \in X$,

$$\mathcal{O}_{X,x} = B_{(x)}.$$

We define $\mathrm{Proj}(B)$ to be *the local ringed space,* $(X, \mathcal{O}_X)$*, just defined*. This is our generalization of a projective variety. Of course, we must first check that $\mathrm{Proj}(B)$ is a scheme.

Proposition 7.1 *Given a non-negatively graded ring B, the local ringed space,* $\mathrm{Proj}(B)$*, is a scheme.*

Proof. All we need to check is that $(X_{(f)}, \mathcal{O}_X \restriction X_{(f)})$ is affine. In fact, we shall show that

$$(X_{(f)}, \mathcal{O}_X \restriction X_{(f)}) \cong \mathrm{Spec}(B_{(f)}).$$

To do this, for each homogeneous ideal, $\mathfrak{B}$, of B, let

$$\theta(\mathfrak{B}) = (B_f \mathfrak{B}) \cap B_{(f)}, \quad \text{an ideal of } B_{(f)}.$$

If $\mathfrak{p} \in X_{(f)}$, then $\theta(\mathfrak{p}) \in |\mathrm{Spec}(B_{(f)})|$, which gives the map on spaces. Now, it should be clear that (DX)

$$B_{(\mathfrak{p})} = (B_{(f)})_{\theta(\mathfrak{p})},$$

and the reader can complete the proof that

$$X_{(f)} \cong \mathrm{Spec}(B_{(f)}), \quad \text{as schemes.} \qquad \square$$

Remarks:

(1) The gluing of Proposition 7.1 shows that $\mathrm{Proj}(B)$ is a separated scheme.

(2) If B is a ggr, then the $X_{(f)}$, where $f \in B_1$, cover $\mathrm{Proj}(B)$. Otherwise, we would have $B_1 \subseteq \mathfrak{p}$, and then, $B^+ \subseteq \mathfrak{p}$, because B is a ggr, contradicting that $\mathfrak{p}$ is relevant.

Of course, as in the general theory, we want to make sheaves of $\mathcal{O}_X$-modules, and of course, the good ones must come from graded B-modules. Recall that a module M is a graded B-module if

$$M = \coprod_{n\in\mathbb{Z}} M_n \quad \text{and} \quad B_r \cdot M_s \subseteq M_{r+s}.$$

The localization $M_{(\mathfrak{p})}$ is defined in the same way as $B_{(\mathfrak{p})}$, namely

$$M_{(\mathfrak{p})} = \left\{ \frac{\xi}{\eta} \mid \xi \in M, \eta \in B, \quad \text{with } \xi, \eta \text{ homogeneous, } \deg(\xi) = \deg(\eta) \text{ and } \eta \notin \mathfrak{p} \right\},$$

and similarly for $M_{(f)}$.

The sheaf, $M^\sharp$, of $\mathcal{O}_X$-modules we make from M is defined as follows:

For any open subset U of X,

$$\Gamma(U, M^\sharp) = \left\{ F\colon U \to \bigcup_{\mathfrak{p}\in U} M_{(\mathfrak{p})} \;\middle|\; \begin{array}{l} (1)\ F(\mathfrak{p}) \in M_{(\mathfrak{p})} \\ (2)\ (\forall \mathfrak{p} \in U)(\exists\,\text{homogeneous}\,\xi \in M)(\exists\,\text{form}\ g \in B) \\ \quad (g \notin \mathfrak{p}\ \text{and}\ \deg(\xi) = \deg(g)) \\ (3)\ (\forall \mathfrak{q} \in X_{(g)} \cap U)\left(F(\mathfrak{q}) = \text{image}\left(\frac{\xi}{g}\right)\ \text{in}\ M_{(\mathfrak{q})}\right). \end{array} \right\}$$

This is a sheaf of $\mathcal{O}_X$-modules, and

$$M^\sharp \upharpoonright X_{(f)} = \widetilde{M_{(f)}},$$

as the reader can check; therefore, $M^\sharp$ is a QC $\mathcal{O}_X$-module.

Remarks:

(1) If B is noetherian and M is f.g., then $M^\sharp$ is coherent as $\mathcal{O}_X$-module.

(2) If B_0 is noetherian, B_1 is a f.g. B_0-module and B is a ggr, then B is noetherian.

(3) We have the ring inclusion $B_0 \hookrightarrow B$, if we localize at f, where f is a form in B^+, we get a morphism
$$\mathrm{Spec}\, B_{(f)} \longrightarrow \mathrm{Spec}\, B_0.$$
These maps patch, and yield the *structure morphism*
$$\mathrm{Proj}\, B \longrightarrow \mathrm{Spec}\, B_0.$$

Example 7.1 *Projective space over a ring.*
Let A be a ring and write $B = A[X_0, \ldots, X_n]$. We have $B_0 = A$. Note that B is a ggr, and let $Z = \mathrm{Proj}\, B$. Then, the $Z_{(X_j)}$'s cover Z for $j = 0, \ldots, n$. We have

$$\begin{aligned} Z_{(X_j)} &= \mathrm{Spec}\, B_{(Z_j)} \\ &= \mathrm{Spec}(A[X_0, \ldots, X_n]_{(X_j)}) \\ &= \mathrm{Spec}\left(\left\{\frac{f}{X_j^r} \mid f \in B,\ f \text{ is a form, } \deg(f) = r\right\}\right), \end{aligned}$$

and since f is a form,

$$\frac{1}{X_j^r} f(X_0, \ldots, X_n) = f\left(\frac{X_0}{X_j}, \ldots, \frac{X_n}{X_j}\right).$$

Thus,

$$Z_{(X_j)} = \mathrm{Spec}\left(A\left[\frac{X_0}{X_j}, \ldots, \frac{X_n}{X_j}\right]\right) = \mathbb{A}^n_A.$$

This Z is what we mean by *projective n-space over A* and we denote it by $\mathbb{P}^n_A$. So,

$$\mathbb{P}^n_A = \mathrm{Proj}(A[X_0, \ldots, X_n]).$$

Remarks:

(1) Note that we have *not* defined projective space over a ring A by saying what are its points (with values anywhere)—projective space is merely defined as the scheme obtained by gluing correctly the right number of affine spaces.

(2) In the case $n = 0$, we get

$$\mathbb{P}^0_A = \mathrm{Proj}(A[X]) = \mathrm{Spec}\, A,$$

as the reader easily sees.

(3) If ξ is a geometric point of $\mathrm{Spec}\, A$, then the fibre of $\mathbb{P}^n_A$ over ξ is just the scheme $\mathbb{P}^n_{\kappa(\xi)}$, that is, the algebraic variety: Projective n-space over $\kappa(\xi)$.

(4) If we start with $A = \mathbb{Z}$, then we make $\mathbb{P}^n_{\mathbb{Z}}$. It is easy to see that (DX)

$$\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \prod_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, A.$$

Later on, we will make $\mathbb{P}^n_X$ where X is any scheme, and in fact, we'll find that

$$\mathbb{P}^n_X \cong \mathbb{P}^n_{\mathbb{Z}} \prod_{\mathrm{Spec}\, \mathbb{Z}} X.$$

A central feature of projective geometry is the shifting of degrees in modules. Let M be a B-module,

$$M = \coprod_{n\in\mathbb{Z}} M_n,$$

then for any $q \in \mathbb{Z}$, we can make the new graded B-module

$$M(q) = \coprod_{n\in\mathbb{Z}} M(q)_n, \quad \text{where} \quad M(q)_n = M_{q+n}.$$

Of course, this gives us $B(q)$, a new module over B.

Note that the tensor product of two graded modules over the graded ring, B, is again a graded module. To see this, let M and N be our graded modules and B be our graded ring. Consider the tensor product $M \otimes_{B_0} N$. Of course, this is the coproduct

$$\coprod_{r,s} M_r \otimes_{B_0} N_s,$$

and it is graded by assigning to each piece $M_r \otimes_{B_0} N_s$ the degree $r + s$. But $M \otimes_B N$ is the quotient of $M \otimes_{B_0} N$ by the submodule generated by the elements

$$\xi \otimes_{B_0} b\eta - b\xi \otimes_{B_0} \eta,$$

with $b \in B$, a form. This submodule is homogeneous, and so the quotient $M \otimes_B N$ is again graded. Note, further, that every $b \in B(q)$ is equal to $b \cdot 1$, where $b \in B$ and $1 \in B(q)_{-q}$. Hence, $M(q) = M \otimes_B B(q)$.

If $X = \mathrm{Proj}\, B$, we can form

(1) $(M(q))^\sharp$.

(2) $(B(q))^\sharp$, *which we denote by* $\mathcal{O}_X(q)$ (As $B^\sharp = \mathcal{O}_X$, this notation is consistent.)

(3) $\mathcal{F}(q)$, for any $\mathcal{O}_X$-module, $\mathcal{F}$, where $\mathcal{F}(q)$ is defined to be $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(q)$. The module, $\mathcal{F}(q)$, is the *Serre q-twist* of $\mathcal{F}$.

Proposition 7.2 *Assume that B is a ggr. Then, the following properties hold:*

(1) $\mathcal{O}_X(q) = (B(q))^\sharp$ *is a line bundle on* $X = \mathrm{Proj}(B)$.

(2) If M and N are graded B-modules, then $(M \otimes_B N)^\sharp = M^\sharp \otimes_{\mathcal{O}_X} N^\sharp$.

(3) $(M(q))^\sharp = M^\sharp(q) = M^\sharp \otimes_{\mathcal{O}_X} \mathcal{O}_X(q)$.

(4) $\mathcal{O}_X(q + q') = \mathcal{O}_X(q) \otimes_{\mathcal{O}_X} \mathcal{O}_X(q') = \mathcal{O}_X(q)(q')$.

(5) Let C be another graded ring, and let $\theta \colon C \to B$ be a graded homomorphism (i.e., $\theta \colon C_i \to B_i$, so, preserves the grading). Then, there is a canonical open set $U \subseteq X = \mathrm{Proj}(B)$ and a morphism of schemes $\varphi \colon U \longrightarrow Y = \mathrm{Proj}(C)$, and U is the maximal open for such a morphism. Further,

$$\varphi_*(\mathcal{O}_U(q)) = (\varphi_*\mathcal{O}_U)(q).$$

(6) $\varphi^*(\mathcal{O}_Y(q)) = \mathcal{O}_X(q) \restriction U = \mathcal{O}_U(q)$.

Proof. (1) We can cover X by its standard opens $X_{(f)}$, where f has degree 1 (because B is a ggr). I claim: $\mathcal{O}_X(q) \restriction X_{(f)}$ is the trivial line bundle.

We have

$$\mathcal{O}_X(q) \restriction X_{(f)} = \widetilde{B(q)_{(f)}},$$

and

$$B(q)_{(f)} = \left\{ \frac{\xi}{f^r} \mid \deg(\xi) = r \text{ in } B(q) \right\} = \left\{ \frac{\xi}{f^r} \mid \xi \in B_{q+r} \right\}.$$

Consider the map $B_{(f)} \longrightarrow B(q)_{(f)}$ *via* $\eta \mapsto f^q \eta$. This is an isomorphism as f is invertible on $X_{(f)}$. But, we have $1 \mapsto f^q$, which implies that f^q is a free generator of $B(q)_{(f)}$, and $\mathcal{O}_X(q) \restriction X_{(f)}$ is trivial.

(2) Look at

$$(M \otimes N)^\sharp \restriction X_{(f)} = (\widetilde{M \otimes N})_{(f)},$$

wich arises from the degree 0 part of $(M \otimes N)_f$, i.e., the degree 0 part of $M_f \otimes N_f$. These are finite linear combinations of elements of the form

$$\frac{m}{f^a} \otimes \frac{n}{f^b}, \quad \text{where } \deg(m) + \deg(n) = a + b.$$

Now, m/f^a need not have degree 0 in M_f, nor n/f^b in N_f. So, let $\deg(m) = \alpha$, then,

$$\frac{m}{f^a} = \frac{mf^\alpha}{f^a f^\alpha} = f^{\alpha - a}\frac{m}{f^\alpha},$$

and similarly for n/f^b. We get

$$\frac{m}{f^a} \otimes \frac{n}{f^b} = \frac{m}{f^\alpha} \otimes \frac{n}{f^{a+b-\alpha}}.$$

As $\deg(n) = a+b-\alpha$, we see that $(M \otimes N)_{(f)} = M_{(f)} \otimes N_{(f)}$. Now, we pass to the associated sheaves on the affine $X_{(f)}$, and on $X_{(f)}$, we get

$$(M^\sharp \otimes N^\sharp) \restriction X_{(f)} = M^\sharp \restriction X_{(f)} \otimes N^\sharp \restriction X_{(f)}.$$

Clearly, these isomorphisms patch, and (2) is proved. We also have

$$M(q)^\sharp = (M \otimes B(q))^\sharp = M^\sharp \otimes \mathcal{O}_X(q) = M^\sharp(q),$$

and

$$\mathcal{O}_X(q + q') = B(q + q')^\sharp = (B(q)(q'))^\sharp = \mathcal{O}_X(q) \otimes \mathcal{O}_X(q'),$$

which proves (3) and (4).

(5) The map $\theta\colon C \to B$ takes C^+ to B^+, but need not be onto. Pick a relevant prime ideal, $\mathfrak{p}$, and look at $\theta^{-1}(\mathfrak{p})$. Observe that

$$C^+ \not\subseteq \theta^{-1}(\mathfrak{p}) \quad \text{iff} \quad \theta(C^+) \not\subseteq \mathfrak{p}.$$

Now, we define φ so that $|\varphi| = \theta^{-1}$, which implies that

$$U = \bigcup_{c \in C^+} X_{(\theta(c))}.$$

The rest of the proof is as in the affine case together with twisting arguments. $\square$

Observe that C_0 and B_0 play no role in defining U. There is a map $C_0 \longrightarrow B_0$, which induces the morphism $\varphi_0\colon \mathrm{Spec}\, B_0 \to \mathrm{Spec}\, C_0$. If $C^+ \longrightarrow B^+$ is surjective, then our map above is defined everywhere and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}\, B \arrow[r, "\varphi"] \arrow[d] & \mathrm{Proj}\, C \arrow[d] \\
\mathrm{Spec}\, B_0 \arrow[r, "\varphi_0"'] & \mathrm{Spec}\, C_0.
\end{tikzcd}$$

We have constructed QC $\mathcal{O}_X$-modules when $X = \mathrm{Proj}(B)$ from graded B-modules. We want to go backwards from $\mathcal{O}_X$-modules to graded modules—in the affine case, this was accomplished merely by taking global sections. It turns out that if we have an $\mathcal{O}_X$-module of the form $M^\sharp$, only the degree 0 part of M is connected to the global sections of $M^\sharp$. Consequently, merely taking global sections is totally insufficient for our purposes. However, it is now clear that we should attempt to use all the Serre twists of our sheaf, and then we might be successful. All these ideas were pioneered by Serre in FAC [47]. So, assume that $\mathcal{F}$ is a QC $\mathcal{O}_X$-module, where $X = \mathrm{Proj}\, B$ and B is a ggr. Following Serre, we define $\mathcal{F}^\flat$ by

$$\mathcal{F}^\flat = \coprod_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)), \quad \text{here} \quad \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Claim. $\mathcal{F}^\flat$ is a graded B-module; of course, the elements of degree n are to be global sections of $\mathcal{F}(n)$.

Proof. Pick $x \in B_d$ and $\xi \in \mathcal{F}^\flat_n = \Gamma(X, \mathcal{F}(n))$. We have $B_d = B(d)_0$. We claim that x gives us a global section of $\mathcal{O}_X(d)$. Look at $X_{(f)}$, where f has degree 1. We have

$$\frac{x}{1} \in B(d)_{(f)} = \Gamma(X_{(f)}, \mathcal{O}_X(d) \upharpoonright X_{(f)}),$$

and these patch (no denominators). Therefore, $x \in \Gamma(X, \mathcal{O}_X(d))$. Now,

$$x \otimes \xi \in \Gamma(X, \mathcal{O}_X(d)) \otimes \Gamma(X, \mathcal{F}(n)),$$

and there is a map

$$\Gamma(X, \mathcal{O}_X(d)) \otimes \Gamma(X, \mathcal{F}(n)) \longrightarrow \Gamma(X, \mathcal{O}_X(d) \otimes \mathcal{F}(n)) \cong \Gamma(X, \mathcal{F}(d+n)) = \mathcal{F}^\flat_{d+n}.$$

Set $x \cdot \xi = \mathrm{image}(x \otimes \xi)$. $\square$

Remarks:

(1) We have seen above that an element of degree 0 in $B(d)$ gives rise to a global section of $\mathcal{O}_X(d)$ $(= B(d)^\sharp)$. The same is true for elements of degree 0 of $M(d)$—these give rise to global sections of $M^\sharp(d)$.

(2) What happens if we do the $\sharp$ construction then the $\flat$, and the opposite order? That is, consider the functors

$$M \rightsquigarrow M^\sharp \rightsquigarrow (M^\sharp)^\flat \quad \text{and} \quad \mathcal{F} \rightsquigarrow \mathcal{F}^\flat \rightsquigarrow (\mathcal{F}^\flat)^\sharp.$$

In general, M is not the same as $(M^\sharp)^\flat$, but $\mathcal{F}$ is the same as $(\mathcal{F}^\flat)^\sharp$, at least for quasi-coherent $\mathcal{F}$.

Obviously, we must investigate the discrepancy between M and $(M^\sharp)^\flat$. It turns out that the discrepancy is "concentrated in low degrees." Consequently, we need some formal language for isolating low degrees. Let us agree that if P is a property of graded modules, we will say that P_n holds for $n >> 0$ if P_n holds for all $n \geq N$ for some given (large) $N \in \mathbb{N}$.

Definition 7.1 Let M be a graded module over a graded ring B.

(1) We say that M is a *(TN)-module* if $M_d = (0)$ for $d >> 0$.

(2) A map $\varphi\colon M \to N$ of graded modules is a *(TN)-isomorphism (resp. (TN)-injection, (TN)-surjection)* if $\mathrm{Ker}\,(\varphi)$ and $\mathrm{Coker}\,(\varphi)$ are (TN) (resp. $\mathrm{Ker}\,(\varphi)$ is (TN), $\mathrm{Coker}\,(\varphi)$ is (TN)).

(3) M is a *(TF)-module* if M is (TN)-isomorphic to a finitely generated B-module.

(4) B is a *special good graded ring (sggr)* if

   (a) B is a ggr.
   (b) B_1 is a finitely generated B_0-module.
   (c) B_0 is a finitely generated k-algebra, for some field k.

Remarks:

(1) $\mathrm{Proj}(B) = \emptyset$ iff B is a graded (TN) A-algebra.

(2) An sggr is noetherian and $\mathbb{Z}$, while a noetherian ggr is *not* an sggr.

Suppose B is a ggr, M is a graded B-module, and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module (with $X = \mathrm{Proj}(B)$). Then there exist canonical maps

$$\alpha\colon M \to (M^\sharp)^\flat \quad \text{and} \quad \beta\colon (\mathcal{F}^\flat)^\sharp \to \mathcal{F}.$$

First, we construct α as follows: Given $\xi \in M_d$, we have $\xi \in M(d)_0$, and this gives us a global section of $M(d)^\sharp$, i.e., we get $\xi' \in \Gamma(X, M(d)^\sharp)$. However, $M(d)^\sharp = M^\sharp(d)$, and so,

$$\xi' \in \Gamma(X, M^\sharp(d)) = ((M^\sharp)^\flat)_d.$$

Set $\alpha(\xi) = \xi'$.

Next, we construct β by patching consistent maps

$$\Gamma(X_{(f)}, (\mathcal{F}^\flat)^\sharp) \longrightarrow \Gamma(X_{(f)}, \mathcal{F}).$$

Observe that $\Gamma(X_{(f)}, (\mathcal{F}^\flat)^\sharp) = \mathcal{F}^\flat_{(f)}$. Pick $\xi \in \mathcal{F}^\flat_{(f)}$. Then,

$$\xi \in \left\{ \frac{\eta}{f^r} \mid \deg(\eta) = r \right\},$$

and of course, $\eta \in \Gamma(X, \mathcal{F}(r))$. Since f^r is invertible in $X_{(f)}$, the element f^{-r} is in $\Gamma(X_{(f)}, \mathcal{O}_X(-r))$. Then, we have

$$\frac{1}{f^r} \otimes \eta \in \Gamma(X_{(f)}, \mathcal{O}_X(-r)) \otimes \Gamma(X, \mathcal{F}(r)),$$

and since there is a canonical map

$$\Gamma(X_{(f)}, \mathcal{O}_X(-r)) \otimes \Gamma(X, \mathcal{F}(r)) \longrightarrow \Gamma(X_{(f)}, \mathcal{O}_X(-r) \otimes \mathcal{F}(r)) = \Gamma(X_{(f)}, \mathcal{F}),$$

we can check that the map:

$$\xi \mapsto \text{image}\left(\frac{1}{f^r} \otimes \eta\right)$$

is well defined and that these maps patch. This gives us our map β.

To prove that the maps α and β have the properties hinted at above, we need a slight generalization of the propositions concerning extensions of sections from an open set and restrictions of sections to an open set (Theorem 3.8(4)). This is:

Theorem 7.3 *(Section theorem) Let X be a quasi-compact scheme, $\mathcal{L}$ a line bundle on X, $f \in \Gamma(X, \mathcal{L})$, and $\mathcal{F}$ a QC $\mathcal{O}_X$-module. The following properties hold:*

(a) Let $\sigma \in \Gamma(X, \mathcal{F})$ and assume that $\sigma \upharpoonright X_f = 0$. Then, there is some $n > 0$ so that

$$\sigma \otimes f^n = 0 \quad \textit{in} \quad \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}).$$

(b) Assume that X is covered (finitely) by affine opens, U_i, so that

(i) $\mathcal{L} \upharpoonright U_i$ is trivial for all i.

(ii) $U_i \cap U_j$ is a again quasi-compact, for all i, j.

Then, the extension property holds, i.e., given any $\tau \in \Gamma(X_f, \mathcal{F})$, there is some $n > 0$ so that $f^n \otimes \tau \in \Gamma(X_f, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$ extends to a global section in $\Gamma(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$.

Proof. Use the same argument as before in Theorem 3.8(4). $\square$

Observe that the hypotheses of Theorem 7.3 are satisfied if X is quasi-compact and separated, or X is noetherian. Of course, $\mathrm{Proj}(B)$ (with B a ggr) is separated. If B_1 is a finitely generated B_0-module, then $\mathrm{Proj}(B)$ is also quasi-compact. Now, we use Theorem 7.3 to prove the following theorem:

Theorem 7.4 *(Serre) Let $X = \mathrm{Proj}\, B$, where B is a ggr, write $\mathcal{F}$ for a QC $\mathcal{O}_X$-module, and M for a graded (TF) B-module. The following properties hold:*

(1) If B is an sggr, then $\alpha\colon M \to (M^\sharp)^\flat$ is a (TN)-isomorphism.

(2) $\beta\colon (\mathcal{F}^\flat)^\sharp \to \mathcal{F}$ is an isomorphism.

(3) If $B = A[T_0, \ldots, T_N]$, then $\alpha\colon B \to (B^\sharp)^\flat = \mathcal{O}_X^\flat$ is an isomorphism.

Proof.

(1) The proof of (2) follows from the section theorem (Theorem 7.3) and is left to the reader. (Use the section theorem).

(2) ** The proof of (1) needs to be supplied. Is it used in (1)? Is the assumption (TF) necessary? **

(3) Let us prove (3), next.

Look at $(\mathcal{O}_X^\flat)_d = \Gamma(X, \mathcal{O}_X(d))$. Any $\sigma \in \Gamma(X, \mathcal{O}_X(d))$ is a collection of local sections

$$\sigma_i = \sigma \restriction X_{(T_i)} \in \Gamma(X_{(T_i)}, \mathcal{O}_X(d)) = B(d)_{(T_i)}.$$

Thus, we have

$$\sigma_i = \frac{\xi_i}{T_i^r}, \text{ with } \xi_i \in B(d)_r = B_{d+r}.$$

Therefore, σ_i is an element of degree d in B_{T_i}. In summary, if $\sigma \in \mathcal{O}_X^\flat$, then σ is determined by $N+1$ local sections, $\sigma_i \in B_{T_i}$, that fit together on $B_{T_iT_j}$. Since the T_j's are non-zero divisors, localization at T_j gives injections

$$B \longrightarrow B_{T_j} \quad \text{and} \quad B_{T_j} \longrightarrow B_{T_iT_j}.$$

Looking in the ring $B_{T_0\cdots T_N}$, we get

$$\mathcal{O}_X^\flat = \bigcap_{j=0}^N B_{T_i} \quad \text{in} \quad B_{T_0\cdots T_N}.$$

The homogeneous elements of $B_{T_0\cdots T_N}$ are of the form

$$\frac{f(T_0, \ldots, T_N)}{T_0^{a_0} \cdots T_N^{a_N}},$$

where $f(T_0, \ldots, T_N)$ is a form in $T_0, \ldots, T_N$. By factoring powers $T_i^{b_i}$ out of $f(T_0, \ldots, T_N)$, since B is a polynomial ring, we see that each homogeneous element of $B_{T_0\cdots T_N}$ has the unique form

$$T_0^{\alpha_0} \cdots T_N^{\alpha_N} g(T_0, \ldots, T_N), \tag{$*$}$$

where $\alpha_j \in \mathbb{Z}$ and no power of T_j divides g for any j. But $(*)$ shows that our element is in B_{T_j} iff $\alpha_l \geq 0$ for all $l \neq j$. Since this has to hold for every j, we must have $\alpha_j \geq 0$ for $j = 0, \ldots, N$, and so our element is in B. $\square$

Our proof yields the following corollary:

Corollary 7.5 *If M is a graded B-module and B is a ggr with generators $T_0, \ldots, T_N$ for B_1 as a B_0-module, then*

$$(M^\sharp)^\flat = \mathrm{Ker}\left(\prod_{j=0}^{N} M_{T_j} \rightrightarrows \prod_{i,j=0}^{N} M_{T_iT_j}\right).$$

If further, the localization maps

$$M \longrightarrow M_{T_j} \quad \textit{and} \quad M_{T_j} \longrightarrow M_{T_iT_j}$$

are all injective, then

$$(M^\sharp)^\flat = \bigcap_{j=0}^{N} M_{T_i} \quad \textit{in} \quad M_{T_0 \cdots T_N}.$$

Remark: Let M, N be two B-modules. Then, any homomorphism $\varphi\colon M \to N$ gives rise to a homomorphism

$$\varphi(\geq d)\colon \coprod_{t \geq d} M_t \to \coprod_{t \geq d} N_t.$$

Of course, if we are given a homomorphism from $\coprod_{t\geq d} M_t$ to $\coprod_{t\geq d} N_t$, then we can get a homomorphism from $\coprod_{t\geq e} M_t$ to $\coprod_{t\geq e} N_t$ for every $e \geq d$. Consequently, the B-modules $\mathrm{Hom}_B\left(\coprod_{t\geq d} M_t, \coprod_{t\geq d} N_t\right)$ form an inductive mapping system. We define

$$((\mathrm{Hom}))_B(M, N) = \varinjlim \mathrm{Hom}_B\left(\coprod_{t\geq d} M_t, \coprod_{t\geq d} N_t\right).$$

This makes graded B-modules into a new category, and in fact the same ideas can be applied to graded rings. Note that φ is a (TN)-isomorphism when $((\varphi)) \in ((\mathrm{Hom}))_B$ is an isomorphism and M is (TF) iff it is $((\mathrm{Hom}))_B$-isomorphic to a finitely generated B-module. Also, M is (TN) iff it is zero in the new category. Therefore, Serre's result says that for an sggr, the functors

$$M \rightsquigarrow M^\sharp \quad \text{and} \quad \mathcal{F} \rightsquigarrow \mathcal{F}^\flat$$

establish an equivalence of categories between the category of graded finitely generated B-modules with ((Hom))-morphisms and the category of coherent $\mathcal{O}_X$-modules, when $X = \mathrm{Proj}(B)$.

Remark: ** Remark from Steve goes here **

As a consequence of Theorem 7.4 (2), we get the following important fact: Let Y be a closed subscheme of $\mathbb{P}^N_A$ over A. Then, Y is defined by a QC ideal sheaf $\mathfrak{I}_Y$ in $\mathcal{O}_{\mathbb{P}^N_A}$. Now, $B = A[T_0, \ldots, T_N]$ is a ggr. Therefore,

$$\mathfrak{I}_Y(n) = \mathfrak{I}_Y \otimes \mathcal{O}_{\mathbb{P}^N_A}(n)$$

and $\mathcal{O}_{\mathbb{P}^N_A}(n)$ is a line bundle. From the sequence $0 \longrightarrow \mathfrak{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^N_A}$, we get the exact sequence

$$0 \longrightarrow \mathfrak{I}_Y(n) \longrightarrow \mathcal{O}_{\mathbb{P}^N_A}(n),$$

by tensoring with $\mathcal{O}_{\mathbb{P}^N_A}(n)$; thus,

$$0 \longrightarrow \Gamma(X, \mathfrak{I}_Y(n)) \longrightarrow \Gamma(X, \mathcal{O}_{\mathbb{P}^N_A}(n)) \quad \text{is exact for all } n.$$

As a result,

$$0 \longrightarrow \mathfrak{I}_Y^\flat \longrightarrow \mathcal{O}^\flat_{\mathbb{P}^N_A} \quad \text{is exact.}$$

By Theorem 7.4 (3), $\mathcal{O}^\flat_{\mathbb{P}^N_A} = B = A[T_0, \ldots, T_N]$. Therefore, $\mathfrak{I}_Y^\flat = \mathfrak{I}$ is a homogeneous ideal of B and as Y is a scheme over A, the ideal $\mathfrak{I}$ is contained in B^+. Now, $B/\mathfrak{I}$ is a ggr, $B \longrightarrow B/\mathfrak{I}$ is surjective and maps $B_1 \longrightarrow (B/\mathfrak{I})_1$. Thus, we get a map

$$\mathrm{Proj}(B/\mathfrak{I}) \longrightarrow \mathrm{Proj}(B) = \mathbb{P}^N_A,$$

a closed immersion. The ideal of this latter closed subscheme is given by the ideal sheaf

$$\mathfrak{I}^\sharp = (\mathfrak{I}_Y^\flat)^\sharp = \mathfrak{I}_Y,$$

by Theorem 7.4 (2). Therefore, $Y = \mathrm{Proj}(B/\mathfrak{I})$, and this proves both statements of the following proposition:

Proposition 7.6 *Given $\mathbb{P}^N_A$ and a closed A-subscheme, Y, of $\mathbb{P}^N_A$, there exists a homogeneous ideal $\mathfrak{I}$ of $B = A[T_0, \ldots, T_N]$, with $\mathfrak{I} \subseteq B^+$, so that*

$$Y = \mathrm{Proj}(B/\mathfrak{I}).$$

A n.a.s.c. that Y, a scheme over A, be a closed subscheme of $\mathbb{P}^N_A$ for some N is that $Y = \mathrm{Proj}(B)$ for a ggr B (over $A = B_0$) with B_1 a finitely generated A-module.

Recall that

$$\mathbb{P}^N_A = \mathbb{P}^N_{\mathbb{Z}} \prod_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A.$$

For any scheme S, *we define* $\mathbb{P}^N_S$ by:

$$\mathbb{P}^N_S = \mathbb{P}^N_{\mathbb{Z}} \prod_{\mathrm{Spec}\,\mathbb{Z}} S.$$

We have a morphism $\pi \colon \mathbb{P}^N_S \to \mathbb{P}^N_{\mathbb{Z}}$, and we set

$$\mathcal{O}_{\mathbb{P}^N_S}(1) = \pi^* \mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1).$$

When $S = \mathrm{Spec}\, A$, it is clear that this definition of $\mathcal{O}_{\mathbb{P}^N_A}(1)$ agrees with the old definition. When $S = \mathrm{Spec}\, A$ and $A\ (= k)$ is a field we can show that $\mathcal{O}_{\mathbb{P}^N_k}(1)$ is just our old $\mathcal{O}_{\mathbb{P}^N}(1)$, the hyperplane bundle ((DX)—the sections are linear forms). Now, on an arbitrary affine scheme, Spec A, nontrivial line bundles, $\mathcal{L} = \widetilde{L}$, usually exist. If B is a ggr over A, then the coproduct $\coprod_{n \geq 0} B_n \otimes_A L^{\otimes n}$ is a new ggr over A. Therefore, we can form Proj of this new ggr. Concerning the two Proj's, we have

Proposition 7.7 *For any ring, A, and any line bundle, $\mathcal{L}$, over* $\mathrm{Spec}\, A$ *there is a canonical, functorial isomorphism*

$$\Theta_{\mathcal{L}}\colon P' = \mathrm{Proj}(B') = \mathrm{Proj}\left(\coprod_{n\geq 0} B_n \otimes_A L^{\otimes n}\right) \xrightarrow{\sim} \mathrm{Proj}(B) = P.$$

Here, B is a ggr over A and $\mathcal{L}$ is the sheaf corresponding to an invertible A-module, L; also, we have set $B' = \coprod_{n\geq 0} B_n \otimes_A L^{\otimes n}$.

Proof. Pick some $n \geq 0$, and an open affine of Spec A where $\mathcal{L}$ is trivial and call c the generator of L over this affine; consider the map

$$B_n \longrightarrow B_n \otimes_A L^{\otimes n}$$

given by taking each generator of B_1 to c times that generator. Since c is invertible over the given affine open, this map is an isomorphism. Since Proj is formed by gluing the ratios x_i/x_j (the x_i being the generators of B_1), we see that our isomorphism

$$B_n \longrightarrow B_n \otimes_A L^{\otimes n} \quad \text{(mult. by } c\text{)}$$

is independent of c. Hence, we get the desired isomorphism of Proj $\left(\coprod_{n\geq 0} B_n \otimes_A L^{\otimes n}\right)$ to Proj(B). $\square$

Note that under the isomorphism, $\Theta_{\mathcal{L}}$, the "fundamental sheaf", $\mathcal{O}_{P'}(1)$, is exactly $\Theta_{\mathcal{L}}^*(\mathcal{O}_P(1)) \otimes_A \pi'^*(\mathcal{L})$, where π and π' are the respective structure morphisms of P and P' over X.

In keeping with the general program of "schemifying and sheafifying" algebraic objects such as modules and rings (instituted in all generality by Grothendieck [EGA]), we need to generalize the notion of Proj(B), so that we can replace $\mathrm{Spec}\, A$ by any scheme. Take X to be some scheme and call a sheaf of $\mathcal{O}_X$-algebras, $\mathcal{B}$, a *quasi-coherent graded $\mathcal{O}_X$-algebra* iff locally (over open affine U) on X, the sheaf $\mathcal{B}$ is isomorphic to a graded A-algebra, where A is the ring of global sections of $\mathcal{O}_X \restriction U$. For such $\mathcal{O}_X$-algebras, $\mathcal{B}$, we can perform the construction of Proj on a covering family of open affines, U_α, of X and they glue together to give us a scheme over X which we call Proj$(\mathcal{B})$. Of course, Proj$(\mathcal{B})$ comes with a fundamental sheaf, $\mathcal{O}_P(1)$, where we have denoted Proj$(\mathcal{B})$ by P. Locally over $\pi^{-1}(U)$, our sheaf $\mathcal{O}_P(1)$ is just the old $\mathcal{O}(1)$ constructed for an affine base, U. Also, we have the notion of good graded $\mathcal{O}_X$-algebra, again denoted ggr, and this means that $\mathcal{B}$ is generated by the $\mathcal{O}_X$-module $\mathcal{B}_1$ as an $\mathcal{O}_X$-algebra. Equivalently, it means that $\mathcal{B}$ is the image of $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{B}_1)$ under the canonical map

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{B}_1) \longrightarrow \mathcal{B}.$$

In particular, if $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$*-module*, we can take $\mathcal{B}$ to be $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})$. The resulting Proj is denoted $\mathbb{P}(\mathcal{E})$ and is called the *projective fibre scheme over X corresponding to $\mathcal{E}$.*

We must now generalize the Serre functors $\sharp$ and $\flat$. For $\sharp$, which is from graded $\mathcal{B}$-modules to sheaves on $\mathrm{Proj}(\mathcal{B})$, we merely restrict our graded module, $\mathcal{M}$, to $\pi^{-1}(U)$, where U is affine open in X, and perform the $\sharp$ on this restriction thereby obtaining $(\mathcal{M} \restriction \pi^{-1}(U))^\sharp$, a sheaf on $\mathrm{Proj}_U(\mathcal{B} \restriction U)$. Then, we observe that these glue together to give the quasi-coherent $\mathcal{O}_P$-module, $\mathcal{M}^\sharp$; where, as usual, $P = \mathrm{Proj}(\mathcal{B})$.

Again, as usual, we will restrict to the case that $\mathcal{B}$ is a ggr as $\mathcal{O}_X$-algebra. In this case, Proposition 7.2 generalizes to the following (of course, if $\mathcal{B} = \coprod_{n\geq 0} \mathcal{B}_n$, then $\mathcal{B}(q) = \coprod_{n\in\mathbb{Z}} \mathcal{B}_{q+n}$):

Proposition 7.8 *For a base scheme, X, assume $\mathcal{B}$ is a ggr as $\mathcal{O}_X$-algebra. Then, if P denotes* $\mathrm{Proj}(\mathcal{B})$*, we have:*

(1) $\mathcal{O}_P(q) = (\mathcal{B}(q))^\sharp$ is a line bundle on P.

(2) If $\mathcal{M}$ and $\mathcal{N}$ are graded $\mathcal{B}$-modules, then $(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})^\sharp = \mathcal{M}^\sharp \otimes_{\mathcal{O}_P} \mathcal{N}^\sharp$.

(3) $(\mathcal{M}(q))^\sharp = \mathcal{M}^\sharp(q) = \mathcal{M}^\sharp \otimes_{\mathcal{O}_P} \mathcal{O}_P(q)$.

(4) $\mathcal{O}_P(q+q') = \mathcal{O}_P(q) \otimes_{\mathcal{O}_P} \mathcal{O}_P(q')$.

(5) If $\mathcal{C}$ is another $\mathcal{O}_X$-ggr and if $\theta\colon \mathcal{C} \to \mathcal{B}$ is a graded homomorphism (preserves degrees), then, there is a canonical open set, $\mathcal{G}(\theta)$, contained in $P =$ $\mathrm{Proj}(\mathcal{B})$ and a morphism of schemes
$$(\mathcal{G}(\theta), \mathcal{O}_P \restriction \mathcal{G}(\theta)) \xrightarrow{\Theta} Q = \mathrm{Proj}(\mathcal{C}),$$
and $\mathcal{G}(\theta)$ is the maximal open for such a morphism. Further,
$$\Theta_*(\mathcal{O}_{\mathcal{G}(\theta)}(q)) = (\Theta_*\mathcal{O}_{\mathcal{G}(\theta)})(q).$$

(6) $\Theta^*(\mathcal{O}_Q(q)) = \mathcal{O}_P(q) \restriction \mathcal{G}(\theta) = \mathcal{O}_{\mathcal{G}(\theta)}(q)$.

The proofs are obtained simply by covering the base, X, by open affines and applying Proposition 7.2 for these affines.

For the functor, $\flat$, we take a QC $\mathcal{O}_P$-module, $\mathcal{F}$, and twist it by $\mathcal{O}(q)$ for each $q \in \mathbb{Z}$. Now recall that the analog of the global sections functor of affine geometry is the direct image functor for general geometry. Hence, we set
$$\mathcal{F}^\flat = \coprod_{n\in\mathbb{Z}} \pi_*(\mathcal{F}(q)),$$
where π is the structure morphism $P \longrightarrow X$ and we have written $\mathcal{F}(q)$ for $\mathcal{F} \otimes \mathcal{O}_P(q)$, as in the proposition above. The sheaf, $\mathcal{F}^\flat$, is a graded $\mathcal{B}$-module, because the action of $\mathcal{B}$ on it can be defined on the various open sets of the form $\pi^{-1}(U)$, where U is affine open in X. Moreover, $\mathcal{F}^\flat$ is quasi-coherent as $\mathcal{O}_X$-algebra.

Just as in the discussion following Proposition 7.2, we have the finiteness notions of (TF) and (TN). Further, globalizing the Serre construction of the maps α and β by the obvious patching, we deduce the existence of canonical maps

$$\alpha\colon \mathcal{M} \mapsto (\mathcal{M}^\sharp)^\flat \quad \text{and} \quad \beta\colon (\mathcal{F}^\flat)^\sharp \mapsto \mathcal{F}.$$

(Of course we always assume $\mathcal{B}$ is a ggr over $\mathcal{O}_X$.) And, again from Theorem 7.3, we deduce the following globalization of Serre's theorem (Theorem 7.4):

Theorem 7.9 *Write $P = \mathrm{Proj}(\mathcal{B})$, where $\mathcal{B}$ is a ggr over $\mathcal{O}_X$ (X a scheme) and let $\mathcal{F}$ be a QC $\mathcal{O}_P$-module and $\mathcal{M}$ a graded (TF) $\mathcal{B}$-module. Then we have:*

(1) If $\mathcal{B}$ is an sggr over $\mathcal{O}_X$ (this just means that $\mathcal{B}_1$ is finitely generated as $\mathcal{B}_0$-module and that $\mathcal{B}_0$ is a coherent $\mathcal{O}_X$-module while X is a finite type k-scheme with k a field), then the map $\alpha\colon \mathcal{M} \to (\mathcal{M}^\sharp)^\flat$ is a (TN)-isomorphism.

(2) The map $\beta\colon (\mathcal{F}^\flat)^\sharp \to \mathcal{F}$ is an isomorphism.

(3) If $\mathcal{B} = \mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[T_0, \ldots, T_N]$, then $\alpha\colon \mathcal{B} \to (\mathcal{B}^\sharp)^\flat = \mathcal{O}_P^\flat$ is an isomorphism.

Remark: If $\mathcal{B}$ is $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[T_0, \ldots, T_N]$, then $\mathrm{Proj}(\mathcal{B})$ is just $\mathbb{P}^N_X$.

7.2 Projective Fibre Bundles

Let us now restrict ourselves, momentarily, to the case that $X = \mathrm{Spec}\, A$, and let E be an A-module. Write $\mathcal{E}$ for the QC sheaf, $\widetilde{E}$, on X. Consider $P = \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_A(E))$. Of course, $E =$ degree one components of $\mathrm{Sym}_A(E)$. If $\mathcal{S}$ is any graded QC $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module, then we know $\pi_*(\mathcal{S}) = \Gamma(\widetilde{\mathbb{P}(\mathcal{E}), \mathcal{S}})$ and we have seen that in such a situation there is a map

$$\mathcal{S}_0 \longrightarrow \pi_*(\mathcal{S}) = \Gamma(\widetilde{\mathbb{P}(\mathcal{E}), \mathcal{S}}).$$

Apply this to the case where $\mathcal{S} = \widetilde{\mathrm{Sym}_A}(E)(1)$; we get

$$\mathcal{E} = \widetilde{E} \longrightarrow \pi_*(\widetilde{\mathrm{Sym}_A}(E)(1)) = \pi_*(\mathcal{O}_P(1)).$$

That is, we obtain a map

$$\pi_P^*(\mathcal{E}) \longrightarrow \mathcal{O}_P(1).$$

However, the latter map is surjective because it corresponds to the map

$$E \otimes \mathrm{Sym}_A(E) \longrightarrow \mathrm{Sym}_A(E)(1)$$

and $\mathrm{Sym}_A(E)$ is a ggr.

Now, consider an A-morphism from an A-scheme, T, into $\mathbb{P}(\mathcal{E})$; call it φ. From the exact sequence

$$\pi_P^*(\mathcal{E}) \longrightarrow \mathcal{O}_P(1) \longrightarrow 0,$$

we obtain the pullback exact sequence

$$\pi_T^*(\mathcal{E}) = \varphi^*(\pi_P^*(\mathcal{E})) \longrightarrow \varphi^*(\mathcal{O}_P(1)) \longrightarrow 0.$$

Let us write $\mathcal{L}$ for $\varphi^*(\mathcal{O}_P(1))$.

We can be more explicit with this map: Suppose $f \in E$ is considered as a generator from $\mathrm{Sym}_A(E)_1$. Then, we can localize $\mathbb{P}(\mathcal{E})$ at f; that is, form $\mathbb{P}(\mathcal{E})_{(f)}$, and this is merely $\mathrm{Spec}(S_{(f)})$ where S stands for $\mathrm{Sym}_A(E)$. Take an open affine, U, in $\varphi^{-1}(\mathbb{P}(\mathcal{E})_{(f)})$, say $U = \mathrm{Spec}(B)$. Then, our map, φ, restricted to U just comes from an A-algebra map $*\colon S_{(f)} \to B$; and the sheaf $\varphi^*(\mathcal{E}) \restriction U$ is just $\widetilde{E \otimes_A B}$. As for $\mathcal{L}$, when we restrict to U it is just $\widetilde{S(1)_{(f)} \otimes_{S_{(f)}} B}$ (remember, B is an $S_{(f)}$-algebra by $(*)$). And now, our map (over U) is induced by a map

$$E \otimes_A B \longrightarrow S(1)_{(f)} \otimes_{S_{(f)}} B$$

and the latter is given by

$$x \otimes 1 \mapsto \frac{f}{1} \otimes * \left(\frac{x}{f}\right). \tag{$\dagger$}$$

Our map $\pi_T^*(\mathcal{E}) \longrightarrow \mathcal{L} \longrightarrow 0$ yields a surjection

$$\mathrm{Sym}_{\mathcal{O}_T}(\pi_T^*(\mathcal{E})) \longrightarrow \mathrm{Sym}_{\mathcal{O}_T}(\mathcal{L})$$

and

$$\pi_T^*(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})) = \mathrm{Sym}_{\mathcal{O}_T}(\pi_T^*(\mathcal{E})),$$

hence we obtain the surjection

$$\pi_T^*(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})) \longrightarrow \coprod_{n \geq 0} \mathcal{L}^{\otimes n}. \tag{$\dagger\dagger$}$$

Once again, we can be more explicit: On affine patches as above, the lefthand member is simply $S_n(E) \otimes_A B$ $(= S_n(E \otimes_A B))$, while the righthand member is $(S(1)_{(f)} \otimes_{S_{(f)}} B)^{\otimes n} = (S_n)_{(f)} \otimes_A B$. The map between the two is just the map induced by $(\dagger)$ on the n-fold symmetric powers, namely

$$s \otimes 1 \mapsto \left(\frac{f}{1}\right)^{\otimes n} \otimes * \left(\frac{s}{f^n}\right).$$

To recapitulate, a T-point of $P = \mathbb{P}(\mathcal{E})$ yields a line bundle, $\mathcal{L}$, on T and a surjection

$$\pi_T^*(\mathcal{E}) \longrightarrow \mathcal{L} \longrightarrow 0. \tag{$\dagger\dagger\dagger$}$$

Conversely, given such a line bundle, $\mathcal{L}$, and surjection (†††), we obtain the surjection (††). Because it is a surjection, we obtain the morphism

$$\mathrm{Proj}(\coprod_{n\geq 0}\mathcal{L}^{\otimes n}) \longrightarrow \mathrm{Proj}(\pi_T^*(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}))) = T \prod_{\mathrm{Spec}(A)} \mathbb{P}(\mathcal{E}) \xrightarrow{pr_2} \mathbb{P}(\mathcal{E}).$$

But $T \cong \mathrm{Proj}(\coprod_{n\geq 0}\mathcal{L}^{\otimes n})$, and so, we finally get a T-point,

$$\varphi_{\mathcal{L}} \colon T = \ (\mathrm{Proj}(\coprod_{n\geq 0}\mathcal{L}^{\otimes n})) \to \mathbb{P}(\mathcal{E}),$$

of P by composition.

Our discussion above has proved most of

Theorem 7.10 *If X is a scheme and $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-module, then for any X-scheme, T, the points, φ, of $P = \mathbb{P}(\mathcal{E})$ with values in T are in one-to-one correspondence with equivalence classes of pairs, $(\mathcal{L}, \psi)$ in which $\mathcal{L}$ is a line bundle over T and ψ is a* **surjection** *of $\pi_T^*(\mathcal{E})$ to $\mathcal{L}$. The equivalence relation on pairs is: $(\mathcal{L}, \psi) \sim (\mathcal{L}', \psi')$ iff there exists an $\mathcal{O}_T$-isomorphism, α, of $\mathcal{L}$ to $\mathcal{L}'$ rendering the diagram*

$$\begin{tikzcd} & \mathcal{L} \arrow[dd, "\alpha"] \\ \pi_T^*(\mathcal{E}) \arrow[ur, "\psi"] \arrow[dr, "\psi'"'] & \\ & \mathcal{L}' \end{tikzcd}$$

commutative.

Proof. First of all, by the usual gluing on affines, we may assume that X is $\mathrm{Spec}\, A$. (The gluing is most easily seen by using the explicit form of the morphisms given above.) In this case, all that remains is to show that equivalent pairs give the same morphism $T \longrightarrow \mathbb{P}(\mathcal{E})$ and conversely. If we have the commutative diagram

$$\begin{tikzcd} & \mathcal{L} \arrow[dd, "\alpha"] \\ \pi_T^*(\mathcal{E}) \arrow[ur, "\psi"] \arrow[dr, "\psi'"'] & \\ & \mathcal{L}' \end{tikzcd}$$

then we get the commutative diagram of graded algebras

$$\begin{tikzcd} & \coprod_{n\geq 0}\mathcal{L}^{\otimes n} \arrow[dd, "\alpha"] \\ \mathrm{Sym}_{\mathcal{O}_T}(\pi_T^*(\mathcal{E})) \arrow[ur, "\psi"] \arrow[dr, "\psi'"'] & \\ & \coprod_{n\geq 0}\mathcal{L}'^{\otimes n}. \end{tikzcd}$$

Now, taking Proj of the latter diagram, we obtain the same morphism of $T \longrightarrow \mathbb{P}(\mathcal{E})$ from either $(\mathcal{L}, \psi)$ or $(\mathcal{L}', \psi')$, because T is identified with $\mathrm{Proj}(\coprod_{n\geq 0}\mathcal{L}^{\otimes n})$ (resp. $\mathrm{Proj}(\coprod_{n\geq 0}\mathcal{L}'^{\otimes n})$) and these identifications agree *via* $\mathrm{Proj}(\alpha)$.

** Proof of the converse needs to be written **

Remark: The manipulations above correspond to the dow–to–earth statement that, in ordinary projective space over a field, the points $(z_0\colon \cdots : z_n)$ and $(\lambda z_0\colon \cdots : \lambda z_n)$ are the same. More precision in this remark will be given below.

There are several important special cases of Theorem 7.10. The first is

Corollary 7.11 *If X is a scheme and $\mathcal{E}$ is a QC $\mathcal{O}_X$-module, then the X-points of $P = \mathbb{P}(\mathcal{E})$ are in one-to-one canonical correspondence with the QC submodules, $\mathcal{F}$, of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is an invertible sheaf.*

Proof. By the theorem, the X-points of P are in one-to-one correspondnce with classes of pairs, $(\mathcal{L}, \psi)$, in which ψ is a surjection from $\mathcal{E}$ to $\mathcal{L}$. Let $\mathcal{F} = \mathrm{Ker}\,\psi$. Note that when $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are equivalent, we get the same $\mathcal{F}$. Conversely, given $\mathcal{F}$, we can use an automorphism of $\mathcal{E}$ mapping $\mathcal{F}$ to itself and the two quotients, $\mathcal{L}, \mathcal{L}'$ are then equivalent. $\square$

The second case is when $\mathcal{E} = \mathcal{O}^{N+1}$. In this case, $\mathbb{P}(\mathcal{E})$ is just $\mathbb{P}^N_X$. Theorem 7.10 yields

Corollary 7.12 *The T-points of $\mathbb{P}^N_X$ are in one-to-one correspondence with pairs, $(\mathcal{L}, \psi)$, in which $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module and ψ is a surjection from $\mathcal{O}_T^{N+1}$ to $\mathcal{L}$.*

Hence, $\mathbb{P}^N_X$ represents the functor

$$T \rightsquigarrow \{(\mathcal{L}, \psi) \mid \mathcal{L} \in \mathrm{Pic}(T) \quad \text{and} \quad \psi\colon \mathcal{O}_T^{N+1} \longrightarrow \mathcal{L} \longrightarrow 0 \quad \text{is exact}\}.$$

For many purposes, the formulation of Corollary 7.11 is more convenient. To formulate it a bit better, introduce the following notation for an X-scheme, T:

$$\mathrm{Hyp}_X(T, \mathcal{E}) = \left\{ \mathcal{F} \;\middle|\; \begin{array}{l} (1)\ \mathcal{F} \text{ is a QC } \mathcal{O}_T\text{-submodule of } \pi_T^*(\mathcal{E}). \\ (2)\ \pi_T^*(\mathcal{E})/\mathcal{F} \text{ is an invertible } \mathcal{O}_T\text{-module.} \end{array} \right\}$$

Fix $\mathcal{E}$ and let T vary over the X-schemes. Say S is another X-scheme and $S \xrightarrow{\beta} T$ is an X-morphism. We have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \pi_T^*(\mathcal{E}) \longrightarrow \mathcal{L} \longrightarrow 0,$$

where $\mathcal{L}$ is invertible. Since β^* is right-exact, we get the exact sequence

$$\beta^*(\mathcal{F}) \longrightarrow \beta^*(\pi_T^*(\mathcal{E})) = \pi_S^*(\mathcal{E}) \longrightarrow \beta^*(\mathcal{L}) \longrightarrow 0.$$

But, pullback of an invertible sheaf is invertible, and so, the map $\mathcal{F} \rightsquigarrow \mathrm{Im}\,(\beta^*(\mathcal{F}) \longrightarrow \pi_S^*(\mathcal{E}))$ gives a map $\mathrm{Hyp}_X(T, \mathcal{E}) \longrightarrow \mathrm{Hyp}_X(S, \mathcal{E})$. Clearly, this shows that $T \rightsquigarrow \mathrm{Hyp}_X(T, \mathcal{E})$ is a cofunctor from X-schemes to $\mathcal{S}$ets. Corollary 7.11 can now be reformulated as

Corollary 7.13 *The scheme $P = \mathbb{P}(\mathcal{E})$ and its invertible sheaf $\mathcal{O}_P(1)$ represent the functor* $\mathrm{Hyp}_X(-, \mathcal{E})$.

Notice that the fixing of the invertible sheaf, $\mathcal{O}_P(1)$, removes the ambiguity of the equivalence relation mentioned in Theorem 7.10. In fact, we can again reformulate Corollary 7.13 as follows:

Corollary 7.14 *There is a one-to-one correspondence between invertible $\mathcal{O}_T$-modules, $\mathcal{L}$, which are quotients of $\pi_T^*(\mathcal{E})$ and morphisms, $\theta\colon T \to P = \mathbb{P}(\mathcal{E})$, so that $\theta^*(\mathcal{O}_P(1)) = \mathcal{L}$. (Similarly, of course, when $\mathcal{E}$ is $\mathcal{O}_X^{N+1}$ and $P = \mathbb{P}_X^N$.)*

Theorem 7.10 shows that for a scheme, X, the projective space $\mathbb{P}_X^N$ is not defined as one might have imagined. Instead, it is defined via line bundles and global sections. The reason for this is that on the base scheme X there are nontrivial line bundles and these are hidden when we just have $X = \mathrm{Spec}\, k$, where k is a field. We now have two definitions of $\mathbb{P}_X^N$ as a functor in the case that $X = \mathrm{Spec}\, k$ with k a field. Do they agree? Our map above (locally given by (†)) is just the map

$$t \in X \mapsto (s_0(t) \colon \cdots \colon s_N(t)) \in \mathbb{P}_k^N \quad \text{(old definition)}$$

and so, in fact, they do agree. This explicit form of our map is used all the time.

But we can be a little more general, yet. All that is necessary is that the line bundles on X all be trivial. For example, if X is Spec(local ring). When this happens, we have the more ordinary description of the X-points of $\mathbb{P}(\mathcal{E})$:

Corollary 7.15 *Assume that* $\mathrm{Pic}(X)$ *is trivial. Write $\widetilde{H}$ for the subset of the $\Gamma(X, \mathcal{O}_X)$-module $\Gamma(X, \mathcal{E}^D)$ corresponding to those homomorphisms $\mathcal{E} \longrightarrow \mathcal{O}_X$ which are surjective. Then, the X-points of $\mathbb{P}(\mathcal{E})$ are in one-to-one correspondence with $\widetilde{H}/\mathbb{G}_m(\Gamma(X, \mathcal{O}_X))$.*

Proof. Since $\Gamma(X, \mathcal{E}^D)$ is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, the elements of $\widetilde{H}$ correspond to surjections

$$\mathcal{E} \xrightarrow{\psi} \mathcal{O}_X \longrightarrow 0.$$

Further, since $\mathrm{Pic}(X)$ is trivial, these ψ just correspond to the pairs $(\mathcal{O}_X, \psi)$ mentioned in Theorem 7.10. The equivalence relation of the theorem is just multiplication by an element of $\mathbb{G}_m(\Gamma(X, \mathcal{O}_X))$, because these invertible sections are the isomorphisms $\mathcal{O}_X \longrightarrow \mathcal{O}_X$ making the diagram

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dd] \\ \mathcal{E} \arrow[ur, "\psi"] \arrow[dr, "\psi'"'] & \\ & \mathcal{O}_X \end{tikzcd}$$

commute. $\square$

Corollary 7.15 supplies the precise form of the remark at the end of the proof of Theorem 7.10. As a final remark in this chain of results, note that the special case of Corollary 7.11 actually implies the full content of Theorem 7.10. To see this, note that the X-morphisms $T \longrightarrow P = \mathbb{P}(\mathcal{E})$ correspond uniquely to their graphs in $T \prod_X P$. The latter are just the T-points of $T \prod_X P$. But, $T \prod_X P$ is just $\mathbb{P}(\pi_T^*(\mathcal{E}))$. According to Corollary 7.11, the T-points of $\mathbb{P}(\pi_T^*(\mathcal{E}))$ are in one-to-one correspondence with the QC $\mathcal{O}_T$-submodules of $\pi_T^*(\mathcal{E})$ whose quotients are invertible. That is, the points of $\mathbb{P}(\pi_T^*(\mathcal{E}))$ correspond to exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \pi_T^*(\mathcal{E}) \longrightarrow \mathcal{L} \longrightarrow 0 \tag{$*$}$$

where $\mathcal{F}$ is a submodule of $\pi_T^*(\mathcal{E})$ and $\mathcal{L}$ is invertible. But, this is just the description of the T-points of $\mathbb{P}(\mathcal{E})$ as in the conclusion of Theorem 7.10, because the equivalence relation corresponds to keeping $\mathcal{F}$ fixed and identifying all the $\mathcal{L}$'s one can get from an exact sequence $(*)$.

The construction of the projective fibre space, $\mathbb{P}(\mathcal{E})$, gives a good mechanism in which to view the generalization of the Segre morphism of Chapter 2. First, observe that if we have two QC $\mathcal{O}_X$-modules, $\mathcal{E}, \mathcal{F}$, and a surjection from $\mathcal{E}$ to $\mathcal{F}$, then we get an obvious closed immersion $\mathcal{E} \longrightarrow \mathcal{F}$ and, as mentioned above, base extension of $\mathbb{P}(\mathcal{E})$, say by $\pi_T \colon T \to X$, just gives $\mathbb{P}(\pi_T^*(\mathcal{E}))$. For the Segre morphism, we have:

Proposition 7.16 *There is a natural closed immersion*

$$\mathbb{P}_X(\mathcal{E}) \prod_X \mathbb{P}_X(\mathcal{F}) \longrightarrow \mathbb{P}_X(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

This is the Segre morphism.

Proof. Let P_1 be $\mathbb{P}_X(\mathcal{E})$ and P_2 be $\mathbb{P}_X(\mathcal{F})$, and write Q for $P_1 \prod_X P_2$. Then, on Q, we have the invertible $\mathcal{O}_Q$-module:

$$\mathcal{O}_{P_1}(1) \otimes_X \mathcal{O}_{P_2}(1) = pr_1^*(\mathcal{O}_{P_1}(1)) \otimes_{\mathcal{O}_Q} pr_2^*(\mathcal{O}_{P_2}(1)).$$

This will play the role of $\mathcal{O}_Q(1)$. Now, we know there are surjective homomorphisms:

$$\pi_1^*(\mathcal{E}) \longrightarrow \mathcal{O}_{P_1}(1), \quad \pi_2^*(\mathcal{F}) \longrightarrow \mathcal{O}_{P_2}(1),$$

where, π_i is the structure morphism, $P_i \longrightarrow X$. Thus, we get a surjection

$$q^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \longrightarrow \mathcal{O}_Q(1), \tag{$*$}$$

where q is the structure morphism of Q over X. But, a surjection from $q^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$ to a line bundle on Q (in this case, $\mathcal{O}_Q(1)$), is exactly a morphism from Q to $\mathbb{P}(\mathcal{E} \otimes \mathcal{F})$, by Theorem 7.10. This is the Segre morphism. What we must show is that it is a closed immersion. However, this is a local question on X; so, we may and do assume that X is affine–call it Spec A.

The modules $\mathcal{E}$ and $\mathcal{F}$ are then $\widetilde{E}$ and $\widetilde{F}$, for some A-modules E and F. To check that the Segre morphism in this case is a closed immersion we will compute it on suitable affine opens of Q. Pick $e \in E$ and $f \in F$ and look at the affine open, $U = P_{1(e)} \prod_X P_{2(f)}$. This is Spec of a ring, and that ring is

$$(\mathrm{Sym}E)_{(e)} \otimes_A (\mathrm{Sym}F)_{(f)},$$

which we will denote by B. Now, our line bundle, $\mathcal{O}_Q(1)$ is merely the sheaf given by the module $(\mathrm{Sym}E)_{(e)} \otimes_A (\mathrm{Sym}F)_{(f)}$ and over our affine open it is generated by $\frac{e}{1} \otimes \frac{f}{1}$. Then, the surjection $(*)$ corresponds to the map

$$x \otimes y \otimes b \mapsto b\left(\frac{x}{1} \otimes \frac{y}{1}\right),$$

where x and y are elements of E, F, respectively and b is an element of B. Note that $\frac{x}{1}$ is indeed the ratio of elements of degree 0 and similarly for $\frac{y}{1}$. Then, for the symmetric algebras the maps one degree-one pieces is just

$$x \otimes y \mapsto \left(\frac{x}{e}\right) \otimes \left(\frac{y}{f}\right).$$

In $\mathbb{P}(\mathcal{E} \otimes \mathcal{F})$ the corresponding affine open into which the Segre morphism maps U is $\mathbb{P}(E \otimes F)_{(e \otimes f)}$. And on the algebras, this map is given by

$$(x \otimes y)_{(e\otimes f)} \mapsto \left(\frac{x}{e}\right) \otimes \left(\frac{y}{f}\right). \tag{$\dagger$}$$

Since the open sets $\mathbb{P}(E \otimes F)_{(e\otimes f)}$ cover $\mathbb{P}(E \otimes F)$ because the $e \otimes f$ generate $\mathrm{Sym}_A(E \otimes F)$, all we need to prove is that the Segre morphism is a closed immersion on these particular affines. But, that will happen if $(\dagger)$ is surjective, and this is certainly true. $\square$

Remarks:

(1) The reader should check that the Segre morphism commutes with base extension.

(2) What about the coproduct of $\mathcal{E}$ and $\mathcal{F}$ as modules? We find that the coproduct of the schemes $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{F})$ has a closed immersion into $\mathbb{P}(\mathcal{E}\coprod\mathcal{F})$.

After the multitude of abstractions of this section, it will be refreshing to have a concrete and constructive example. We will restrict ourselves to the case where $X = \mathrm{Spec}(k)$ where k is a field and $\mathbb{P}(\mathcal{E})$ is just $\mathbb{P}^N$.

Example 7.2 *Linear systems; hypersurface embedding.*

(1) Take $Z = \mathbb{P}^n_k$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(q)$ with $q > 0$. A basis for the global sections of $\mathcal{L}$ is the set of monomials $M_0, \ldots, M_N$ of degree q in $n+1$ variables, and

$$N = \binom{n+q}{q} - 1.$$

According to (††), we have a map

$$q\text{−uple}\colon \mathbb{P}^n_k \to \mathbb{P}^N_k$$

given by

$$(X_0\colon \cdots \colon X_n) \mapsto (M_0(X)\colon \cdots \colon M_N(X)), \quad \text{where } X = (X_0, \ldots, X_n).$$

This is the *q-uple embedding*. We know that $(q\text{−uple})^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(q)$. This shows that hyperplanes in $\mathbb{P}^N_k$ (zeros of sections of $\mathcal{O}_{\mathbb{P}^N}(1)$) correspond to the zeros of sections of $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(q)$. The latter are hypersurfaces of degree q in $\mathbb{P}^n_k$. Hence, our map, q−uple, "straightens out" hyper*surfaces* of degree q in $\mathbb{P}^n_k$ and makes them into hyper*planes* in $\mathbb{P}^N_k$.

Recall that if $\mathcal{L} \in \mathrm{Pic}(Z)$ and σ is one of its global sections then we get a Cartier divisor, $Z(\sigma)$, so that $\mathcal{O}_Z(Z(\sigma)) \cong \mathcal{L}$ (cf. Proposition 5.31). In our case, when we have a morphism $\theta\colon Z \to \mathbb{P}^N_S$, it corresponds to $N+1$ sections of a line bundle, $\mathcal{L}$, on Z (so that the $N+1$ sections generate everywhere). Consequently, we have $N+1$ Cartier divisors $Z(s_0), \ldots, Z(s_N)$, and the condition that $s_0, \ldots, s_N$ generate $\mathcal{L}$ is exactly that no point of Z is in the intersection of these divisors. Of course, we recall also that two divisors D and E are linearly equivalent when and only when their line bundles $\mathcal{O}_Z(D)$ and $\mathcal{O}_Z(E)$ are isomorphic. So, in the present situation, all the Cartier divisors $Z(s_i)$, $Z(s_j)$ are linearly equivalent. The same is true for $Z(s)$, where s is any linear combination of $s_1, \ldots, s_N$. This means that what we have is a family of linearly equivalent Cartier divisors on Z. The fact that: Given a point $t \in Z$, at least one divisor from our family does not pass through t (which is the condition that the morphism $\theta\colon Z \to \mathbb{P}^N_S$ is everywhere defined) has a special name. For a given family of linearly equivalent divisors on Z, say $\mathcal{F}$, a point t in Z is a *base point* of $\mathcal{F}$ if all the divisors of $\mathcal{F}$ pass through t. Thus, θ is everywhere defined iff our family, $\mathcal{F}$, has no base points.

The collection of sections of $\mathcal{L}$ spanned by $s_0, \ldots, s_N$ is a linear space, V. The corresponding projective space (i.e., the space of hyperplanes of our linear space) is just our family, $\mathcal{F}$, of linearly equivalent divisors. Consequently, $\mathbb{P}^N_k$ is best written $\mathbb{P}_k(V)$, and our map θ takes Z to $\mathbb{P}_k(V)$. Each divisor from $\mathcal{F}$ corresponds to a hyperplane in $\mathbb{P}_k(V)$. So for each hyperplane, H, of $\mathbb{P}_k(V)$, the divisor $\theta^* H$ is an effective Cartier divisor on Z lying in $\mathcal{F}$.

Let C be a curve of degree d in $\mathbb{P}^n$, and let $C(q)$ be the image of C under the q-uple embedding. Write $\mathcal{H}$ for a hyperplane in $\mathbb{P}^N$, and look at $\mathcal{H} \cdot C(q)$. This corresponds to the divisor on $C(q)$ corresponding to the line bundle $\mathcal{O}_{\mathbb{P}^N}(1) \restriction C(q)$. But then, $(q\text{−uple})^*(\mathcal{H} \cdot C(q))$ is a divisor of the line bundle $(q\text{−uple})^*(\mathcal{O}_{\mathbb{P}^N}(1) \restriction C)$; that is, it is equal to the divisor of $\mathcal{O}_{\mathbb{P}^n}(q) \restriction C$, which is just $qH \cdot C$. Here, H is a hyperplane in $\mathbb{P}^n$. Thus,

$$\#(\mathcal{H} \cdot C(q)) = q(\#(H \cdot C)) = qd,$$

and

$$\deg(C(q)) = qd = q \deg(C).$$

(2) Take $Z = \mathbb{P}^n_k$, again, and $\mathcal{L}$ a line bundle on Z. However, don't take enough sections to generate. What happens?

Say, we take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(1)$ and use the sections $s_0, \ldots, s_{n-1}$. Then, the set

$$\widetilde{Z} = \{t \in T \mid s_j(t) = 0,\ 0 \leq j \leq n-1\}$$

is a closed set where θ is not defined, namely the set consisting of the point

$$P = (0\colon \cdots : 0\colon 1).$$

Then, the open set $U = Z - \widetilde{Z}$ is just $\mathbb{P}^n_k - \{P\}$, and θ is defined on U and given by

$$(X_0\colon \cdots : X_n) \mapsto (X_0\colon \cdots : X_{n-1}).$$

So, we see that $\theta\colon (\mathbb{P}^n_k - \{P\}) \to \mathbb{P}^{n-1}_k$ is simply the projection from P.

Generally, $\widetilde{Z} \neq \emptyset$, and if $(s_0, \ldots, s_n)$ are chosen in $\Gamma(Z, \mathcal{L})$ but don't generate, then $\mathrm{codim}(\widetilde{Z})$ is $n+1$, in general, and we get the morphism

$$\theta_{\mathcal{L}\restriction U}\colon U \to \mathbb{P}(W((s))),$$

where $W((s))$ is the subspace of $\Gamma(T, \mathcal{L})$ generated by $(s) = (s_0, \ldots, s_n)$. It follows that our line bundle, $\mathcal{L}$, and the "inadequate" collection of its sections $(s_0, \ldots, s_n)$ gives us a rational map $\theta\colon Z \dashrightarrow \mathbb{P}(W((s)))$. So we see that a linear system with base points gives rise to a rational map from Z into the appropriate projective space.

Remember that given a line bundle, $\mathcal{L}$, the sections of $\mathcal{L}$ give effective divisors, all equivalent to one another. Given a subspace, $V \subseteq \Gamma(X, \mathcal{L})$, if $s_0, \ldots, s_N$ form a basis of V, the map $\varphi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}(V)$ given by

$$\xi \mapsto (s_0(\xi) : \cdots : s_N(\xi)),$$

which, we shall assume to be a morphism, sends the effective divisors, D, with $\mathcal{O}_X(D) = \mathcal{L}$, to hyperplanes in V. Each effective divisor goes to a distinct hyperplane, all the hyperplanes are covered; when the divisor is given by the linear combination $\sum_{j=0}^{N} \alpha_j s_j$, the corresponding hyperplane is just given by the equation $\sum_{j=0}^{N} \alpha_j s_j = 0$. From this, we see that the points of $\mathbb{P}(V)$ are in one-to-one correspondence with the linear system of these effective divisors and from now on, we will make this identification. When $V = \Gamma(X, \mathcal{L})$, the linear system is called a *complete linear system*. The ususal notation for a complete linear system, one of whose divisors is D is $|D|$.

Consider $\mathbb{P}^2_k$, where k is algebraically closed, and six points $P_1, \ldots, P_6$ in general position. This means that no three of our points are collinear, and not all lie on a conic. Examine the complete linear system given by

$$W = \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(3)),$$

so that W is spanned by all cubic monomials in three variables T_0, T_1, T_2. There are 10 such monomials, and thus the dimension of the linear system $\mathbb{P}(W)$ is $9\,(= 10 - 1)$. Note that our linear system consists exactly of the divisors on $\mathbb{P}^2$ which are the zeros of linear combinations of cubic monomials. That is, our linear system consists exactly of the cubic curves in $\mathbb{P}^2$.

Look at the subsystem, V, consisting of all of the cubics through $P_1, \ldots, P_6$. As these points are in general position, the subspace, V, has dimension $4 = 10 - 6$ (DX), and $\mathbb{P}(V)$ has dimension 3. There are no other base points besides $P_1, \ldots, P_6$. Letting $X = \mathbb{P}^2 - \{P_1, \ldots, P_6\}$, we get a morphism

$$\Phi : X \to \mathbb{P}^3.$$

Take $P, Q \in X$, with $P \neq Q$. Then, we have seven points $P, P_1, \ldots, P_6$, and the space of cubics through these seven points has dimension at least 2, i.e., it is at least a $\mathbb{P}^2$. The cubics through the eight points $P, Q, P_1, \ldots, P_6$ form at least a $\mathbb{P}^1$. If the cubics through the seven points form a $\mathbb{P}^3$, then the cubics through the eight points form a $\mathbb{P}^2$, and, if the cubics through the seven points form a $\mathbb{P}^2$, then the cubics through the eight points form a $\mathbb{P}^1$. In any case, there is an open set of cubics through $P, P_1, \ldots, P_6$ and not through Q. Let $P^* = \Phi(P)$, $Q^* = \Phi(Q)$. Then, there exists an open set of hyperplanes through P^* not through Q^*, which implies that $P^* \neq Q^*$. Therefore, Φ is injective (set-theoretically). Consider the case $k = \mathbb{C}$. From complex geometry, Φ is an embedding of manifolds iff it is injective on tangent spaces (actually, this is also true in the algebraic case). This means that, given any $P \in X$ and any pair of tangent vectors $(\overrightarrow{v}, \overrightarrow{w})$ at P, there exist curves C_1 and C_2 through P with C_1 having tangent vector $\overrightarrow{v}$ at P and C_2 having tangent vector $\overrightarrow{w}$ at

P. But, to give a cubic curve, C, through $P_1, \ldots, P_6$ and having a given tangent vector, $\overrightarrow{v}$, at P, is to give eight conditions on te cubic, namely: The six of passing through $P_1, \ldots, P_6$, the seventh of passing through P and the eighth of having a given tangent vector at P. Consequently such curves form a hyperplane in our linear system $\mathbb{P}(V)$. So, given P, $\overrightarrow{v}$, $\overrightarrow{w}$, we can consider the nonempty open subset of $\mathbb{P}(V)$ off the hyperplane determined by P and $\overrightarrow{v}$ and this open intersects the hyperplane determined by P and $\overrightarrow{w}$ (of course, $\overrightarrow{v} \neq \overrightarrow{w}$). Any curve, C_1, in the first hyperplane and any curve, C_2, in the second hyperplane and in the open will do. Therefore, we get an embedding (in fact, a closed immersion)

$$X \hookrightarrow \mathbb{P}^3.$$

Consider the blowup, $B_{\mathbb{P}^2, P_1, \ldots, P_6}$ of $\mathbb{P}^2$ at the six points $P_1, \ldots, P_6$ (also denoted B, for short), and write E_i for the exceptional line corresponding to the point P_i in the blowup. Given a point, P_i', on E_i, i.e., a tangent vector, $\overrightarrow{v}$, at P_i, there is a curve, C, in the blowup, B, passing through P_i'. Let us consider the proper transform of all the curves from our linear system, $\mathbb{P}(V)$, on B. These form a linear system on B (what is the corresponding line bundle?) and it is easy to see that it gives a morphism $\widetilde{\Phi}$ taking B to $\mathbb{P}^3$. We obtain the commutative diagram

$$\begin{tikzcd} B \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\Phi}"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\Phi"'] & \mathbb{P}^3, \end{tikzcd}$$

where we have written the embedding $\Phi \colon X \to \mathbb{P}^3$ as a rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$. The new map, $\widetilde{\Phi} \colon B \to \mathbb{P}^3$ clearly separates points because the only problem might arise for points on an exceptional divisor and here to separate points is merely to separate tangent directions at the corresponding P_i. It also separates tangent directions. In fact, the only question arises from a point, P_i', on an exceptional divisor, E_i. Here, our curves in $\mathbb{P}^2$ all pass through P_i, which gives six conditions, and we fix the tangent direction, $\overrightarrow{v}$, at P_i, corresponding to the point P_i'–this gives a seventh condition. Of course, just as above, we now have room in our family to separate the next higher order contact at P_i among our cubic curves.

And so, we obtain an embedding

$$\widetilde{\Phi} \colon B_{\mathbb{P}^2, P_1, \ldots, P_6} \to \mathbb{P}^3.$$

In fact, as $B_{\mathbb{P}^2, P_1, \ldots, P_6}$ is proper we deduce that $\widetilde{\Phi}$ is a closed immersion. Now B is smooth, so its image $\Sigma = \mathrm{Im}\,(\widetilde{\Phi})$ is a smooth surface in $\mathbb{P}^3$. What is $\deg(\Sigma)$?

Take two generic hyperplanes $\mathcal{H}, \mathcal{H}'$ in $\mathbb{P}^3$, and consider

$$\mathcal{H} \cap \mathcal{H}' \cap \Sigma.$$

Then, $\mathcal{H}$ corresponds to a cubic, C, and $\mathcal{H}'$ corresponds to a cubic, C', and both C and C' pass through each exceptional line $E_1, \ldots, E_6$. The image, $\widetilde{\Phi}(C)$, of C cuts the exceptional line E_i once, in general. For, to pass through twice means either that we have two distinct tangent directions at P_i (i.e., C has a node at P_i) or that we have a tangent direction and that C has order of contact 2 with it (i.e., C has a cusp at P_i). But, an open set of these curves has neither of these conditions and so we can choose a curve C from this open set which goes through each P_i exactly once. Pick $\mathcal{H}'$ to miss all six points $\widetilde{\Phi}(C)\cap E_1, \ldots, \widetilde{\Phi}(C)\cap E_6$. Since C and C' have distinct tangent vectors at $P_1, \ldots, P_6$, and since $\#(C \cdot C') = 9$, by Bezout's theorem for curves in the plane, the six intersections of C and C' in $P_1, \ldots, P_6$ yield as a remainder only three further intersections. Consequently,

$$\mathcal{H} \cap \mathcal{H}' \cap \Sigma = \Phi((C \cdot C') - \{P_1, \ldots, P_6\}),$$

which implies that

$$\#(\mathcal{H} \cap \mathcal{H}' \cap \Sigma) = 3.$$

Therefore, Σ is a nonsingular cubic surface.

What is the dimension of the space of such cubic surfaces? We can apply PGL(3) to $\mathbb{P}^3$ and get a new Σ. We can also vary the points $P_1, \ldots, P_6$. We have $\dim(\mathrm{PGL}(3)) = 15$, and the choice of $P_1, \ldots, P_6$ gives two parameters for each P_i (coordinates), and thus, 12 further parameters. This looks like 27 degrees of freedom. However, we could use PGL(2) on $\mathbb{P}^2$ and not change Σ. Since $\dim(\mathrm{PGL}(3)) = 8$, we expect $19 = 27 - 8$ degrees of freedom. The cubic surfaces form a $\mathbb{P}^{19}$ and the nonsingular ones form an open subset in $\mathbb{P}^{19}$. Since we have a family of dimension 19, our Σ's are almost all the nonsingular cubics in $\mathbb{P}^3$. In fact, they are all of them.

What about the 27 lines on a nonsingular cubic?

Let $L_i = \widetilde{\Phi}(E_i)$, $i = 1, ..., 6$. We get six curves. Given i and j, the line L_{ij} determined by P_i and P_j has a proper transform on B, call it L_{ij}, again. Let $M_{ij} = \widetilde{\Phi}(L_{ij})$, for $i \neq j$, $i, j = 1, \ldots, 6$. We get fifteen more curves. Any five points in $\mathbb{P}^2$ (in general position) determine a conic. Let Q_j be the conic through $\{P_1, \ldots, P_6\} - \{P_j\}$, and write, Q_j, again for the proper transform of Q_j on B. Now, let $N_j = \widetilde{\Phi}(Q_j)$. This yields six more curves.

Claim: All the curves L_i, M_{ij}, N_j, are lines on Σ.

Pick a hyperplane, $\mathcal{H}$, in $\mathbb{P}^3$. We know that $\mathcal{H}$ corresponds to a cubic curve passing through $P_1, \ldots, P_6$. A point on E_i is just a tangent vector at P_i and we can find an open set of the cubics having this tangent vector and no higher-order contact at P_i. This means that on B, the proper transform of our cubic cuts E_i just once; hence, $\mathcal{H} \cdot L_i = 1$. Thus, L_i is a line. For the curves M_{ij}, we observe that the line P_iP_j gives vectors, $\overrightarrow{v_i}$, at P_i and, $\overrightarrow{v_j}$, at P_j. Now, an open set of the cubics, C, passing through $P_1, \ldots, P_6$ has neither the vector $\overrightarrow{v_i}$ nor the vector $\overrightarrow{v_j}$ as tangents at P_i, respectively P_j. Such cubics cut the line P_iP_j in three points, two of which are P_i and P_j and their proper transforms miss each other on E_i and E_j respectively. This leaves exactly one intersection on B for the proper transform of C and L_{ij}. So, on Σ, the hyperplane, $\mathcal{H}$, corresponding to C cuts M_{ij} just once; and M_{ij} is then a line.

Consider a cubic curve through all the points $P_1, \ldots, P_6$. Pick a conic, Q_j, through the points $P_1, \ldots, P_6$, omitting P_j. This conic determines five tangent vectors, namely its tangents at the five points of the $P_1, \ldots, P_6$ it goes through. For each of those tangent vectors we get a closed subvariety of all the cubics through $P_1, \ldots, P_6$, namely those also having the given tangent vector at the given point. The union of these five closed varieties fails to exhaust the irreducible variety of cubics through $P_1, \ldots, P_6$. So, for any C in the open set of cubics remaining, the intersection of C and Q_j—which consists of six points with multiplicities—is actually six distinct points: The original five and one further point. When we blow up, the proper transform of the conic and our cubic go through distinct points on the exceptional lines corresponding to the five chosen points. Hence, on B, the proper transform of Q_j and C intersect just once. Therefore, the hyperplane, $\mathcal{H}$, corresponding to our cubic cuts N_j just once; so, N_j is a line.

We have seen that the 27 seven curves: the E_i, the M_{ij} and the N_j are all lines in $\mathbb{P}^3$ contained in Σ. These are the 27 lines in Σ. The reader can consider their geometry—all will follow from our description of them as images of proper transforms on B.

The cubic, Σ, is isomorphic to the blowup surface, B. Hence $\mathrm{Pic}(\Sigma)$ is isomorphic to $\mathrm{Pic}(B)$. But the only new divisors on B are the exceptional loci, $E_1, \ldots, E_6$. It is not hard to show (DX) that these are distinct in $\mathrm{Pic}(\Sigma)$, and so $\mathrm{Pic}(\Sigma)$ is $\mathbb{Z}^7$ because we have the further divisor class of proper transforms of ordinary lines in $\mathbb{P}^2$. This is a further example of the fact that the map $\mathrm{Pic}(\mathbb{P}^3) \longrightarrow \mathrm{Pic}(\Sigma)$ corresponding to the injection $\Sigma \hookrightarrow \mathbb{P}^3$ need not be surjective.

7.3 Projective Morphisms

Recall that in Chapter 2, we defined projective varieties as closed subvarieties of $\mathbb{P}^N_k$. Obviously, the correct definition in our now more general case of schemes, X, over a base, S, is that such a scheme is projective over S when it is a closed S-subscheme of $\mathbb{P}_S(\mathcal{E})$ for some $\mathcal{E}$. Of course, this is a relative notion, referring as it does to the morphism $X \longrightarrow S$. Here for the record, is the official definition.

Definition 7.2 If X is a scheme over S, then the morphism, $X \longrightarrow S$, is a *projective morphism* (we also say X *is projective over* S) iff there exists a *closed* S-immersion of X to $\mathbb{P}_S(\mathcal{E})$, for some f.g. QC $\mathcal{O}_S$-module, $\mathcal{E}$. The morphism $X \longrightarrow S$ is *quasi-projective* iff we merely have an S-immersion to $\mathbb{P}_S(\mathcal{E})$.

There is an important generalization of the notion of projective morphism, namely:

Definition 7.3 If X is a scheme over S, then the morphism, $X \longrightarrow S$, is a *proper morphism* (we also say X *proper over* S) iff

(1) X is separated over S.

(2) $X \longrightarrow X$ is a finite-type morphism.

(3) The map $X \longrightarrow X$ is *universally closed*, that is, for every T over S, the morphism $pr_2 \colon X \prod_S T \to T$ is a closed map.

Of course, if X is projective or even quasi-projective over S, then the morphism $X \longrightarrow S$ is a separated morphism. However, projective morphisms have a further crucial property: They are proper. We proved this in Chapter 2 (Section 2.5, Theorem 2.36) and the proof there is sufficiently general for us to merely modify it slightly to give our assertion. Here it is:

Theorem 7.17 *If $X \longrightarrow S$ is a projective morphism, then for any scheme, T, over S, the morphism $pr_2 \colon X \prod_S T \to T$ is a closed map.*

Proof. The statement is local on S, so we may and do assume that S is affine, say $S = \operatorname{Spec} A$, and then if we cover T by affine opens we may even assume T is affine. Now, X is $\operatorname{Proj}(\mathcal{S})$, where $\mathcal{S}$ is a good graded A-algebra. Consequently, the above remarks reduce us to proving that the morphism $\operatorname{Proj}(\mathcal{S}) \longrightarrow \operatorname{Spec} A$ is a closed map (if $T = \operatorname{Spec} B$, then $pr_2 \colon \operatorname{Proj}(\mathcal{S}) \prod_{\operatorname{Spec} A} \operatorname{Spec} B \to \operatorname{Spec} B$ is just the map $\operatorname{Proj}(\mathcal{S}_B) \longrightarrow \operatorname{Spec} B$.) As a last reduction, we need only prove that the image of $\operatorname{Proj}(\mathcal{S})$ itself in $\operatorname{Spec} A$ is closed. For, if C is a closed subset of $\operatorname{Proj}(\mathcal{S})$, then C possesses a scheme structure so that, as scheme, C is $\operatorname{Proj}(\mathcal{S}')$. But then, the image of C would be closed, as required.

We now face the essential case: The image of $\operatorname{Proj}(\mathcal{S})$ in $\operatorname{Spec} A$, where $\mathcal{S}$ is a good graded A-algebra, is closed in $\operatorname{Spec} A$. A point z in $\operatorname{Spec} A$ is in the image iff $\pi^{-1}(z)$ is nonempty (of course, π is the map $\operatorname{Proj}(\mathcal{S}) \longrightarrow \operatorname{Spec} A$). But the fibre $\pi^{-1}(z)$ is just $\operatorname{Proj}(\mathcal{S} \otimes_A \kappa(z))$, and so, $\pi^{-1}(z)$ is empty iff the algebra $\mathcal{S} \otimes_A \kappa(z)$ is a (TN)-algebra over $\kappa(z)$. This means $\mathcal{S} \otimes_A \kappa(z) = (0)$ iff $n >> 0$; now $\mathcal{S}_n$ is a f.g. A-module, so by Nakayama's lemma, we find $(\mathcal{S}_n)_z = (0)$ for $n >> 0$ iff $z \notin \operatorname{image}(\pi)$. Write $\mathfrak{A}_n$ for the annihilator of $\mathcal{S}_n$ as A-module, then our condition is that $\mathfrak{A}_n$ is an irrelevant ideal when tensored up to A_z. But, as $\mathcal{S}$ is a ggr we find that $\mathcal{S}_n \cdot \mathcal{S}_1 = \mathcal{S}_{n+1}$ for $n >> 0$, which means that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$. Write $\mathfrak{A}$ for $\bigcup_{n>>0} \mathfrak{A}_n$. Then, $\pi^{-1}(z) = \emptyset$ iff $z \notin V(\mathfrak{A})$. Therefore, the image of $\operatorname{Proj}(\mathcal{S})$ is exactly $V(\mathfrak{A})$. $\square$

Remark: It is very instructive for the reader to compose the above proof with the proof of Theorem 2.36 in Chapter 2, Section 2.5. They are the same proof but the extra details and precision in the proof of Theorem 2.36 come about because varieties are simpler than schemes.

Now, we face a problem: How can one tell, by looking at X itself (over S) whether X is projective or quasi-projective? Observe that if we could embed X into $\mathbb{P}_S(\mathcal{E})$, more generally even if there were just a morphism from X to $\mathbb{P}_S(\mathcal{E})$, the pullback of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ would be a distinguished line bundle on X. This suggests that we examine the line bundles on X to check if they might be a pullback of some $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. By experience, the correct concepts are what are called ample and very ample line bundles on X. These are the bundles to which we now turn our attention.

Definition 7.4 Let X be a scheme and $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *ample on* X iff for all coherent $\mathcal{O}_X$-modules, $\mathcal{F}$, there is some $N(\mathcal{F})$ so that for every $n \geq N(\mathcal{F})$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

Remark: The definition of ampleness makes no reference to a base scheme S. Thus, it is an absolute notion as opposed to a notion relative to the morphism $X \longrightarrow S$. That this notion is a step in the correct direction is the content of the following theorem of J.P. Serre FAC [47].

Theorem 7.18 *(Serre) Let X be a projective scheme over* Spec A*, where A is noetherian and the fibre bundle into which X is embedded is given by a coherent A-module, $\mathcal{E}$. Write $\mathcal{L} = \mathcal{O}_X(1)$ for the pullback of $\mathcal{O}_{\mathbb{P}_A(\mathcal{E})}(1)$ under the closed immersion $i\colon X \hookrightarrow \mathbb{P}_A(\mathcal{E})$. Then, $\mathcal{L}$ is ample on X.*

Proof. The proof proceeds in two steps.

(1) *Reduction to the case where $X = \mathbb{P}_A(\mathcal{E})$.*

Let $\mathcal{F}$ be a coherent sheaf on X. Then, as i is a closed immersion, $i_*\mathcal{F}$ is coherent on $\mathbb{P}_A(\mathcal{E})$ (c.f. Proposition 4.21). Now, we know that

$$(i_*\mathcal{F})(n) = i_*(\mathcal{F}(n)),$$

since A is noetherian. Now,

$$\Gamma(\mathbb{P}_A(\mathcal{E}), (i_*\mathcal{F})(n)) = \Gamma(\mathbb{P}_A(\mathcal{E}), i_*(\mathcal{F}(n))) = \Gamma(X, \mathcal{F}(n)) = \Gamma(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F}).$$

So, if the sequence

$$\mathcal{O}^M_{\mathbb{P}_A(\mathcal{E})} \longrightarrow i_*\mathcal{F}(n) \longrightarrow 0 \quad \text{is exact for for } n >> 0,$$

then the sequence

$$\mathcal{O}^M_X \longrightarrow \mathcal{L}^{\otimes n} \otimes \mathcal{F} \longrightarrow 0 \quad \text{is also exact for } n >> 0,$$

and the reduction is achieved.

(2) *Now, we may assume that $X = \mathbb{P}_A(\mathcal{E})$.*

Since $\mathbb{P}_A(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_A(E))$, where $\widetilde{E} = \mathcal{E}$, there are standard opens, $U_i = \mathbb{P}_A(\mathcal{E})_{(f_i)}$, where $f_1, \ldots, f_N$ are generators of E. (Remember, we assumed $\mathcal{E}$ is coherent, so, E is f.g.) Then, since A is noetherian, $\mathcal{F} \restriction U_i$ is the tilde of a f.g. A_i-module where

$$A_i = (\mathrm{Sym}_A(E))_{(f_i)} = A\left[\frac{f_0}{f_i}, \ldots, \frac{f_N}{f_i}\right].$$

We can write $\mathcal{F} \restriction U_i = \widetilde{M_i}$, and M_i has generators $\beta_j^{(i)}$, for $j = 1, \ldots, z(i)$ (as A_i-module). Each $\beta_j^{(i)}$ is a section of $\mathcal{F} \restriction U_i$. By Serre's extension lemma, there is some N_i so that $f_i^{N_i} \otimes \beta_j^{(i)}$ extends to a global section of $\mathcal{L}^{\otimes N_i} \otimes \mathcal{F}$, where f_i is a section of $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Pick $\mathcal{N}$ which works for all $i = 0, \ldots, N$. Then, $f_i^{\mathcal{N}} \otimes \beta_j^{(i)}$ extends to a global section t_{ij} of $\mathcal{L}^{\otimes \mathcal{N}} \otimes \mathcal{F}$, and

$$t_{ij} \restriction U_i = f_i^{\mathcal{N}} \otimes \beta_j^{(i)}.$$

Now, $(\mathcal{L}^{\otimes \mathcal{N}} \otimes \mathcal{F}) \restriction U_i$ is equal to $\widetilde{\mathcal{M}_i}$ for some module, $\mathcal{M}_i$. We know from previous work that the map

$$\xi \mapsto f_i^{\mathcal{N}} \otimes \xi$$

takes M_i isomorphically onto $\mathcal{M}_i$. Pick $x \in \mathbb{P}_A(\mathcal{E})$, then there is some U_i so that $x \in U_i$, and on U_i, the global sections $t_{rs} \restriction U_i$ have among them the generators for $(\mathcal{L}^{\otimes \mathcal{N}} \otimes \mathcal{F}) \restriction U_i$. Thus, at x, they generate the stalk, and therefore, $\mathcal{N}$ will do for $\mathcal{F}$. This proves that $\mathcal{L}$ is ample. $\square$

Remarks:

(1) If S is a noetherian scheme, then S is covered by finitely many affines, each of which is noetherian, and so, by applying the argument to these affines and taking $\mathcal{N}$ large enough, it is clear that we obtain the

Corollary 7.19 *Assume $X \longrightarrow S$ is a projective morphism, where S is a noetherian scheme and $\mathcal{E}$ is coherent. Write $\mathcal{L} = \mathcal{O}_X(1)$ for the pullback of $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ under the closed immersion $X \longrightarrow \mathbb{P}_S(\mathcal{E})$. Then, $\mathcal{L}$ is ample on X.*

(2) Suppose X is an S-scheme and $\mathcal{L}$ is a line bundle on X. Further assume that we choose a finite number of global sections of $\mathcal{L}$, say R of these. Then we get the not necessarily surjective map of $\mathcal{O}_X$-modules

$$\pi^*(\mathcal{O}_S)^R \longrightarrow \mathcal{L}$$

where π is the structure morphism $X \longrightarrow S$. By Theorem 7.10 and its corollaries, we obtain a rational map:

$$X = \mathrm{Proj}(\mathrm{Sym}(\mathcal{L})) \dashrightarrow \mathbb{P}_S^R.$$

If we can arrange that $\mathcal{L}$ is ample, then we find that some power of $\mathcal{L}$ is generated by its global sections, at least when X is locally noetherian (for: $\mathcal{L}$ is finitely generated, coherence is a local property, and f.g. modules over noetherian rings are coherent.)

If, in addition, X is actually noetherian, then its power, $\mathcal{L}^{\otimes N}$, is generated by a finite number of its global sections, and replacing the original ample $\mathcal{L}$ by this power, we find the morphism

$$X = \mathrm{Proj}(\mathrm{Sym}(\mathcal{L}^{\otimes N})) \dashrightarrow \mathbb{P}^R_S,$$

where R is now the number of sections needed to generate $\mathcal{L}^{\otimes N}$. Hence, from an ample $\mathcal{L}$ on a noetherian X, we deduce a closed immersion, $X \longrightarrow \mathbb{P}^R_S$, for some R. This property of ample line bundles will be discussed further when we consider the notion of being *very ample*.

Proposition 7.20 *Let X be a scheme, then properties (1)–(3) below are equivalent for a line bundle, $\mathcal{L}$, on X. Furthermore, we also have property (4).*

(1) $\mathcal{L}$ is ample.

(2) $\mathcal{L}^{\otimes m}$ is ample for all $m > 0$.

(3) $\mathcal{L}^{\otimes m}$ is ample for some $m > 0$.

(4) If $\mathcal{M}$ is another line bundle and $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Proof. (1) $\Rightarrow$ (2) follows from the definition. (2) $\Rightarrow$ (3) is trivial. Now, for (3) $\Rightarrow$ (1). Assume that $\mathcal{L}^{\otimes m}$ is ample for some m, and that $\mathcal{F}$ is coherent. Write

$$\mathcal{F}_j = \mathcal{F} \otimes \mathcal{L}^{\otimes j},$$

for $j = 0, \ldots, m-1$. Then, we know that there is some N_j so that for all $t \geq N_j$, the sheaf $\mathcal{F}_j \otimes (\mathcal{L}^{\otimes m})^{\otimes t}$ is generated by its sections. Let

$$\mathcal{N} = m \cdot \max_{0 \leq j \leq m-1} \{N_j\}.$$

Take $t \geq \mathcal{N}$. We can write

$$t = m\theta + k, \quad \text{where } 0 \leq k \leq m-1.$$

Then, we have

$$\mathcal{F} \otimes \mathcal{L}^{\otimes t} = (\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \otimes (\mathcal{L}^{\otimes m})^{\otimes \theta} = \mathcal{F}_k \otimes (\mathcal{L}^{\otimes m})^{\otimes \theta}.$$

Since $t \geq \mathcal{N}$, we find $\theta \geq \max\{N_j\}$, and thus, the righthand side is generated by its sections, which implies that $\mathcal{L}$ is ample.

(4) Assume that $\mathcal{L}$ and $\mathcal{M}$ are ample. Apply ampleness of $\mathcal{L}$ to $\mathcal{L}$ itself. Thus, there is some N so that for all $s \geq N$, the sheaf $\mathcal{L}^{\otimes (s+1)}$ is generated by its sections. Pick a coherent sheaf, $\mathcal{F}$, and consider

$$\mathcal{F}_k = \mathcal{F} \otimes \mathcal{L}^{\otimes k},$$

where $0 \leq k \leq N-1$. Write N_k for the integer for $\mathcal{F}_k$ that works for the ample sheaf $\mathcal{M}$. Let

$$\mathcal{N} = (N+1) \cdot \max_{0 \leq k \leq N-1} \{N_k\}.$$

We must prove that for $t \geq \mathcal{N}$, the sheaf $(\mathcal{L} \otimes \mathcal{M})^{\otimes t} \otimes \mathcal{F}$ is generated by its sections. Now, as $t \geq \mathcal{N}$, we can write

$$t = (N+1)\theta + k, \quad \text{with } 0 \leq k \leq N \text{ and } \theta \geq \max\{N_j\}.$$

But,

$$\begin{aligned} (\mathcal{L} \otimes \mathcal{M})^{\otimes t} \otimes \mathcal{F} &= (\mathcal{L}^{\otimes k} \otimes \mathcal{F}) \otimes \mathcal{L}^{\otimes (N+1)\theta} \otimes \mathcal{M}^{\otimes t} \\ &= \mathcal{F}_k \otimes \mathcal{M}^{\otimes t} \otimes \left(\mathcal{L}^{\otimes (N+1)}\right)^{\otimes \theta}. \end{aligned}$$

The sheaf $\mathcal{F}_k \otimes \mathcal{M}^{\otimes t}$ is generated by its sections by choice of t, and the other sheaf, $\left(\mathcal{L}^{\otimes (N+1)}\right)^{\otimes \theta}$, is also generated by its sections by the above. Hence, their tensor product is generated by its sections and we are done. $\square$

Serre's Theorem (Theorem 7.18) has several important corollaries.

Corollary 7.21 *(Serre's generation theorem) Let X be a projective scheme over the scheme S, where S is assumed noetherian and the $\mathcal{O}_S$-module, $\mathcal{E}$, for which X is contained in $\mathbb{P}_S(\mathcal{E})$ is coherent. If $\mathcal{F}$ is a coherent sheaf on X, then there is a vector bundle, $\mathcal{V}$, so that*

(1) $\mathcal{V} = \mathcal{L}^m$, for some line bundle, $\mathcal{L}$, and

(2) there is a surjection, $\mathcal{V} \longrightarrow \mathcal{F}$.

Proof. The sheaf $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$ is generated by finitely many sections if $n >> 0$, because our hypotheses guarantee that X is a noetherian scheme and Theorem 7.18 implies that $\mathcal{O}_X(1)$ is then ample. Thus, we have an exact sequence

$$\mathcal{O}_X^m \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Twisting by $\mathcal{O}_X(-n)$, we get the exact sequence

$$(\mathcal{O}_X(-n))^m \longrightarrow \mathcal{F} \longrightarrow 0,$$

and we let $\mathcal{V} = (\mathcal{O}_X(-n))^m$, for $n >> 0$. $\square$

Corollary 7.22 *Under the same hypotheses for X as in Corollary 7.21, if $\mathcal{F}$ is a coherent sheaf on X, then there is the Syzygy resolution*

$$\cdots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0, \qquad (\mathrm{Syz}(\mathcal{F}))$$

where each $\mathcal{E}_i$ is a vector bundle of the form $\mathcal{E}_i = \mathcal{L}_i^{m_i}$, for some line bundle, $\mathcal{L}_i$, and some $m_i > 0$.

Proof. By Corollary 7.21, we have

$$\mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\mathcal{E}_0$ of the required form. Let $\mathcal{K}_0 = \mathrm{Ker}\,(\mathcal{E}_0 \longrightarrow \mathcal{F})$. Then, $\mathcal{K}_0$ is coherent. We can apply Corollary 7.21 again to get

$$\mathcal{E}_1 \longrightarrow \mathcal{K}_1 \longrightarrow 0.$$

We finish the proof by induction. $\square$

In order to state the third corollary, we need the definitions of the Grothendieck groups $K_{\mathrm{coh}}(X)$ and $K_{\mathrm{vect}}(X)$. We define $K_{\mathrm{coh}}(X)$, the definition of $K_{\mathrm{vect}}(X)$ being similar. We let $K_{\mathrm{coh}}(X)$ be the quotient of the free group generated by all the coherent sheaves on X by the subgroup generated by expressions of the form $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, where

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \quad \text{is exact.}$$

Corollary 7.23 *(Hilbert–Serre) Let X be a projective scheme over* $\mathrm{Spec}\, k$*, with k a field. If X is smooth over* $\mathrm{Spec}\, k$*, then the Syzygy resolution* $(\mathrm{Syz}(\mathcal{F}))$ *stops after* $\dim(X)$ *terms. Thus,*

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \quad \textit{is exact,}$$

where each $\mathcal{E}_i$ is a vector bundle and for $i = 0, \ldots n-1$, the bundle $\mathcal{E}_i$ is of the form $\mathcal{E}_i = \mathcal{L}_i^{m_i}$, for a line bundle, $\mathcal{L}_i$, and some $m_i > 0$. Hence, the natural map

$$K_{\mathrm{vect}}(X) \longrightarrow K_{\mathrm{coh}}(X)$$

is an isomorphism.

Proof. This is just Hilbert's theorem on chains of Syzygies (1893): If M is a f.g. graded module over $k[X_0, \ldots, X_n]$ and the ring $k[X_0, \ldots, X_n]$ is nonsingular (which means that all homogeneous localizations are regular), then the Syzygy sequence

$$0 \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

stops, where F_d is locally free and $d \leq \dim k[X_0, \ldots, X_n]$. $\square$

We have been dealing with closed subschemes of $\mathbb{P}_S(\mathcal{E})$ for coherent $\mathcal{E}$ and noetherian S. The finiteness conditions are placed on $\mathcal{E}$ and S in order that there is some bounded power of the ample $\mathcal{L}$ which will tensor a given coherent $\mathcal{F}$ into another coherent but generated by a finite number of its sections. However, we have been begging the question of when a given S-scheme, X, can actually be embedded as a closed subscheme of $\mathbb{P}_S(\mathcal{E})$ for some $\mathcal{E}$. For this, we make the definition:

Definition 7.5 If X is a scheme over S and $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}$ is *very ample over S* iff there is an immersion $X \xrightarrow{i} \mathbb{P}_S(\mathcal{E})$, with $\mathcal{E}$ QC as $\mathcal{O}_S$-module and $\mathcal{L} = i^*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1))$.

Of course, all this definition does is to change the question above to: When is a given line bundle on a given X over S very ample? In his paper FAC [47], Serre gave one reduction of this question:

Theorem 7.24 *(Serre) Let X be a finite-type scheme over a noetherian scheme, S. Choose a line bundle, $\mathcal{L}$ on X, then, $\mathcal{L}$ is ample on X iff some tensor power, $\mathcal{L}^{\otimes m}$, of $\mathcal{L}$ is very ample over S. (Here, $m > 0$.)*

Proof. First, assume that $S = \mathrm{Spec}\, A$, with A a noetherian ring and that $\mathcal{L}^{\otimes m}$ gives an immersion into $\mathbb{P}^N_A$ (Of course, as we are assuming $\mathcal{L}^{\otimes m}$ is very ample, it really gives an immersion into $\mathbb{P}_S(\mathcal{E})$, for some QC $\mathcal{E}$. Thus, our assumption is a special case.) Let $\overline{X}$ be the closure of X in $\mathbb{P}^N_A$. By hypothesis, $j\colon X \to \overline{X}$ is an open immersion. Pick a coherent sheaf, $\mathcal{F}$, on X, then $j_*\mathcal{F}$ is QC on $\overline{X}$. Now, there is some $\mathcal{F}' \subseteq j_*\mathcal{F}$ so that

(1) $\mathcal{F}'$ is coherent on $\overline{X}$, and

(2) $\mathcal{F}' \restriction X = \mathcal{F}$.

By Serre's ampleness's theorem (Theorem 7.18), there is some $q >> 0$ so that $\mathcal{F}' \otimes \mathcal{O}_{\overline{X}}(q)$ is generated by its global sections. But then,

$$\mathcal{L} \otimes \mathcal{O}_X(q) = \mathcal{F}' \otimes \mathcal{O}_{\overline{X}}(q) \restriction X$$

is generated by sections, and $(\mathcal{L}^{\otimes m})^{\otimes q}$ does the job.

Continue with the assumption that $S = \mathrm{Spec}\, A$, as above. Assume now that $\mathcal{L}$ is ample. Pick $x \in X$, let U be some affine open with $x \in U$, where $\mathcal{L} \restriction U$ is trivial, and let $Y = X - U$. Then, Y is given by a QC-ideal, $\mathfrak{I}_Y$, chosen so that Y has the reduced induced structure. The assumptions on X imply that $\mathfrak{I}_Y$ is coherent. Then, there is some $q >> 0$ so that $\mathfrak{I}_Y \otimes \mathcal{L}^{\otimes q}$ is generated by its global sections; thus, there is some $s \in \Gamma(X, \mathfrak{I}_Y \otimes \mathcal{L}^{\otimes q})$ so that $s(x) \neq 0$, i.e., $s(x) \notin \mathfrak{m}_x(\mathfrak{I}_Y \otimes \mathcal{L}^{\otimes q})$. Look at X_s. This is an open set, and $x \in X_s$. The sequence $0 \longrightarrow \mathfrak{I}_Y \longrightarrow \mathcal{O}_X$ is exact, and thus, the sequence

$$0 \longrightarrow \mathfrak{I}_Y \otimes \mathcal{L}^{\otimes q} \longrightarrow \mathcal{L}^{\otimes q} \quad \text{is exact,}$$

which implies that

$$s \in \Gamma(X, \mathfrak{I}_Y \otimes \mathcal{L}^{\otimes q}) \hookrightarrow \Gamma(X, \mathcal{L}^{\otimes q}).$$

Since s vanishes on Y, we have $X_s \subseteq U$. Look at $s \restriction U$. We have

$$s \restriction U \in \Gamma(U, (\mathcal{L} \restriction U)^{\otimes q}) = \Gamma(U, \mathcal{O}_U),$$

as $\mathcal{L} \restriction U$ is trivial. The fact that

$$f = s \restriction U \in \Gamma(U, \mathcal{O}_U)$$

implies that $X_s = U_f$, and thus, X_s is affine. To recapitulate the above few lines of argument, we have proved: For every $x \in X$, there is some $q(x) >> 0$ and some $s \in \Gamma(X, \mathcal{L}^{\otimes q(x)})$ such that

(1) X_s is open affine.

(2) $\mathcal{L} \upharpoonright X_s$ is trivial.

(3) $x \in X_s$.

Since X is quasi-compact, X is covered by finitely many of the X_{s_i}'s. Now, for any $k \in \mathbb{N}$, $X_s = X_{s^{\otimes k}}$. Hence, we may replace $\mathcal{L}^{\otimes q(x)}$ and s by some fixed q and finitely many sections $s_1, \ldots, s_r$ of $\mathcal{L}^{\otimes q}$. Write A_i for the ring $\Gamma(X_i, \mathcal{O}_{X_i})$, this is a finitely generated A-algebra. So, there exist some $b_j^{(i)}$, so that $b_j^{(i)}$ generate A_i as an A-algebra, with

$$b_j^{(i)} \in \Gamma(X_i, \mathcal{O}_{X_i}) = \Gamma(X_i, \mathcal{L} \upharpoonright X_i).$$

By Serre's extension lemma, there is some N_i so that $s_i^{N_i} \otimes b_j^{(i)}$ extends to a global section t_{ij} of $\mathcal{L}^{\otimes N_i} \otimes \mathcal{L} = \mathcal{L}^{\otimes(N_i+1)}$, for each j. In the usual way, we may assume that all N_i are equal, say equal to N. Consider all the global sections s_i^{N+1}, t_{ij} and use them to define a morphism

$$X \longrightarrow \mathbb{P}^M_A.$$

(We know that the X_i cover X and hence that the map is a morphism.) Let T_i and T_{ij} be the homogeneous coordinates corresponding to s_i^{N+1} and t_{ij}. Our map is given by

$$\theta_i \colon A\left[\frac{T_j}{T_i}, \frac{T_{ij}}{T_i}\right] \to A_i,$$

via

$$\frac{T_j}{T_i} \mapsto \frac{s_j^{N+1}}{s_i^{N+1}} \quad \text{and} \quad \frac{T_{ij}}{T_i} \mapsto \frac{s_i^N}{s_i^{N+1}}\, b_j^{(i)} = \frac{b_j^{(i)}}{s_i},$$

and thus, θ_i is a surjection. Therefore, each θ_i is a closed immersion

$$\theta_i \colon X_i \to U_i,$$

where

$$U_i = \operatorname{Spec} A\left[\frac{T_j}{T_i}, \frac{T_{ij}}{T_i}\right].$$

Hence, our map, θ, is the composition

$$\theta \colon X = \bigcup_{i=1}^{r} X_i \hookrightarrow \bigcup_{i=1}^{r} U_i \longrightarrow \mathbb{P}^M_A,$$

where the first map is a closed immersion and the second map is open.

We have completed the proof of the equivalence in the special case that S is $\operatorname{Spec} A$ and that when $\mathcal{L}^{\otimes q}$ is very ample we embed in $\mathbb{P}^N_A$. To get the general case, first consider the assumption on S. Since S is noetherian, it is covered by finitely many open affines, Spec A, with A noetherian. Write π for the structure morphism, $X \longrightarrow S$, and X_j for $\pi^{-1}(V_j)$, where $V_1, \ldots, V_t$ are the open affines which cover S. If $\mathcal{L}$ is ample on X, then $\mathcal{L}$ is ample on each $\pi^{-1}(V_j)$ and so by taking the maximum of the numbers, q_i, that work for each $\pi^{-1}(V_i)$ we obtain a single tensor power, $\mathcal{L}^{\otimes q}$, so that each $\mathcal{L}^{\otimes q} \upharpoonright \pi^{-1}(V_j)$ is very ample. Then, $\mathcal{L}^{\otimes q}$ itself very ample over S according to the following lemma whose proof will be given at the close of this proof:

Lemma 7.25 *If $\pi \colon X \to S$ is a quasi-compact morphism and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $\mathcal{L}$ is very ample over X iff the following conditions hold:*

(a) $\pi_*\mathcal{L}$ is QC on S,

(b) $\pi^*\pi_*\mathcal{L} \longrightarrow \mathcal{L}$ is surjective,

(c) The morphism, $X \longrightarrow \mathbb{P}_S(\pi_*(\mathcal{L}))$, induced by (b) is an immersion.

However, as the reader will see in the proof of this lemma the definition of very ample involves immersions into $\mathbb{P}(\mathcal{E})$. We have treated the special case $\mathcal{E} = \mathcal{O}_S^N$. So now, we must show that, in our situation, we actually get a morphism to $\mathbb{P}_S^N$. Now, given $\mathcal{E}$, it is the direct limit of its coherent submodules. Our immersion corresponds to a surjection of sheaves

$$\pi^*(\mathcal{E}) \longrightarrow \mathcal{L}^{\otimes m} \longrightarrow 0.$$

Now, $\pi^*(\mathcal{E})$ is $\varinjlim_{\alpha} \pi^*(\mathcal{E}_\alpha)$, where the $\mathcal{E}_\alpha$'s are the sheaves corresponding to the finitely generated submodules, E_α, of E. Also, π is the structure morphism, $X \longrightarrow S$. Since X is noetherian, we can cover it by finitely many affine opens, say $U_1, \ldots, U_r$. On each of these, we obtain the surjection

$$\pi^*(\mathcal{E} \upharpoonright U_j) \longrightarrow \mathcal{L}^{\otimes m} \upharpoonright U_j \longrightarrow 0.$$

Moreover, we can choose the U_j's as a trivializing cover for the bundle $\mathcal{L}^{\otimes m}$. Therefore, on U_j, the bundle $\mathcal{L}^{\otimes m} \upharpoonright U_j$ is generated by one element and this element is in the image of $\pi^*(\mathcal{E}_\alpha \upharpoonright U_j)$, for some α. Since the U_j are finite in number, there is one α which works for all of X. This means that in the diagram

$$\begin{tikzcd}
\pi^*(\mathcal{E}) \arrow[r] & \mathcal{L}^{\otimes m} \arrow[r] \arrow[d, equal] & 0 \\
\pi^*(\mathcal{E}_\alpha) \arrow[r] \arrow[u] & \mathcal{L}^{\otimes m} &
\end{tikzcd}$$

the lower horizontal arrow is surjective. Hence, the immersion $X \longrightarrow \mathbb{P}_A(\mathcal{E})$ actually is an immersion $X \longrightarrow \mathbb{P}_A(\mathcal{E}_\alpha)$. Now, $\mathcal{E}_\alpha = \widetilde{E_\alpha}$ and E_α is a homomorphic image of A^{n_α}. It follows that $\mathbb{P}_A(\mathcal{E}_\alpha)$ is embedded in $\mathbb{P}_A^{n_\alpha}$. Thus, we may and do assume that our $\mathcal{L}^{\otimes m}$ gives an immersion $X \longrightarrow \mathbb{P}_A^N$.

We have just given the full proof that ampleness implies that some tensor power, $\mathcal{L}^{\otimes q}$, of $\mathcal{L}$ is very ample over S. There remains the converse statement, which we know to be true when $S = \mathrm{Spec}\, A$. So, cover S by affine opens, $V_1, \ldots, V_t$, and write X_i for $\pi^{-1}(V_i)$. Then, by the converse statement, for each $\mathrm{Spec}\, A_i$ $(= V_i)$, there is some tensor power, $(\mathcal{L}^{\otimes m})^{\otimes q_i}$, which works for any given coherent sheaf, $\mathcal{F}_i$, on X_i. That is, $\mathcal{F}_i \otimes (\mathcal{L}^{\otimes m})^{\otimes q_i}$ is generated by its global sections. If $\mathcal{F}$ is a given coherent sheaf on X, write $\mathcal{F}_i$ for $\mathcal{F} \restriction X_i$ and take $\widetilde{q}$ to be the maximum of the q_i's. Then, $\mathcal{F} \otimes (\mathcal{L}^{\otimes m})^{\otimes \widetilde{q}} \restriction X_i$ is generated by its sections and, in the usual way, using Serre's extension of section lemma, we may increase $\widetilde{q}$ to some q so that $\mathcal{F} \otimes (\mathcal{L}^{\otimes m})^{\otimes q}$ is generated by its global sections over X. Hence, $\mathcal{L}$ is ample. $\square$

Proof of Lemma 7.25. Assume that $\mathcal{L}$ is very ample over S, then $\mathcal{L}$ induces an immersion, $j \colon X \to \mathbb{P}_S(\mathcal{E})$, for some QC $\mathcal{O}_X$-module, $\mathcal{E}$. Then, we have a surjection $\pi^*(\mathcal{E}) \longrightarrow \mathcal{L}$. However, there is a canonical factorization

$$\pi^*(\mathcal{E}) \longrightarrow \pi^*\pi_*(\mathcal{L}) \longrightarrow \mathcal{L}, \tag{$\dagger$}$$

and so, the homomorphism $\pi^*\pi_*(\mathcal{L}) \longrightarrow \mathcal{L}$ is surjective. Moreover, as j is an immersion, X is separated over S, hence, π is both separated and quasi-compact. But then, $\pi_*(\mathcal{L})$ is QC. And, lastly, condition (c) follows because the surjection ($\dagger$) gives rise to a surjection

$$\pi^*\mathrm{Sym}_S(\mathcal{E}) \longrightarrow \pi^*\mathrm{Sym}_S(\pi_*(\mathcal{L})) \longrightarrow \coprod_{n \geq 0} \mathcal{L}^{\otimes n}.$$

Hence, we obtain the diagram

$$\begin{tikzcd} & \mathbb{P}_S(\pi_*(\mathcal{L})) \arrow[dd] \\ X \arrow[ur] \arrow[dr] & \\ & \mathbb{P}_S(\mathcal{E}) \end{tikzcd}$$

showing that $X \longrightarrow \mathbb{P}_S(\pi_*(\mathcal{L}))$ is an immersion. Conversely, set $\mathcal{E} = \pi_*(\mathcal{L})$, which is QC by (a). Then, by (b) and (c) we obtain an immersion $j \colon X \to \mathbb{P}_S(\pi_*(\mathcal{L})) = \mathbb{P}_S(\mathcal{E})$. But then, by Theorem 7.10, we obtain the surjection

$$\pi^*\pi_*(\mathcal{L}) = \pi^*(\mathcal{E}) \longrightarrow \mathcal{L},$$

and $\mathcal{L}$ is $j^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. So, $\mathcal{L}$ is very ample. $\square$

7.4 Some Geometric Applications

In this section, we shall work over a given field, k. Whenever necessary, we shall assume that k is algebraically closed and perhaps of characteristic zero.

Take a scheme, X, over k and consider two Cartier divisors D and E on X. Let's assume that D and E are effective. When is it that D is linearly equivalent to E ($D \sim E$) for effective divisors on X?

Recall that $D \sim E$ means that

$$D - E = (f),$$

where $f \in \mathcal{M}\mathrm{er}(X)$ is some meromorphic function $f\colon X \to \mathbb{P}^1$. For simplicity, let's assume that X is irreducible over k. The graph, Γ_f, of f is then a Cartier divisor on $X \prod \mathbb{P}^1$ (Here, we have omitted the notation that our product is taken over $\mathrm{Spec}\, k$). Consider the projection $pr_2\colon X \prod \mathbb{P}^1 \to \mathbb{P}^1$. It can be checked that if f is nonconstant, then the graph Γ_f (as scheme) is flat over $\mathbb{P}^1$. (In fact, this is a local question on both X and $\mathbb{P}^1$ so we can assume X is irreducible and affine, say $X = \mathrm{Spec}\, A$. Then, $\Gamma_f = \mathrm{Spec}(A[T]/(T - f))$, where T is the coordinate on $\mathbb{A}^1$, and we see that $T - f$ is the local equation for Γ_f as Cartier divisor. But $pr_2\colon \Gamma_f \to \mathbb{P}^1$ is given algebraically by the map

$$k[T] \longrightarrow k[f(\xi_1, \ldots, \xi_N)]$$

(here, $\xi_1, \ldots, \xi_N$ are coordinates on X), and $k[f(\xi_1, \ldots, \xi_N)]$ is torsion-free as $k[T]$-module because f is nonconstant. But $k[T]$ is a P.I.D., so Γ_f is flat.) We have the closed immersion

$$X_{(0)} \hookrightarrow X \prod \mathbb{P}^1,$$

and $X_{(0)}$ is a C-divisor on $X \prod \mathbb{P}^1$ (with local equation $T = 0$), and similarly for

$$X_{(\infty)} \hookrightarrow X \prod \mathbb{P}^1.$$

Therefore, we have the intersection cycles $X_{(0)} \cdot \Gamma_f$ and $X_{(\infty)} \cdot \Gamma_f$, in which we consider these cycles as divisors on $X_{(0)}$ and $X_{(\infty)}$, respectively, whose local equations are f and $1/f$, respectively. We find that

$$f^{-1}(0) = X_{(0)} \cdot \Gamma_f,$$

and similarly,

$$f^{-1}(\infty) = X_{(\infty)} \cdot \Gamma_f.$$

So, $D - E = (f)$ implies that there is some divisor, Γ, on $X \prod \mathbb{P}^1$ with $D = X_{(0)} \cdot \Gamma$ and $E = X_{(\infty)} \cdot \Gamma$. (Of course, $\Gamma = \Gamma_f$.) The picture is shown in Figure 7.1.

Consequently, we find that $D - E$ is the boundary, $\partial\Gamma$, of Γ in the sense of homology. Linear equivalence is a special case of homology with linear base $\mathbb{P}^1$.

We can use $\mathbb{P}^N$ instead of $\mathbb{P}^1$. In this case, if f is a morphism, $X \longrightarrow \mathbb{P}^N$, and Γ_f is its graph as Cartier divisor in the scheme $X \prod \mathbb{P}^1$, then we may choose two points P_0 and P_∞, in $\mathbb{P}^N$ and consider the line $P_0 P_\infty$ in $\mathbb{P}^N$ to get a $\mathbb{P}^1$. By restricting our graph Γ_f to the subscheme $X \prod \mathbb{P}^1 \hookrightarrow X \prod \mathbb{P}^N$, we find the linear equivalence of the divisors $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$. Obviously, this geometric notion is susceptible of generalization: Take a morphism,

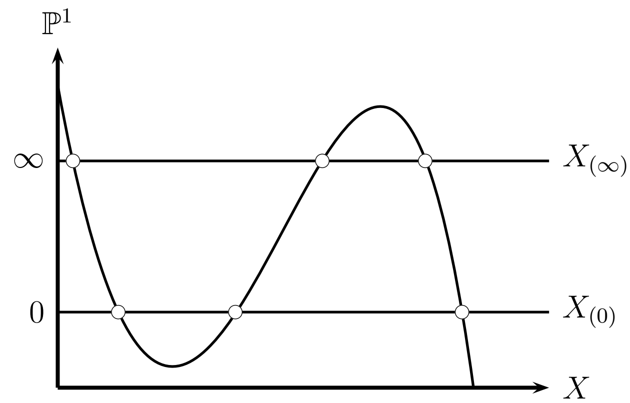

Figure 7.1: Interpreting linear equivalence as a special case of homology

$f\colon X \to S$, where X is a scheme (irreducible) and S is a curve. We have the graph Γ_f, again a C-divisor on $X \prod S$. We can say that

$$D \approx E \quad \text{iff} \quad X_P \cdot \Gamma_f = D \quad \text{and} \quad X_Q \cdot \Gamma_f = E,$$

for some $P, Q \in S$, where $X_P = pr_2^{-1}(P)$ and $X_Q = pr_2^{-1}(Q)$. This is *algebraic equivalence*. Note that linear equivalence implies algebraic equivalence; which, in turn, implies homological equivalence.

We can even look at a wider equivalence, *torsion equivalence*. Say that

$$D \approxeq E \quad \text{iff} \quad (\exists n > 0)(nD \approx nE).$$

Clearly, torsion equivalence and algebraic equivalence are the same after we tensor with $\mathbb{Q}$.

There is yet another equivalence: *Numerical equivalence*. For two divisors D and E on X, we say that D is *numerically equivalent* to E, denoted $D \equiv E$, iff

$$\deg(D \cdot C) = \deg(E \cdot C) \quad \text{for all curves } C \text{ in } X.$$

Recall that $D \cdot C$ means the Cartier divisor corresponding to the line bundle $\mathcal{O}_X(D) \restriction C$. And the degree $(\deg(D \cdot C))$ is just the degree of this line bundle.

Generally, we have strict implications (no converse implications)

$$D \sim E \Rightarrow D \approx E \Rightarrow D \approxeq E \Rightarrow D \equiv E,$$

and D numerically equivalent to E implies homological equivalence of D and E.

It is instructive to view all these equivalence relations for line bundles. Choose two line bundles, $\mathcal{L}$ and $\mathcal{M}$ on X. We know that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic means exactly the same as linear equivalence for divisors. For algebraic equivalence, consider the scheme $X \prod S$ (S is a curve, as above) and pick two points P and Q on S. Of course, X_P and X_Q will denote $pr_2^{-1}(P)$ (resp. $pr_2^{-1}(Q)$) and both are isomorphic to X itself. Then, $\mathcal{L}$ will be called algebraically equivalent to $\mathcal{M}$ iff there is some line bundle, $\mathcal{N}$, on $X \prod S$ so that under the isomorphisms $X \cong X_P$ and $X \cong X_Q$, we have

$$\mathcal{L} \cong \mathcal{N} \restriction X_P \quad \text{and} \quad \mathcal{M} \cong \mathcal{N} \restriction X_Q.$$

Write $\mathrm{Pic}^0(X)$ for the set of line bundles on X algebraically equivalent to 0. Torsion equivalence is now quite easy. Namely, $\mathcal{L} \approxeq \mathcal{M}$ iff there exists $m >> 0$ so that $\mathcal{L}^{\otimes m} \approx \mathcal{M}^{\otimes m}$. That is, torsion equivalence is the same as algebraic equivalence on a curve. Write $\mathrm{Pic}^\tau(X)$ for the set of all line bundles torsion equivalent to 0.

For numerical equivalence the situation is as sketched above. That is, $\mathcal{L}$ is numerically equivalent to $\mathcal{M}$ iff for all curves in X, say C, we have

$$\deg(\mathcal{L} \restriction C) = \deg(\mathcal{M} \restriction C).$$

The above notions of equivalence give rise to the decreasing filtration of $\mathrm{Pic}(X)$ by subgroups:

$$\mathrm{Pic}(X) \supseteq \mathrm{Pic}^\tau(X) \supseteq \mathrm{Pic}^0(X).$$

It turns out that in characteristic zero the group, $\mathrm{Num}(X)$, defined as

$$\mathrm{Num}(X) = \mathrm{Pic}(X)/\{\mathcal{L} \in \mathrm{Pic}(X) \mid \mathcal{L} \equiv 0\}$$

is also given by $\mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$.

Theorem 7.26 *(Néron-Severi) In characteristic zero, for a proper irreducible smooth variety, X, (reduced structure) over an algebraically closed field, k, the group* $\mathrm{Num}(X)$ *is a finitely generated abelian group (called the Néron-Severi group).*

Proof. We have not really discussed proper maps but the following proof using, as it does the notions of algebraic topology, permits us to also use the method of analysis because our field, k, may be assumed to be the complex numbers.[1] In this case, we have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

and by cohomology we get the exact sequence

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

Now, as X is proper over $\mathbb{C}$, it is compact as a topological space in the norm topology and it is a connected smooth manifold, by hypothesis. Suppose $\mathcal{L} \in \mathrm{Pic}(X)$ goes to zero under the

[1] The latter reduction is a consequence of what is sometimes called the "Lefschetz Principle": Since X is defined by finitely many polynomials on its finitely many affine open patches, by adjoining all the coefficients of these polynomials to the rationals, we obtain a field embeddable into the complex numbers over which X is defined. Consequently we may replace X by its base extension to $\mathbb{C}$–it remains irreducible because of our hypotheses and as we are in characteristic zero base extension incurs no nilpotent elements.

connecting homomorphism $\mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z})$. Remember that by Poincaré duality, the finitely generated group $H^2(X, \mathbb{Z})$ is isomorphic to $H_{2n-2}(X, \mathbb{Z})$, where n is the dimension of X. The connecting homomorphism associates to $\mathcal{L}$ the homology class of the divisor of $\mathcal{L}$. Since $\mathcal{L}$ goes to zero, this homology class is zero and by Poincaré duality the intersection of this class with any embedded (real) surface in X is zero (homologically). But then, for any algebraic curve, C, of X, we can consider C as a real surface in X and the degree of the intersection of this surface with the homology class of the divisor of $\mathcal{L}$ is just the degree of $\mathcal{L} \upharpoonright C$. As the homology class of the divisor of $\mathcal{L}$ vanishes, so does the degree of $\mathcal{L} \upharpoonright C$. This means that $\mathcal{L}$ is numerically equivalent to zero. We have proved that $\mathrm{Num}(X)$ is embedded in $H^2(X, \mathbb{Z})$. By compactness, $H^2(X, \mathbb{Z})$ is finitely generated and therefore so is $\mathrm{Num}(X)$. $\square$

Remark: The above argument is due to Severi. The contribution of Néron was to remove the analytic aspect of the argument and prove the theorem in much wider generality.

When X is a curve over $\mathbb{C}$, we can use the exponential sequence and just observe that the connecting homomorphism is the degree map. For, on a connected curve there is just one generator for $H^2(X, \mathbb{Z})$. Moreover, for curves, it turns out that $\approx$, $\approxeq$, $\equiv$ are all the same. Therefore

$$\mathrm{Pic}^\tau = \mathrm{Pic}^0 = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

Now, $H^1(X, \mathbb{Z})$ is a $2g$-dimensional lattice, where g is the (topological) genus of X. That is, when we view X as a real closed and orientable surface, g is the number of its handles. It follows that Pic^0 is an abelian Lie group, indeed it is a torus and further that $\mathrm{Num}(X) = \mathbb{Z}$.

In the case under consideration, namely when X is a proper irreducible variety over an algebraically closed field, we can be more precise about embedding X in projective space. That is, we can be more precise about when a line bundle (equivalently its linear system) is very ample on X over k. Let us restrict attention at first to linear systems of divisors because of the direct intuitive geometric feelings we obtain from them.

Definition 7.6 Let $\mathcal{D}$ be a linear system on X. We say that $\mathcal{D}$ *separates points* if for all pairs of closed points P, Q, there is some $D \in \mathcal{D}$ so that

$$P \in \mathrm{Supp}(D) \quad \text{and} \quad Q \notin \mathrm{Supp}(D).$$

In terms of line bundles, if our linear system is given by a subspace, $V \subseteq \Gamma(X, \mathcal{L})$, there must be some $\sigma \in \Gamma(X, \mathcal{L})$ so that

$$\sigma(P) = 0 \quad \text{and} \quad \sigma(Q) \neq 0.$$

We say that $\mathcal{D}$ *separates tangent vectors* if given any closed point $P \in X$ and any nonnull vector $\overrightarrow{t} \in T_P(X)$, there is some $D \in \mathcal{D}$ with $P \in D$ and $\overrightarrow{t} \notin T_P(D)$.

In terms of the line bundle, $\mathcal{L}$, and the subspace, $V \subseteq \Gamma(X, \mathcal{L})$, we need: $\sigma \in V$; $\sigma(P) \in \mathfrak{m}_P \mathcal{L}_P$ (i.e., $\sigma(P) = 0$), and yet, $\sigma(P) \notin \mathfrak{m}_P^2 \mathcal{L}_P$. More generally, the map $V \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2$ given by $\sigma \mapsto \sigma(P)$ should be surjective.

This second condition can be explained more intuitively as follows: Let P be any closed point, and let $D_1 \in \mathcal{D}$ so that $P \in D_1$. Pick some $\overrightarrow{t} \in T_P(D_1) \subseteq T_P(X)$ (where $\overrightarrow{t} \neq \overrightarrow{0}$). Then, there is to be some $D_2 \in \mathcal{D}$ so that $P \in D_2$, and yet, $\overrightarrow{t} \notin T_P(D_2)$. In other words, D_1 and D_2 are not tangent at P.

Notice that the conditions (1) and (2) are extremely local, namely almost punctual, on X. In fact, condition (2) just involves the "first infinitesimal neighborhood" of P. They give rise to Theorem 7.27:

Theorem 7.27 *(Embedding Criterion) Let X be a proper scheme over an algebraically closed field k, and let $\mathcal{D}$ be a linear system on X. Then, the following are equivalent.*

(1) $\mathcal{D}$ yields a closed immersion $X \hookrightarrow \mathbb{P}_k^n$ (for some n).

(2) (A) $\mathcal{D}$ separates points of X.

(B) $\mathcal{D}$ separates tangent vectors.

We will prove the embedding criterion a bit later, but now we want to give its application in the case that X is a curve over k. First, recall that $|D|$ consists of the projective space

$$\{g \in \mathcal{M}\mathrm{er}(X) \mid D + (g) \geq 0\}/(\text{mult. by nonzero constants}).$$

Instead of $|D|$, for questions of dimension, we may examine the vector space

$$\Gamma(X, \mathcal{O}_X(D)) = L(D) = \{g \in \mathcal{M}\mathrm{er}(X) \mid D + (g) \geq 0\} \cup \{0\}.$$

Now, in a natural way, $L(D-P)$ appears as subspace of $L(D)$. Namely, choose $g \in L(D-P)$, then $(g) + D - P \geq 0$. Hence,

$$(g) + D - P + P = (g) + D \geq 0,$$

i.e., $g \in L(D)$. Moreover, $g \in L(D)$ is not in $L(D-P)$ iff $(g) + D$ when expressed as a nonnegative sum of points has no P in it. I claim: Either $L(D-P) = L(D)$ or else their dimension differs by one.

This claim can be proved very simply from the Riemann-Roch theorem (below). However, it is instructive to prove it directly. Given two elements h and $\widetilde{h}$ of $L(D)$ with neither in $L(D-P)$, we know that $(h) + D \geq 0$ and $(\widetilde{h}) + D \geq 0$. As h (resp. $\widetilde{h}$) is not in $L(D-P)$, the divisors $(h) + D$ and $(\widetilde{h}) + D$ do not contain P. So,

$$\mathrm{ord}_P(h) + \mathrm{ord}_P(D) = \mathrm{ord}_P(\widetilde{h}) + \mathrm{ord}_P(D) = 0;$$

hence, $\mathrm{ord}_P(h) = \mathrm{ord}_P(\widetilde{h})$. The ratio $\widetilde{h}/h$ is then a local unit at P, hence near P. Let λ be the value of $\widetilde{h}/h$ at P, this is a nonzero complex number.

Consider the function $\widetilde{h}-\lambda h$ and pick any point Q of X. We know $\mathrm{ord}_Q(\lambda h)+\mathrm{ord}_Q(D) \geq 0$ and $\mathrm{ord}_Q(\widetilde{h}) + \mathrm{ord}_Q(D) \geq 0$. It follows that

$$\mathrm{ord}_Q(\widetilde{h} - \lambda h) + \mathrm{ord}_Q(D) \geq 0, \quad \text{for any } Q \text{ in } X.$$

However, by our choice of λ, $\mathrm{ord}_P(\widetilde{h} - \lambda h) > \mathrm{ord}_P(h)$, and we find

$$\mathrm{ord}_P(\widetilde{h} - \lambda h) + \mathrm{ord}_P(D) > \mathrm{ord}_P(h) + \mathrm{ord}_P(D) \geq 0,$$

which proves that P appears in the effective divisor $(\widetilde{h} - \lambda h) + D$. That is, $\widetilde{h} - \lambda h = f$ lies in $L(D - P)$. But then, $\widetilde{h} = \lambda h + f$ and so the dimension of $L(D)/L(D - P) = 1$.

In a similar (slightly more cumbersome) argument, we can prove that

$$\dim(L(D)/L(D - P - Q)) \leq 2$$

(even when $P = Q$). From these inequalities we deduce that the conditions (2)A and (2)B of Theorem 7.27 read in this case:

(A) For every closed point $P \in X$,

$$\dim|D - P| = \dim|D| - 1.$$

(B) For all closed points $P, Q \in X$ (where $P = Q$ is possible),

$$\dim|D - P - Q| = \dim|D| - 2.$$

The problem with (A) and (B), at our present state of knowledge, is that we have no criterion, yet, for deciding the truth of (A) and (B) in terms of D. This problem evaporates when we make use of the Riemann-Roch theorem for curves. We shall give a general proof of the Riemann-Roch theorem for projective varieties in Chapter 9 and another proof for curves in Section 7.6. Here, we just want to state and use the theorem for curves. Let X be a proper smooth connected curve over an algebraically-closed field k, and D be any divisor on X. We define the *Euler characteristic* of $\mathcal{O}_X(D)$ by

$$\chi(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)).$$

Then, the Serre duality theorem (c.f. Section 7.6) implies that

$$\chi(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \omega_X \otimes \mathcal{O}_X(-D)),$$

and *the statement of the Riemann-Roch theorem is that*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = \deg(D) + 1 - g,$$

where $g = p_g(X)$.

The origin of the theorem is with Riemann (1857) who proved that if $\deg(D) > 0$, then

$$\dim H^0(X, \mathcal{O}_X(mD)) = m\deg(D) + 1 - g, \quad \text{when } m >> 0.$$

It was his student Roch who supplied the error term betwen $H^0(X, \mathcal{O}_X(D))$ and $\deg(D) + 1 - g$. Note that we can restate the RR theorem as

$$\dim H^0(X, \mathcal{L}) - \dim H^0(X, \omega_X \otimes \mathcal{L}^D) = \deg(\mathcal{L}) + 1 - g, \quad \text{where } \mathcal{L} \text{ is a line bundle.}$$

Applying RR to $\mathcal{L} = \mathcal{O}_X$ and remembering that $H^0(X, \mathcal{O}_X) \cong k$, we find that

$$1 - \dim H^0(X, \omega_X) = 0 + 1 - g.$$

Consequently,

$$\dim H^0(X, \omega_X) = g = p_g.$$

If we let $\mathcal{L} = \omega_X$, since $\omega_X^D \otimes \omega_X \cong \mathcal{O}_X$, we get

$$\dim H^0(X, \omega_X) - 1 = \deg(\omega_X) + 1 - g.$$

Hence,

$$\deg(\omega_X) = 2g - 2.$$

Further,

$$\deg(T_X) = \deg(\omega_X^D) = 2 - 2g.$$

If we choose $\mathcal{L}$ with $\deg(\mathcal{L}) < 0$, we have $H^0(X, \mathcal{L}) = (0)$. Otherwise, there is some $\sigma \in H^0(X, \mathcal{L})$ so that $\sigma \neq 0$, and if $\mathcal{L} = \mathcal{O}_X(D)$, our σ corresponds to a function $F \in \mathcal{M}\mathrm{er}(X)$, so that $(F) + D \geq 0$. But

$$\deg((F) + D) = \deg(F) + \deg(D) = \deg(D) = \deg(\mathcal{L}) < 0.$$

This is impossible because $(F) + D \geq 0$.

Now apply this to the case when $\deg(\mathcal{L}) > 2g - 2$. In this case,

$$\deg(\omega_X \otimes \mathcal{L}^D) = 2g - 2 - \deg(\mathcal{L}) < 0,$$

and from the above we get the vanishing theorem:

Theorem 7.28 *(Vanishing theorem) On a proper smooth connected curve X of genus g, if $\deg(\mathcal{L}) > 2g - 2$, then*

(1) $$H^0(X, \omega_X \otimes \mathcal{L}^D) = (0)$$

and

(2) $$\dim H^0(X, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

Again, write $\mathcal{L}$ as $\mathcal{O}_X(D)$ and apply the vanishing theorem to the complete linear system $|D|$ (recall that $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$. Let's look at $\deg(D) \geq 2g$. Let $P \in X$, then,

$$\deg(D - P) \geq 2g - 1 > 2g - 2,$$

and the vanishing theorem implies that

$$\dim|D| = \deg(D) - g \quad \text{and} \quad \dim|D - P| = \deg(D - P) - g = \dim|D| - 1. \tag{$*$}$$

Thus, $|D|$ separates points, i.e., $|D|$ has no base points; we get a morphism $X \longrightarrow \mathbb{P}^{\deg(D)-g}$. If we assume slightly more, namely $\deg(D) \geq 2g + 1$, then not only do we have conclusions $(*)$ but also

$$\dim|D - P - Q| = \dim|D| - 2$$

because of

$$\deg(D - P - Q) \geq 2g - 1 > 2g - 2.$$

Therefore, D is very ample and $|D|$ gives an embedding into $\mathbb{P}^{\deg(D)-g}$. The degree of X as a subvariety of projective space in this embedding is $\deg(D)$.

Because of its importance, let's restate our conclusion as

Theorem 7.29 *(Embedding theorem for curves) If X is a proper smooth connected curve over an algebraically closed field (char. 0) and if D is a divisor of degree $\geq 2g + 1$ on X, then*

(1) D is very ample on X over k.

(2) $|D|$ embeds X into $\mathbb{P}^{\deg(D)-g}$.

(3) The degree of the image curve is exactly $\deg(D)$.

Before proving the embedding criterion we want to give some classical terminology from the theory of curves and further discuss the meaning of the cohomology of $\mathbb{P}^n$ with coefficients in a Serre twist of an ideal sheaf of $\mathcal{O}_{\mathbb{P}^n}$.

On a curve, X, a linear system of degree d and projective dimension r is called (according to Halphen and Max Noether around 1880) a g^r_d. For example, when a curve possesses a g^1_2 without base points, we get a morphism $X \longrightarrow \mathbb{P}^1$, which makes X into a degree 2 cover of $\mathbb{P}^1$. In this case, X is referred to as a *hyperelliptic* curve. If X possesses a g^1_n that separates points, then X is called an *n-gonal* curve and the g^1_n makes X into an n-fold cover of $\mathbb{P}^1$.

Let's work over an affine base, $X = \operatorname{Spec} A$. Assume that $Y \hookrightarrow \mathbb{P}^n_A$ is a closed embedding. We have the exact sequence

$$0 \longrightarrow \mathfrak{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Twisting by d, we obtain the exact sequence

$$0 \longrightarrow \mathfrak{I}_Y(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_Y(d) \longrightarrow 0.$$

Taking cohomology, we get

$$\begin{array}{l} 0 \longrightarrow H^0(\mathbb{P}^n, \mathfrak{I}_Y(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d)) \longrightarrow \\ \longrightarrow H^1(\mathbb{P}^n, \mathfrak{I}_Y(d)) \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \end{array} \tag{$\dagger$}$$

Recall that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ consists of forms of degree d in the ring $A[T_0, \ldots, T_n]$, and their zero's are the hypersurfaces of degree d. We have $\sigma \in H^0(\mathbb{P}^n, \mathfrak{I}_Y(d))$ iff σ is a form of degree d and $\sigma \upharpoonright Y = 0$. Thus,

$$H^0(\mathbb{P}^n, \mathfrak{I}_Y(d)) \neq (0)$$

iff Y is contained in some hypersurface of degree d. Consequently, Y *is nondegenerate (not contained in any hypersurface) iff*

$$H^0(\mathbb{P}^n, \mathfrak{I}_Y(d)) = (0).$$

Recall that $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)))$ is the complete linear system of hypersurfaces of degree d in $\mathbb{P}^n$. We have

$$\mathcal{O}_Y(d) = \mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(d) \upharpoonright Y.$$

This says that $\mathbb{P}(H^0(Y, \mathcal{O}_Y(d)))$ is the complete linear system on Y corresponding to the bundle $\mathcal{O}_Y(d)$ and the image in $\mathbb{P}(H^0(Y, \mathcal{O}_Y(d)))$ of the divisors in $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)))$ is the set of divisors coming from the intersection with divisors in $\mathbb{P}^n$ of the complete linear system $\mathcal{O}_{\mathbb{P}^n}(d)$. We will see in the next section that for $n \geq 2$,

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0)$$

for all d. From the exact sequence ($\dagger$) it follows that the sequence

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d)) \longrightarrow H^1(\mathbb{P}^n, \mathfrak{I}_Y(d)) \longrightarrow 0 \quad \text{is exact.}$$

Therefore, *the obstruction to the trace of the complete linear system of degree d on $\mathbb{P}^n$ (i.e., the intersection of the complete linear system with Y) being complete on Y is $H^1(Y, \mathfrak{I}_Y(d))$.*

Terminology.

(1) We say that Y is *d-normal in $\mathbb{P}^n$* if

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d)) \quad \text{is surjective.}$$

(2) We say that Y is *linearly normal* if it is 1-normal.

(3) We say that Y is *arithmetically normal (projectively normal in $\mathbb{P}^n$)* if it is d-normal for every $d \geq 0$.

If Y is smooth and A is an algebraically closed field, k, we proved in Section 7.1 that

$$\mathcal{O}_Y^\flat = \coprod_{d \geq 0} H^0(Y, \mathcal{O}_Y(d))$$

is the integral closure of the projective coordinate ring $k[T_0, \ldots, T_n]/\mathfrak{I}_Y^\flat$ in its fraction ring $= \mathcal{M}\mathrm{er}(Y)$. But then, the exact sequence

$$0 \longrightarrow \mathfrak{I}_Y^\flat \longrightarrow \mathcal{O}_{\mathbb{P}^n}^\flat \longrightarrow \mathcal{O}_Y^\flat \longrightarrow \coprod_{d \geq 0} H^1(\mathbb{P}^n, \mathfrak{I}_Y(d)) \longrightarrow 0$$

shows that $k[T_0, \ldots, T_n]/\mathfrak{I}_Y^\flat$ is isomorphic to $\mathcal{O}_Y^\flat =$ integral closure$(k[T_0, \ldots, T_n]/\mathfrak{I}_Y^\flat)$ iff Y is d-normal for every $d \geq 0$. Consequently, *Y is projectively normal (Y being smooth) iff its projective coordinate ring is integrally closed*. As localizations (at prime ideals) of integrally closed rings are themselves integrally closed, we see that *projective normality implies normality*. For smooth hypersurfaces, Y, their ideal sheaves, $\mathfrak{I}_Y$, are line bundles; so, we have

$$H^1(\mathbb{P}^n, \mathfrak{I}_Y) = (0) \quad \text{when } Y \text{ is a smooth hypersurface.}$$

Remarks:

(1) When k is an algebraically closed field and $n \geq 2$, then $H^1(\mathbb{P}^n, \mathfrak{I}_Y) = (0)$ iff Y is connected. To see this, consider the exact sequence

$$0 \longrightarrow \mathfrak{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and apply cohomology. We get

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^n, \mathfrak{I}_Y) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \longrightarrow H^0(Y, \mathcal{O}_Y)& \\ \longrightarrow H^1(\mathbb{P}^n, \mathfrak{I}_Y) \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = (0).& \end{aligned}$$

The last vanishing is because $n \geq 2$ and $\mathcal{O}_{\mathbb{P}^n}$ is a line bundle, as we will see in the next section. Now, $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$ and $H^0(\mathbb{P}^n, \mathfrak{I}_Y)$ consists of those elements of k (constants) which vanish on Y, i.e., 0. So, we deduce the exact sequence

$$0 \longrightarrow k \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^1(\mathbb{P}^n, \mathfrak{I}_Y) \longrightarrow 0.$$

Of course, $H^0(Y, \mathcal{O}_Y)$ contains the constants, k. The exact sequence shows that $H^0(Y, \mathcal{O}_Y)$ is exactly k when and only when $H^1(\mathbb{P}^n, \mathfrak{I}_Y) = (0)$. Thus, Y is connected iff $H^1(\mathbb{P}^n, \mathfrak{I}_Y) = (0)$.

In particular, hypersurfaces in $\mathbb{P}^n$ are connected. Next, complete intersections of connected varieties are connected (DX). This gives us the Bertini connectivity statement (of Theorem 6.13).

(2) Look at a linear system, $\mathcal{D} = \mathbb{P}(V)$, on X and the separation condition on tangent vectors. It says that if $V \subseteq \Gamma(X, \mathcal{L})$ (where $\mathcal{L} = \mathcal{O}_X(D)$), then the map

$$V \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2$$

is surjective for all points $P \in X$. Since V consists of the pullback of the linear forms on $\mathbb{P}(V)$, the statement: $V \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2$ onto, implies that

$$\mathfrak{m}_{\mathbb{P}(V),P} \longrightarrow \mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$$

is also onto. As these are cotangent spaces, this says: For every $P \in X(k)$ (here, k is an algebraically closed field), where P is a closed point, we have an embedding

$$T_P(X) \hookrightarrow T_P(\mathbb{P}(V))$$

and $X(k)$ is embedded in $\mathbb{P}(V)$, as in differential geometry.

We now return to the embedding criterion.

Proof of Theorem 7.27. If the linear system, $\mathcal{D}$, embeds X in $\mathbb{P}_k^N$, then the divisors of $\mathcal{D}$ correspond to the hyperplanes in $\mathbb{P}_k^N$ (DX). But, the collection of all hyperplanes clearly separates points and separates tangent vectors.

For the converse, assume that (A) and (B) hold. We know that (A) implies that $\mathcal{D}$ gives a morphism $\varphi\colon X \to \mathbb{P}_k^N$. Since X is proper, the image is closed.[2] We know from (A) that the morphism induced by $\mathcal{D}$ separates all closed points; thus, it is injective on closed points. Since X is proper, it is separated, and closed points are dense, which implies that $\varphi\colon X \to \mathbb{P}_k^N$ is injective. Then, $\varphi\colon X \to \mathbb{P}_k^N$ is an injective, continuous, closed map, and thus, it is a homeomorphism onto its image.

We still have to prove that it is an embedding. In the complex analytic case, (B) would finish the proof, by the implicit function theorem. In the algebraic case, we have to show that

$$\mathcal{O}_{\mathbb{P}^N} \longrightarrow \varphi_*\mathcal{O}_X$$

is surjective. This can be checked locally at P, for every closed point P. By the finiteness theorem (to be proved later), $\varphi_*\mathcal{O}_X$ is coherent as $\mathcal{O}_{\mathbb{P}^N}$-module. So, we have the following for $A = \mathcal{O}_{\mathbb{P}^N,P}$ and $B = (\varphi_*\mathcal{O}_X)_P = \mathcal{O}_{X,P}$ (because φ is a homeomorphism of X onto its image):

(1) $\overline{\varphi}\colon \kappa(A) \to \kappa(B)$ is an isomorphism, since

$$\kappa(A) = A/\mathfrak{m}_A \cong k \cong B/\mathfrak{m}_B = \kappa(B).$$

[2]One of the aspects of a proper morphism is that it is universally closed. However, since we are dealing with geometry over an algebraically closed field, the reader can consult Chapter 2 as well.

(2) $\mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective (by (B)).

(3) B is a finitely generated A-module (coherence of $\varphi_*\mathcal{O}_X$).

We will use (1)–(3) to prove an algebraic substitute for the desired consequence of the absent implicit function theorem.

Lemma 7.30 *Let A, B be noetherian local rings and $\theta\colon A \to B$ be a local homomorphism. Assume that*

(1) $\overline{\theta}\colon \kappa(A) \to \kappa(B)$ is an isomorphism.

(2) $\mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

(3) B is a finitely generated A-module.

Then, θ is surjective.

Proof. We give two proofs. The first proof uses Nakayama twice.

Proof 1. Consider the inclusion $\mathfrak{m}_A B \hookrightarrow \mathfrak{m}_B$. By (2), the map $\mathfrak{m}_A B \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is onto, and thus, $\mathfrak{m}_A B$ generates $\mathfrak{m}_B$ mod $\mathfrak{m}_B^2$. By Nakayama (for the ring B and module $\mathfrak{m}_A B$), we get

$$\mathfrak{m}_A B = \mathfrak{m}_B.$$

Now, (1) yields the isomorphism

$$\overline{\theta}\colon \kappa(A) \to B/\mathfrak{m}_A B,$$

since $\kappa(B) = B/\mathfrak{m}_B \cong B/\mathfrak{m}_A B$. Consider $1 \in A$, and look at B as A-module. By (3), it is f.g. But, by our reformulated property (1), the element 1 generates B modulo $\mathfrak{m}_A B$. By Nakayama (module B, ring A, generating element 1), we get that 1 generates B. This shows that $\theta\colon A \to B$ is onto. $\square$

We can apply Lemma 7.30 to finish the proof of Theorem 7.27. Now, for the second proof of Lemma 7.30.

Proof 2. We use the formal implicit theorem (Theorem 2.19). This is no surprise as we know the complex implicit function theorem is needed in the complex case. Complete A and B, getting $\widehat{A}$ and $\widehat{B}$. We have

$$\widehat{B} = \widehat{A} \otimes_A B$$

because B is f.g. We can express $\widehat{A}$ and $\widehat{B}$ as quotients of formal power series, where $\xi_1, \ldots, \xi_s$ generate $\mathfrak{m}_A$ and $\eta_1, \ldots, \eta_t$ generate $\mathfrak{m}_B$, and by (2), we get a map $\widehat{\theta}$, as shown below:

$$\begin{tikzcd} k[[\xi_1, \ldots, \xi_s]] \arrow[r] \arrow[d, "\widehat{\theta}"'] & \widehat{A} \arrow[r] \arrow[d] & 0 \\ k[[\eta_1, \ldots, \eta_t]] \arrow[r] & \widehat{B} \arrow[r] & 0 \end{tikzcd}$$

By the formal implicit function theorem (Theorem 2.19), using hypothesis (2), we find $\widehat{\theta}$ is surjective. Thus, the map $\widehat{A} \longrightarrow \widehat{B}$ is surjective. We have

$$\begin{CD} A @>>> \widehat{A} \\ @VVV @VVV \\ B @>>> \widehat{B} \end{CD}$$

where the right vertical map is surjective. As in Section 2.2 (using Krull's intersection theorem), we can prove that

$$(\text{image } \widehat{A}) \cap B = \text{image } A.$$

However, $(\text{image } \widehat{A}) = \widehat{B}$ implies that $(\text{image } A) = B$, and θ is onto. $\square$

Because of its importance we can reformulate, in terms of dimensions, the criteria for base point freeness and very ampleness on a curve:

Theorem 7.31 *Let C be a proper, smooth curve over an algebraically closed field k. If D is a Cartier divisor on C, then:*

(1) $|D|$ has no base point iff for every $P \in C(k)$,

$$\dim |D - P| = \dim |D| - 1.$$

(2) $|D|$ is very ample iff for all $P, Q \in C(k)$,

$$\dim |D - P - Q| = \dim |D| - 2.$$

Proof. Consider any closed point $P \in C(k)$. Then, $\mathcal{O}_C(-P)$ is the ideal sheaf of P and $\mathcal{O}_P$, the sheaf of functions at P, is the skyscraper sheaf, $\kappa(P)$, at P. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow \mathcal{O}_C \longrightarrow \kappa(P) \longrightarrow 0.$$

Twist it by $\mathcal{O}_C(D)$. We get the exact sequence

$$0 \longrightarrow \mathcal{O}_C(D - P) \longrightarrow \mathcal{O}_C(D) \longrightarrow \kappa(P) \longrightarrow 0.$$

Apply cohomology, to get

$$0 \longrightarrow H^0(C, \mathcal{O}_C(D - P)) \longrightarrow H^0(C, \mathcal{O}_C(D)) \longrightarrow \kappa(P).$$

Now, $\kappa(P) = k$ because k is algebraically closed; so

$$\dim H^0(C, \mathcal{O}_C(D - P)) \geq \dim H^0(C, \mathcal{O}_C(D)) - 1.$$

That is,

$$\dim |D| \leq \dim |D - P| + 1. \tag{$*$}$$

Just as in the discussion following Theorem 7.27, the map $|D - P| \longrightarrow |D|$ is given by $\Delta \mapsto \Delta + P$, and $|D - P|$ consists exactly of those $D' \in |D|$ so that $P \in D'$. Our condition on dimensions is equivalent to $|D - P| \neq |D|$ by $(*)$ and just as before, this means P is not a base point of $|D|$. This proves (1). Note that we have already proved this directly in the discussion following Theorem 7.27.

To prove (2), we first assume that $P \neq Q$. The condition on dimension holds iff P and Q each make the dimension drop by exactly one. By (1), this holds if Q is not a base point for $|D - P|$. But, Q is not a base point for $|D - P|$ iff there is some $D' \in |D - P|$, with $P \in D'$ but $Q \notin \mathrm{Supp}(D')$. This means that $|D|$ separates P and Q.

Now, assume that $P = Q$. Again, P is not a base point for $|D - P|$. This means that there is some $D' \in |D|$ so that P appears in D' with multiplicity 1.

Claim. The point P has multiplicity 1 in D' iff $\dim(T_P(D')) = 0$.

Let f be the local equation for D' at P. So, f must vanish at P and have a nonzero linear term at P iff the multiplicity of P is 1 in D'. The tangent space $T_P(D')$ is cut out from $T_P(C)$ by the vanishing of the linear term of f (i.e., df). Thus, there is a nonzero linear term iff

$$\dim(T_P(C)) = \dim(T_P(D')) + 1.$$

But C is a smooth curve, so that $\dim(T_P(C)) = 1$, and thus

$$\dim(T_P(D')) = 0.$$

Therefore, our condition on multiplicity 1 means that any nonzero $\overrightarrow{t} \in T_P(C)$ is not in $T_P(D')$, i.e., tangent vectors are separated, and the proof is complete. $\square$

Corollary 7.32 *The linear system $|D|$ is ample on C iff $\deg(D) > 0$.*

7.5 Finiteness Theorems for Projective Morphisms

Having studied the various properties of projective varieties and projective schemes which do not explicitly use cohomology, but, as seen with the statement of Riemann-Roch, really do involve cohomology, it is time to face squarely the issue of cohomology for projective schemes. At first, we deal with the simplest case: That of projective n-space over a ring. Here, the results of Chapter 4, Sections 4.1, 4.2 and 4.3; most especially Proposition 4.7 and Corollary 4.15 are the main tools. The reader is urged to review this material, now.

However, a slight generalization of these results is necessary in order to take into account the grading. This generalization is analogous to the generalization mentionned in Theorem 7.3. In this case, we shall give it in more explicit detail.

We fix a scheme, X, a line bundle, $\mathcal{L}$, on X and we form the $\mathcal{O}_X$-module

$$\mathcal{L}^\flat = \coprod_{n\in\mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n}) \stackrel{\mathrm{def}}{=} B.$$

Pick elements, $f_0, \ldots, f_r$ in B_1 (we could pick the f_i of any degree d_i, but the case $d_i = 1$ for all i is the most important), and write $U_i = X_{f_i}$, and $U = \bigcup_i U_i$. We examine the open cover $\{U_i \longrightarrow U\}$ and for every quasi-coherent $\mathcal{O}_X$-module, $\mathcal{F}$, we set

$$\begin{aligned} H^p(U, \mathcal{F}(*)) &= \coprod_{n\in\mathbb{Z}} H^p(U, \mathcal{F}\otimes\mathcal{L}^{\otimes n}) \\ H^p(\{U_i \longrightarrow U\}, \mathcal{F}(*)) &= \coprod_{n\in\mathbb{Z}} H^p(\{U_i \longrightarrow U\}, \mathcal{F}\otimes\mathcal{L}^{\otimes n}). \end{aligned}$$

Notice that each of $H^p(U, \mathcal{F}(*))$ and $H^p(\{U_i \longrightarrow U\}, \mathcal{F}(*))$ is a graded B-module. We also obtain the graded B-module

$$\coprod_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}\otimes\mathcal{L}^{\otimes n}) \stackrel{\mathrm{def}}{=} M.$$

(Observe that when X is $\mathrm{Proj}(C)$ where C is a graded ring, we could take $\mathcal{L} = \mathcal{O}_X(1)$ and then M would just be $\mathcal{F}^\flat$, Also, we have taken the elements $f_0, \ldots, f_r$ from B_1, which in the projective case would just be C_1 and one sees that this is exactly how we generalize the Serre section theorem (Theorem 7.3).)

If X is nerve-finite, we find (as usual, cf. Proposition 4.6) that

$$\Gamma(U_{i_0\cdots i_p}, \mathcal{F}(*)) = H^0(U_{i_0\cdots i_p}, \mathcal{F}(*)) = M_{i_0\cdots i_p},$$

and, as in Chapter 4,

$$M_{i_0\cdots i_p} = \varinjlim_n M^{(n)};$$

but, now this is an isomorphism of *graded B-modules*. To see this, we need to define the degree of elements in $M^{(n)}$. Say, ξ is the image in $\varinjlim_n M^{(n)}$ of an element, x, of degree d from $M^{(n)} = M$ with its usual grading. Remember that $M^{(n)}$ maps to $M_{i_0\cdots i_p}$ *via*

$$x \mapsto \frac{x}{(f_{i_0}\cdots f_{i_p})^n}.$$

On the righthand side, the image of x has obvious degree $d - n(p+1)$; hence, when ξ comes from x in $M^{(n)}$, we give ξ the degree: $\deg(x) - n(p+1)$. One checks that this is well-defined. Then, by construction, the map

$$\varinjlim_n M^{(n)} \longrightarrow M_{i_0\cdots i_p}$$

is a global isomorphism. Similarly, $\varinjlim_{n} C^p_n(M)$ is a graded B-module and there is a graded isomorphism

$$C^p(\{U_i \longrightarrow U\}, \mathcal{F}(*)) \cong \varinjlim_{n} C^p_n(M).$$

For the Koszul complex $\varinjlim_{n} K^{\bullet}(\overrightarrow{f^n}, M)$, we again define degrees as above: If $g(e_{i_0} \wedge \cdots \wedge e_{i_p})$ is a cochain whose values lies in the degree d part of m, we give g the degree $d - n(p+1)$. Once, made, this is well-defined. There results the graded isomorphism

$$C^p(\{U_i \longrightarrow U\}, \mathcal{F}(*)) \cong C^{p+1}((\overrightarrow{f}), M) = \varinjlim_{n} K^{p+1}(\overrightarrow{f^n}, M).$$

Of course, this is a chain map and so from Chapter 4 (Proposition 4.7 and Corollary 4.15), we get:

Proposition 7.33 *If X is a nerve-finite scheme and B, $\mathcal{F}$ and M are as given above, then there exist canonical, functorial isomorphisms (of degree 0) of graded B-modules*

$$H^p(\{U_i \longrightarrow U\}, \mathcal{F}(*)) \cong H^{p+1}((\overrightarrow{f}), M), \quad \textit{for all } p \geq 1.$$

and a functorial exact sequence of graded B-modules

$$0 \longrightarrow H^0((\overrightarrow{f}), M) \longrightarrow M \longrightarrow H^0(\{U_i \longrightarrow U\}, \mathcal{F}(*)) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0.$$

Moreover, if each X_{f_i} is affine, then the above isomorphisms and exact sequence become

$$H^p(U, \mathcal{F}(*)) \cong H^{p+1}((\overrightarrow{f}), M), \quad \textit{for all } p \geq 1.$$

and

$$0 \longrightarrow H^0((\overrightarrow{f}), M) \longrightarrow M \longrightarrow H^0(U, \mathcal{F}(*)) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0$$

(again, as graded, degree 0, maps).

Corollary 7.34 *If B is a ggr and $f_0, \ldots, f_r \in B_1$ are elememnts which generate B over B_0, set $X = \mathrm{Proj}(B)$, and choose a graded B-module, M, then*

$$H^p(X, M^{\sharp}(*)) \cong H^{p+1}((\overrightarrow{f}), M), \quad \textit{for all } p \geq 1$$

and

$$0 \longrightarrow H^0((\overrightarrow{f}), M) \longrightarrow M \longrightarrow H^0(U, M^{\sharp}(*)) \longrightarrow H^1((\overrightarrow{f}), M) \longrightarrow 0 \qquad (\dagger)$$

is exact. Here, $H^p(X, M^{\sharp}(*))$ means $\coprod_n H^p(X, M(n)^{\sharp})$.

Proof. We take $\mathcal{F} = M^\sharp$ in our proposition, observe that the X_{f_i} are indeed affine, and as the $f_{i_0}, \ldots, f_{i_r}$ generate, we find $X = \bigcup_i X_{f_i}$. $\square$

Note that as the functors $M \rightsquigarrow \Gamma(U_{i_0\cdots i_p}, M^\sharp(*))$ are exact, and as our isomorphisms of the cochain complexes are chain maps, we actually get isomorphisms which make the obvious diagrams in the long exact cohomology sequence commute.

Finally, we can apply our corollary to prove:

Theorem 7.35 *(Serre) Let A be a ring and let $X = \mathbb{P}^N_A$. Then, the following properties hold:*

(1) For every d,

$$H^r(X, \mathcal{O}_X(d)) = (0) \quad \textit{for all } r,\ 0 < r < N.$$

(2) There is a natural isomorphism $t\colon H^N(X, \omega_X) \to A$, called the trace map.

(3) For every d, the A-modules $H^0(X, \mathcal{O}_X(d))$ and $H^N(X, \mathcal{O}_X(d))$ are finitely generated free modules and the natural morphism

$$H^0(X, \mathcal{O}_X(d)) \otimes H^N(X, \mathcal{O}_X(-d) \otimes \omega_X) \longrightarrow H^N(X, \omega_X) \cong A$$

is a perfect duality of A-modules.

Proof. Consider $B = A[T_0, \ldots, T_N]$, then $\mathbb{P}^N_A = \mathrm{Proj}(B)$ and we take for M the module B itself. Of course, we take $f_j = T_j$ and what we must compute is

$$H^\bullet((\overrightarrow{T}), B).$$

But, the sequence $T_0, \ldots, T_N$ is regular for B; hence, by Koszul result (Proposition 4.3)

$$H^p((\overrightarrow{T}), B) = (0) \quad \text{if } p \neq N+1, \quad H^{N+1}(\overrightarrow{T^n}, B) = B/(T_0^n, \ldots, T_N^n)B.$$

Observe that $H^{N+1}(\overrightarrow{T^n}, B)$ is a free A-module on the monomials $T_0^{j_0}, \ldots, T_N^{j_N}$, where $0 \leq j_l \leq n$ and $0 \leq l \leq N$. Now, we take the limit as $n \mapsto \infty$. Remember that

$$B/(T_0^m, \ldots, T_N^m)B \longrightarrow B/(T_0^n, \ldots, T_N^n)B$$

is given by multiplication by $(T_0^{n-m}, \ldots, T_N^{n-m})$. To identify the limit, the easiest thing to do is to observe that we have an isomorphism

$$B/(T_0^m, \ldots, T_N^m)B \xrightarrow{\sim} \frac{1}{T_0^m \cdots T_N^m} B/B, \tag{$*$}$$

where $\frac{1}{T_0^m \cdots T_N^m} B$ is considered a submodule of $B\left[\frac{1}{T_0}, \ldots, \frac{1}{T_N}\right]$. Pick any tuple, $(p_0, \ldots, p_N)$ of *positive* integers, take $N \geq \max\{p_j\}$ and set

$$\xi^{(n)}_{p_0\cdots p_N} = \text{image of } T_0^{n-p_0} \cdots T_N^{n-p_N} \text{ in } B/(T_0^n, \ldots, T_N^n)B.$$

On the one hand, the degree of this element in $B^{(n)}$ is $(N+1)n - \sum_{i=0}^{N} p_i - (N+1)n$, and on the other hand, the degree of its image $(1/(T_0^{p_0}\cdots T_N^{p_N}))$ in $\frac{1}{T_0^n\cdots T_N^n}B/B$ is manifestly $-\sum_{i=0}^{N} p_i$. What this shows is that the map

$$\varinjlim_n B/(T_0^n\cdots T_N^n) \longrightarrow B\left[\frac{1}{T_0},\ldots,\frac{1}{T_N}\right]/B$$

is an isomorphism (of degree 0) of graded B-modules, where on the lefthand side we use the "correct" notion of degree. Consequently, $H^{N+1}((\overrightarrow{T}), B)$ is free on the generators $1/(T_0^{p_0}\cdots T_N^{p_N})$, where each $p_j \geq 0$ and the degree of this generator is its obvious degree: $-\sum_{i=0}^{N} p_i$.

And now, by untwisting the components degree by degree, we obtain the conclusion:

$$H^p(X, \mathcal{O}_X(d)) = (0) \quad \text{if} \quad p \neq 0, N. \tag{$\dagger$}$$

The canonical map

$$\alpha\colon B \longrightarrow \mathcal{O}_X^\flat = H^0(X, \mathcal{O}_X(*))$$

is bijective (that the Koszul sequence (†) gives the map α is a consequence of the fact that Čech cohomology computes the "real" cohomology). Consequently, $H^0(X, \mathcal{O}_X(*))$ is a free A-module and $H^0(X, \mathcal{O}_X(d))$ is free on the basis $T_0^{p_0}\cdots T_N^{p_N}$, where $0 \leq p_j \leq d$ and $p_0 + \cdots + p_N = d$.

For $p = N+1$, our argument above shows that

$$H^{N+1}(X, \mathcal{O}_X(d)) = \begin{cases} (0) & \text{if } d \geq -N \\ \text{free with basis } \frac{1}{T_0^{p_0}\cdots T_N^{p_N}}, \text{ where} & \\ p_j \geq 0 \text{ and } p_0 + \cdots + p_N = |d| & \text{if } d \leq -(N+1). \end{cases}$$

Since $\omega_X \cong \mathcal{O}_X(-(N+1))$ (cf. Chapter 6), the module $H^{N+1}(X, \omega_X)$ is free on one generator, $\frac{1}{T_0\cdots T_N}$. Notice that, this generator is exactly what one gets from the Euler sequence which computes $\Omega_X = \mathcal{O}_X(-(N+1))$. Consequently, the map announced in statement (2) is indeed a natural isomorphism.

And finally, the modules $H^0(X, \mathcal{O}_X(d))$ (free on $T_0^{i_0}\cdots T_N^{i_N}$, where $i_0 + \cdots + i_N = d$) and $H^N(X, \omega_X \otimes \mathcal{O}_X(-d))$ (free on $\frac{1}{(T_0\cdots T_N)}\left(\frac{1}{(T_0^{q_0}\cdots T_N^{q_N})}\right)$, where $q_j \geq 0$ and $q_0 + \cdots + q_N = d$) are obviously dual under the pairing

$$T_0^{i_0}\cdots T_N^{i_N} \otimes \frac{1}{(T_0\cdots T_N)}\left(\frac{1}{(T_0^{i_0}\cdots T_N^{i_N})}\right) \mapsto \frac{1}{(T_0\cdots T_N)}.$$

That this pairing is the cup-product is a simple computation. $\square$

Remark: When $\mathcal{F}$ is a locally free sheaf and X is a projective nonsingular scheme over an algebraically closed field k, then

$$\mathcal{F}^D \otimes \omega_X$$

is called the *Serre dual* of $\mathcal{F}$. We will sometimes denote it as $\mathcal{F}^{(SD)}$.

Recapping statement (3) of the above theorem, we have the most basic case of Serre's duality theorems: If $X = \mathbb{P}^N_A$, then the natural pairing

$$H^0(X, \mathcal{O}_X(d)) \otimes H^N(X, \mathcal{O}_X(d)^{(SD)}) \longrightarrow H^N(X, \omega_X) \cong A,$$

a perfect duality of free, finitely generated A-modules.

Using Theorem 7.35, we get Serre's form of the finiteness theorem for projective morphisms:

Theorem 7.36 *(Serre) Let S be a locally noetherian scheme and X be a projective schexme over S, with structure morphism $\pi\colon X \to S$. Write $\mathcal{O}_X(1)$ for the pullback of $\mathcal{O}_{\mathbb{P}^N_S}(1)$ under the embedding $X \hookrightarrow \mathbb{P}^N_S$. Then, the following properties hold:*

(1) (Finiteness) For every coherent sheaf $\mathcal{F}$ on X, the derived functors, $R^p\pi_*\mathcal{F}$, are coherent on S, for all p.

(2) If S is noetherian,

$$\pi^*\pi_*\mathcal{F}(n) \longrightarrow \mathcal{F}(n) \quad \text{is surjective for all } n >> 0.$$

(3) Assume that S is noetherian, and let $\mathcal{L}$ be ample on X. For every coherent sheaf, $\mathcal{F}$, on X, there exists some $n_0(\mathcal{F})$ so that for all $n \geq n_0(\mathcal{F})$, we have

$$R^p\pi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = (0) \quad \text{if } p > 0 \text{ (the “vanishing theorem”).}$$

Proof. All statements are local on S. Thus, we may assume that $S = \operatorname{Spec} A$, where A is noetherian.

(1) We have a closed immersion $i\colon X \to \mathbb{P}^N_A$, and $\mathcal{F}$ is coherent on X. Since i is a closed immersion, we know that $i_*\mathcal{F}$ is also coherent (c.f. Proposition 4.21). By the Leray spectral sequence, we have

$$H^p(\mathbb{P}^N_A, R^q i_*\mathcal{F}) \Longrightarrow H^\bullet(X, \mathcal{F}).$$

But i is affine, which implies that the spectral sequence degenerates (c.f. Chapter 4, Corollary 4.12) and thus

$$H^p(\mathbb{P}^N_A, i_*\mathcal{F}) \cong H^p(X, \mathcal{F}), \quad \text{for all } p \geq 0. \tag{\dagger}$$

Now, we also showed that

$$R^p\pi_*\mathcal{F} = \widetilde{H^p(X, \mathcal{F})}.$$

Thus, we need only show that $H^p(X, \mathcal{F})$ is a finitely generated module, and (†) says that we may assume that $X = \mathbb{P}^N_A$. By Corollary 7.19, there is a vector bundle $\mathcal{E}$ such that $\mathcal{E} = \coprod_M \mathcal{O}_X(-q)$ and the sequence

$$\coprod_M \mathcal{O}_X(-q) = \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{is exact if } q >> 0.$$

Let $\mathcal{K} = \mathrm{Ker}\,(\mathcal{E} \longrightarrow \mathcal{F})$. From the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{††}$$

and the fact that $\mathcal{F}$ and $\mathcal{E}$ are coherent, we find that $\mathcal{K}$ is also coherent. We will finish the proof by using descending induction on p. If $p > N$, we get

$$H^p(X, \mathcal{F}) = (0),$$

which is obviously f.g. Assume by induction that for all coherent sheaves $\mathcal{G}$, the module $H^p(X, \mathcal{G})$ is f.g. over A. Apply cohomology to (††). We get

$$\cdots \longrightarrow H^{p-1}(X, \mathcal{E}) \longrightarrow H^{p-1}(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{K}) \longrightarrow \cdots \tag{$*$}$$

However,

$$H^r(X, \mathcal{E}) = \coprod_M H^r(X, \mathcal{O}_X(-q)),$$

and by (1) and (3) of Theorem 7.35, we find that $H^r(X, \mathcal{E})$ is f.g. for all r. By the induction hypothesis, $H^p(X, \mathcal{K})$ is f.g., and since A is noetherian, this implies $H^{p-1}(X, \mathcal{F})$ is f.g., and completes the induction.

(2) We have

$$\pi_* \mathcal{F}(n) = H^0(\widetilde{X, \mathcal{F}(n)}), \quad \text{as } A\text{-module}.$$

We obtain the exact sequence

$$\coprod_{H^0(X, \mathcal{F}(n))} \mathcal{O}_X \longrightarrow \pi^* \pi_* \mathcal{F}(n) \longrightarrow 0.$$

However, if n is large enough, $\mathcal{F}(n)$ is generated by its sections, which says that

$$\coprod_{H^0(X, \mathcal{F}(n))} \mathcal{O}_X \longrightarrow \mathcal{F}(n) \longrightarrow 0 \quad \text{is exact, for } n >> 0.$$

But now, the diagram

$$\begin{tikzcd} \coprod_{H^0(X,\mathcal{F}(n))} \mathcal{O}_X \arrow[r] & \pi^*\pi_*\mathcal{F}(n) \arrow[r] \arrow[d] & 0 \\ \coprod_{H^0(X,\mathcal{F}(n))} \mathcal{O}_X \arrow[r] & \mathcal{F}(n) \arrow[r] & 0 \end{tikzcd} \qquad n >> 0$$

shows immediately that $\pi^*\pi_*\mathcal{F}(n) \longrightarrow \mathcal{F}(n)$ is also surjective for $n >> 0$.

(3) We first show (3) when $\mathcal{L} = \mathcal{O}_X(1)$, which is very ample. First, I claim that if n is sufficiently large,
$$H^n(X, \mathcal{F}(n)) = (0).$$
To see this, note that, for large n, the sheaf $\mathcal{F}(n)$ is generated by its global sections, that is, $\coprod_M \mathcal{O}_X \longrightarrow \mathcal{F}(n) \longrightarrow 0$ is exact, where M is some finite set. We can twist even further and deduce
$$\coprod_M \mathcal{O}_X(r) \longrightarrow \mathcal{F}(n+r) \longrightarrow 0 \quad \text{is exact, for all } r \geq 0. \tag{$*$}$$
Now, apply cohomology to the exact sequence which results from $(*)$ when we adjoin the kernel, K, and look at the highest dimension term:
$$\coprod_M H^N(X, \mathcal{O}_X(r)) \longrightarrow H^N(X, \mathcal{F}(n+r)) \longrightarrow H^{N+1}(X, K) = (0).$$
However, by Theorem 7.35, the lefthand side of the above exact sequence is dual to
$$\coprod_M H^0(X, \mathcal{O}_X(-r) \otimes \omega_X^D).$$
But, if r is large, the zeroth cohomology will vanish as negative degree bundles never have global sections. Hence our assertion is true for N. That is, we have proved there exists an in integer, $\nu(N)$, so that if $N \geq \nu(N)$, then $H^n(X, \mathcal{F}(n)) = (0)$.

Now, by descending induction we will prove: If if $p > 0$, there exists an integer $\nu(p)$ so that $H^p(X, \mathcal{F}(n)) = (0)$ whenever $n \geq \nu(p)$. The case $p = N$ has been established. If it is true for all p and coherent sheaves, $\mathcal{F}$, we apply cohomology to the exact sequence
$$0 \longrightarrow K \longrightarrow \coprod_M H^0(X, \mathcal{O}_X(r)) \longrightarrow \mathcal{F}(n+r) \longrightarrow 0$$
and look at the terms
$$\cdots \longrightarrow \coprod_M H^{p-1}(X, \mathcal{O}_X(r)) \longrightarrow H^{p-1}(X, \mathcal{F}(n+r)) \longrightarrow H^p(X, K) \longrightarrow \cdots.$$
Here, $p - 1 \geq 1$, and the lefthand side vanishes if $r > \nu(p)$, the function ν depending on K. Consequently, we deduce $H^{p-1}(X, \mathcal{F}(n+r)) = (0)$. We have only finitely dimensions involved, so take $n_0(\mathcal{F}) = \max\{\nu(1), \ldots, \nu(N)\}$, then $H^p(X, \mathcal{F}(n)) = (0)$ if $p > 0$ and $n \geq n_0(\mathcal{F})$.

We now consider any ample line bundle $\mathcal{L}$. There is some m so that $\mathcal{L}^{\otimes m}$ is very ample. We can repeat the above argument, and we get the vanishing if we twist by powers of $\mathcal{L}^{\otimes m}$. Apply this to the coherent sheaves $\mathcal{F}_k = \mathcal{F} \otimes \mathcal{L}^{\otimes k}$, for $k = 0, \ldots, m-1$. Then, by using Serre's argument involving the division algorithm (c.f. the proof of Proposition 7.20), we complete the proof. $\square$

It turns out that the vanishing is characteristic of ampleness.

Proposition 7.37 *Let A be a noetherian ring, X a proper scheme over* $\mathrm{Spec}\, A$, *and $\mathcal{L}$ a line bundle over X. Then, $\mathcal{L}$ is ample on X iff the vanishing statement holds, i.e., for every coherent sheaf $\mathcal{F}$ on X, there is some $n_0(\mathcal{F})$ so that*

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) = (0) \quad \textit{for } p > 0 \textit{ and all } t \geq n_0(\mathcal{F}).$$

Proof. We know by part (3) of the previous theorem that ampleness implies the vanishing statement; so, all we need prove is the converse. That is, assuming vanishing of the higher cohomology and given any coherent sheaf, $\mathcal{F}$, on X we must find an integer, $n_0(\mathcal{F})$, so that for all $n \geq n_0(\mathcal{F})$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global secions.

The beginning of the argument is in fact a repetition of Serre's argument from the characterization of affine schemes by cohomology (cf. Theorem 4.22).

Pick a closed point, P, of X and let $\mathfrak{I}_P$ be the ideal of $\{P\}$. From the exact sequence

$$0 \longrightarrow \mathfrak{I}_P \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{I}_P = k(P) \longrightarrow 0$$

we get by tensoring with $\mathcal{F}$ the exact sequence

$$0 \longrightarrow \mathfrak{I}_P\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow k(P) \otimes \mathcal{F} \longrightarrow 0. \tag{$*$}$$

Here, $k(P)$ is a skyscraper sheaf and $\mathfrak{I}_P\mathcal{F}$ is the image of $\mathfrak{I}_P \otimes \mathcal{F}$ in $\mathcal{F}$. Now, tensor the sequence $(*)$ with $\mathcal{L}^{\otimes n}$, which leaves the sequence exact because $\mathcal{L}$ is locally free:

$$0 \longrightarrow \mathfrak{I}_P\mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow k(P) \otimes \mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow 0$$

and apply cohomology. As $\mathfrak{I}_P\mathcal{F}$ is coherent, there is an $n_0(\mathfrak{I}_P\mathcal{F})$ so that $n \geq n_0(\mathfrak{I}_P\mathcal{F})$ implies that

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(X, k(P) \otimes \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow 0 \quad \text{is exact}$$

(here, we have used the vanishing hypothesis). As $k(P)$ is a skyscraper sheaf, the A-module $\Gamma(X, k(P) \otimes \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is the module $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_P \otimes_{\mathcal{O}_{X,P}} k(P)$. By Nakayama's lemma, the stalk $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_P$ is generated by the global sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0(\mathfrak{I}_P\mathcal{F})$. Now, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is coherent, so there exists a neighborhood, $U(P, n)$, of P (depending on n) so that the global sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generate the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \restriction U(P, n)$; for all $n \geq n_0(\mathfrak{I}_P\mathcal{F})$ (cf. Appendix A, Section A.7, Proposition A.26 or Proposition A.18). Apply this argument to the case $\mathcal{F} = \mathcal{O}_X$. Then, if we write $n_0(P)$ for $n_0(\mathfrak{I}_P\mathcal{F})$ and $V(P)$ for $U(P, n_0(P))$, we find that

$$\mathcal{L}^{\otimes n_0(P)} \restriction V(P) \quad \text{is generated by its global sections.}$$

Secondly, apply the argument above successively to the sheaves $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$, for $r = 0, 1, \ldots,$ $n_0(P) - 1$. We obtain the neighborhoods, $U(P, r)$, where $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$ is generated by its global sections. Consequently, on

$$U(P) = V(P) \cap U(P, 0) \cap U(P, 1) \cap \cdots \cap U(P, n_0(P) - 1)$$

all the sheaves:

$$\mathcal{L}^{\otimes n_0(P)}, \mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{\otimes (n_0(P)-1)}$$

are generated by their global sections. And now the familiar argument with the division algorithm will help us finish the proof. Namely, given $n \geq n_0(P)$, we write

$$n = k\, n_0(P) + r, \quad 0 \leq r \leq n_0(P) - 1,$$

so that

$$\mathcal{F} \otimes \mathcal{L}^{\otimes n} = (\mathcal{F} \otimes \mathcal{L}^{\otimes r}) \otimes (\mathcal{L}^{\otimes n_0(P)})^{\otimes k}. \tag{$\dagger$}$$

Each of the sheaves on the righthand side of (†) is generated by its global sections on $U(P)$. Thus, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections on $U(P)$ for all $n \geq n_0(P)$.

Lastly, the open sets $U(P)$ cover X and, as X is finite type over $\mathrm{Spec}\, A$, it is quasi-compact. For the finitely many $U(P)$ necessary to cover X we take $n_0(\mathcal{F})$ to be the supremum of the various $n_0(P)$. This $n_0(\mathcal{F})$ clearly works. $\square$

Remark: If we put together all the results of this section we se that we proved the

Theorem 7.38 *Suppose $\pi\colon X \to S$ is a proper morphism and S is a noetherian scheme. Then for any line bundle, $\mathcal{L}$, on X the following are equivalent:*

(1) $\mathcal{L}$ is ample on X.

(2) There exists an n so that $\mathcal{L}^{\otimes n}$ is very ample on X.

(3) There exists N so that for every $n \geq N$, the sheaf $\mathcal{F}^{\otimes n}$ is very ample on X.

(4) For every coherent sheaf $\mathcal{F}$ on X there is an integer $n_0(\mathcal{F})$ so that

$$R^p\pi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes t}) = (0), \quad \text{for } p > 0 \text{ and all } t \geq n_0(\mathcal{F}).$$

7.6 Serre Duality Theorem, Some Applications And Complements

Let X be a proper scheme over a field, k, and assume that $\dim X = n$. Note that $H^n(X, -)$ is a right exact functor. Indeed, given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

we get

$$H^n(X, \mathcal{F}') \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}'') \longrightarrow H^{n+1}(X, \mathcal{F}') = (0).$$

Look at the cofunctor
$$\mathcal{F} \mapsto H^n(X, \mathcal{F})^D.$$
This is a left exact cofunctor. Is it representable? In other words, is there a coherent sheaf ω_X^0 and some $t \in H^n(X, \omega_X^0)^D$ (i.e., $t\colon H^n(X, \omega_X^0) \to k$, a *trace map*), so that
$$\mathrm{Hom}\,(\mathcal{F}, \omega_X^0) \cong H^n(X, \mathcal{F})^D \quad \text{functorially (via } t\text{)}.$$

From general facts about representable functors, if it exists, (ω_X^0, t) is unique up to unique isomorphism. Grothendieck proved, in the sixties, that (ω_X^0, t) always exists for X proper over $\mathrm{Spec}\, k$ and a quicker proof was given by Pierre Deligne. The sheaf ω_X^0 is called the *dualizing module* for X, and t is called the *trace map*. The reason for this terminology will become apparent, soon. For now, observe that (3) of Serre's computation of the cohomology of $\mathbb{P}^n_k$ (Theorem 7.35) appears to imply that $\omega_{\mathbb{P}^n_k}$ is the dualizing module for $\mathbb{P}^n_k$. We say it appears to show it because in our formulation of Theorem 7.35, the duality is proved only for $\mathcal{F} = \mathcal{O}_X(d)$. We can repair that immediately:

Theorem 7.39 *(Serre Duality for $\mathbb{P}^n_k$) Let $X = \mathbb{P}^n_k$, with k a field, then*

(1) There is an isomorphism $H^n(X, \omega_X) \overset{t}{\cong} k$.

(2) For every coherent $\mathcal{O}_X$-module, $\mathcal{F}$, there is a functorial (in $\mathcal{F}$)) pairing
$$\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \otimes_k H^{n-l}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \cong k$$
which is a perfect duality of finite dimensional vector spaces.

Proof. Statement (1) follows immediately from (2) of Serre's computation for projective space (Theorem 7.35).

To prove (2), first examine the case $l = 0$. It asserts that there is a functorial pairing
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \otimes_k H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \cong k$$
whic is a perfect duality of finite dimensional vector spaces. For the existence of the pairing, note that if $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$, then $H^n(\varphi)\colon H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \cong k$ is a linear functional on $H^n(X, \mathcal{F})$. Hence, our pairing is
$$(\varphi, \xi) \mapsto t(H^n(\varphi)(\xi)) \in k.$$
When $\mathcal{F} = \mathcal{O}_X(q)$, we find that
$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(q), \omega_X) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(q)^D \otimes \omega_X) \\
&= \Gamma(X, \mathcal{O}_X(q)^D \otimes \omega_X) \\
&= H^0(X, \mathcal{O}_X(-q) \otimes \omega_X).
\end{aligned}$$

And then, part (3) of Serre's computation of the cohomology of projective space shows that the duality is perfect for $l = 0$ and $\mathcal{F} = \mathcal{O}_X(q)$. Obviously, it is therefore valid when $l = 0$ and $\mathcal{F}$ is a coproduct of $\mathcal{O}_X(q)$'s. Now, by the corollary of Serre's generation theorem (Corollary 7.22) there is an exact sequence

$$(\mathcal{O}_X(-q'))^r \longrightarrow (\mathcal{O}_X(-q))^s \longrightarrow \mathcal{F} \longrightarrow 0.$$

We already know that $H^n(X, -)^D$ is a left-exact cofuncor, so apply it to the above exact sequence, and obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^n(X,\mathcal{F})^D \arrow[r] & H^n(X,(\mathcal{O}_X(-q))^s)^D \arrow[r] & H^n(X,(\mathcal{O}_X(-q'))^r)^D \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{O}_X}((\mathcal{O}_X(-q))^s,\omega_X) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{O}_X}((\mathcal{O}_X(-q'))^r,\omega_X) \arrow[u]
\end{tikzcd}$$

in which the righthand two vertical arrows are isomorphisms by what has been proved. The five-lemma new completes the case: $l = 0$, any $\mathcal{F}$.

I claim the functors

$$\mathcal{F} \rightsquigarrow \mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$$

are coeffaceable cofunctors for $l > 0$. Once again, this follows from the corollary of Serre's generation theorem. Namely, we know there is an exact sequence

$$(\mathcal{O}_X(-q))^r \longrightarrow \mathcal{F} \longrightarrow 0, \quad \text{for } q >> 0 \text{ and some } r,$$

hence

$$\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{O}_X(-q), \omega_X) = H^l(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-q), \omega_X)).$$

(cf. Proposition 5.5). However, ω_X is coherent, so by Serre's vanishing theorem (Theorem 7.36 part (3)) the cohomology groups $H^l(X, \omega_X(q))$ vanishes if $q >> 0$. But, our cohomology group $H^l(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-q), \omega_X))$ is just $H^l(X, \omega_X(q))$. Thus, $\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{O}_X(-q), \omega_X)$ vanishes when $q >> 0$. This proves the coeffeability of $\mathrm{Ext}^l_{\mathcal{O}_X}(-, \omega_X)$. Therefore, the cofunctor

$$\mathcal{F} \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$$

is a universal δ-cofunctor, and for such functors any map in degree 0 to another δ-cofunctor extends uniquely to all degrees. But, for the δ-cofunctor $H^n(X, -)^D$, we have a map from the case $l = 0$ proved above. This gives the functorial map

$$\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \longrightarrow H^{n-l}(X, \mathcal{F})^D.$$

If we now prove that the δ-cofunctor $H^n(X, -)^D$ is itself universal, then we find a map in the opposite direction

$$H^{n-l}(X, \mathcal{F})^D \longrightarrow \mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$$

inverting the previous map—that is, the theorem will be proved.

Once again coeffaceability follows from Serre's generation theorem: We have the exact sequence

$$(\mathcal{O}_X(-q))^r \longrightarrow \mathcal{F} \longrightarrow 0, \quad \text{for } q >> 0 \text{ and some } r,$$

and the consequent map

$$H^{n-l}(X, \mathcal{F})^D \longrightarrow (H^{n-l}(X, \mathcal{O}_X(-q))^r)^D$$

in which the latter group is (0) if $l > 0$ and $q >> 0$. This is the coeffaceability of our cofunctor. $\square$

Corollary 7.40 *(Of the proof) If X is a projective scheme over a field, k, then the existence of a dualizing module, ω_X^0, implies that there are functorial maps*

$$\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \longrightarrow H^{n-l}(X, \mathcal{F})^D.$$

Proof. All we used in the above argument for the existence of these maps when $X = \mathbb{P}^n_k$ was that the cohomology of $\mathcal{O}_X(q)$ is zero in positive degrees when $q >> 0$ and Serre's generation theorem for X, plus the functorial property of dualizing modules (namely, that it represents the left-exact cofunctor $\mathcal{F} \rightsquigarrow H^n(X, \mathcal{F})^D$). $\square$

Our problem now is to prove the existence of dualizing modules for projective schemes over fields. This is:

Theorem 7.41 *If $X \longrightarrow \mathbb{P}^N_k$ is a closed immersion and X has codimension r in $\mathbb{P}^N_k$, then the sheaf*

$$\omega_X^0 = \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$$

is a dualizing sheaf for X.

Proof. First, we prove $\mathcal{E}xt^l_{\mathbb{P}^N_k}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$ vanishes for $l < r$, for every coherent $\mathcal{O}_X$-module, $\mathcal{F}$. Write $\mathcal{G}$ for the latter sheaf (for fixed $l < r$), and observe that $\mathcal{G}$ is coherent. By the generation theorem for $q >> 0$, the sheaf $\mathcal{G}(q)$ will be generated by its sections. If then all global sections of $\mathcal{G}(q)$ are zero for $q >> 0$, we would find $\mathcal{G}(q) = (0)$. But, $\mathcal{G}(q) = \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^N_k}(q)$, and $\mathcal{O}_{\mathbb{P}^N_k}(q)$ is locally free, so $\mathcal{G}$ would have to be zero. Therefore, we are reduced to proving that the global sections of $\mathcal{G}(q)$ vanish when $q >> 0$.

Now,

$$\Gamma(\mathbb{P}^N_k, \mathcal{G}(q)) = H^0(\mathbb{P}^N_k, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(q))).$$

I claim that

$$H^0(\mathbb{P}^N_k, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(q))) = \mathrm{Ext}^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(q)).$$

To see this, first consider the case of $\mathcal{F} = \mathcal{O}_{\mathbb{P}^N_k}(z)$, for some integer z; this is not a sheaf on X, but because of its form we can analyze both sides and then make the argument for $\mathcal{F}$ below. For simplicity of notation, we abbreviate $\mathbb{P}^N_k$ as $\mathbb{P}$.

The sheaf $\mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q))$ vanishes for all $l > 0$ and all q and z because $\mathcal{O}_{\mathbb{P}}$ is locally free. Hence, the spectral sequence

$$H^p(\mathbb{P}, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q))) \Longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q))$$

degenerates for all q and z. We obtain the isomorphisms

$$H^p(\mathbb{P}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q))) \cong \mathrm{Ext}^p_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q)). \tag{$*$}$$

Now,

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q)) = \omega_{\mathbb{P}}(q - z),$$

so that if $q >> 0$ the groups

$$H^p(\mathbb{P}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q)))$$

all vanish for $p > 0$. We have proved that

$$\mathrm{Ext}^p_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(z), \omega_{\mathbb{P}}(q)) = (0), \quad \text{if } p > 0 \text{ and } q >> 0.$$

So, our claim is proved for $l > 0$ and the special sheaf $\mathcal{O}_{\mathbb{P}}$, for any z. We need to check the case $l = 0$, when $p = 0$. But, $(*)$ yields our equation at once.

To treat the case of an arbitary coherent sheaf on X we will use induction on l. The case $l = 0$ is trivial. Now, the sheaf $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}}$-module as well as a coherent $\mathcal{O}_X$-module. By the corollary of Serre's generation theorem (Corollary 7.22), we have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{P}}(-z)^s \longrightarrow \mathcal{F} \longrightarrow 0, \tag{$\dagger$}$$

for some z and some $s \geq 1$. If we know our equation for all coherent $\mathcal{F}$ in the cases $< l$, then applying $\mathcal{E}xt^{\bullet}_{\mathcal{O}_{\mathbb{P}}}$ to $(\dagger)$, we get

$$\mathcal{E}xt^t(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)) \to \mathcal{E}xt^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \to \mathcal{E}xt^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q)) \to \mathcal{E}xt^{t+1}(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)).$$

When $t \geq 1$, the ends vanish if $q >> 0$, so

$$\mathcal{E}xt^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \cong \mathcal{E}xt^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q))$$

and if $t = 0$ we have

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow \mathcal{H}om(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)) \longrightarrow \mathcal{H}om(\mathcal{K}, \omega_{\mathbb{P}}(q)) \\ \longrightarrow \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow (0) \end{aligned} \tag{A}$$

is exact when $q >> 0$. For the global Ext, $(\dagger)$ yields

$$\mathrm{Ext}^t(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)) \to \mathrm{Ext}^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \to \mathrm{Ext}^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q)) \to \mathrm{Ext}^{t+1}(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)).$$

When $t \geq 1$, the ends vanish if $q >> 0$ (as proved above) and we get

$$\mathrm{Ext}^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \cong \mathrm{Ext}^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q)), \quad \text{for } t \geq 1 \text{ and } q >> 0.$$

When $t = 0$, we get

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\,(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow \mathrm{Hom}\,(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)) \longrightarrow \mathrm{Hom}\,(\mathcal{K}, \omega_{\mathbb{P}}(q)) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow (0) \qquad (B) \end{aligned}$$

if $q >> 0$.

Look at the induction step for $t \geq 1$, first. The sheaf $\mathcal{K}$ is coherent and we have the commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}, \mathcal{E}xt^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \arrow[r, "\sim"] \arrow[d] & H^0(\mathbb{P}, \mathcal{E}xt^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q)) \arrow[d] \\ \mathrm{Ext}^t(\mathcal{K}, \omega_{\mathbb{P}}(q)) \arrow[r, "\sim"] & \mathrm{Ext}^{t+1}(\mathcal{F}, \omega_{\mathbb{P}}(q)) \end{tikzcd}$$

and on the lefthand side, by induction, the vertical arrow is an isomorphism; so, the induction step works provided $t \geq 1$.

In the case $t = 0$, the exact sequence (A) can be split into two exact sequences

$$0 \longrightarrow \mathcal{H}om\,(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow \mathcal{H}om\,(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q)) \longrightarrow \mathrm{cok} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{cok} \longrightarrow \mathcal{H}om\,(\mathcal{K}, \omega_{\mathbb{P}}(q)) \longrightarrow \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}}(q)) \longrightarrow (0).$$

We apply cohomology to these two exact sequences (over $\mathbb{P} = \mathbb{P}^N_k$) and twist a little more to kill the terms involving $H^1(\mathbb{P}, -)$. Then, we can resplice the cohomology sequences and obtain

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}, \mathcal{H}om\,(\mathcal{F}, \omega_{\mathbb{P}}(q))) \longrightarrow H^0(\mathbb{P}, \mathcal{H}om\,(\mathcal{O}_{\mathbb{P}}(-z)^s, \omega_{\mathbb{P}}(q))) \\ \longrightarrow H^0(\mathbb{P}, \mathcal{H}om\,(\mathcal{K}, \omega_{\mathbb{P}}(q))) \longrightarrow H^0(\mathbb{P}, \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}}(q))) \longrightarrow (0), \qquad q >> 0. \qquad (C) \end{aligned}$$

For the global Hom, we have (B), which, combined with (C) yields an obvious commutative diagram which provides the induction step from 0 to 1. So, finally, our claim

$$\Gamma(\mathbb{P}^N_k, \mathcal{G}(q)) = H^0(\mathbb{P}^N_k, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(q))) = \mathrm{Ext}^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(q))$$

for all $l \geq 0$ and $q >> 0$ is proved.

By Serre Duality for $\mathbb{P}^N_k$, the group on the righthand side is dual to $H^{N-l}(\mathbb{P}^N_k, \mathcal{F}(-q))$. However, $\mathcal{F}(-q)$ has support on the closed set X, therefore we have the isomorphism

$$H^{N-l}(\mathbb{P}^N_k, \mathcal{F}(-q)) \cong H^{N-l}(X, \mathcal{F}(-q)).$$

But, $l < r$, so $N - l > N - r = \dim(X)$; so, the last cohomology group vanishes. This proves

$$\mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) = (0), \quad \text{for } l < r. \tag{$*$}$$

Consider the isomorphism of functors

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{G}, \mathcal{H})) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}).$$

Set $\mathcal{G} = \mathcal{O}_X$, then we obtain

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{H})) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{H}). \tag{$**$}$$

The lefthand side of $(**)$ is a composed functor and so $(**)$ yields the spectral sequence of composed functors

$$E_2^{p,l} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{H})) \Longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{H}).$$

When $\mathcal{H} = \omega_{\mathbb{P}}$, equation $(*)$ shows $E_2^{p,l} = (0)$ if $l < r$. So, a picture of level 2 of the spectral sequence is:

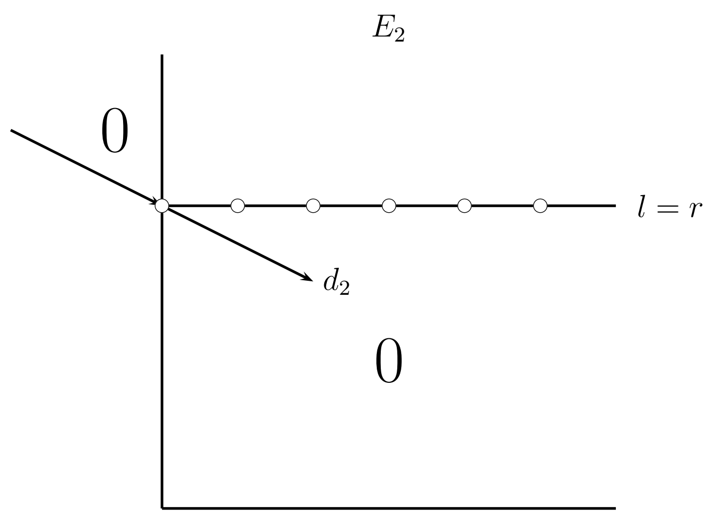

Figure 7.2: The E_2 level of the spectral sequence

Hence, $E_2^{0,r} = E_3^{0,r} = \cdots = E_\infty^{0,r}$.

Now, look at the ∞-level of the spectral sequence (see Figure 7.3).

All the dots on the line $p + l = r$ with $l < r$ are zero. But, the dots on this line are the filtration quotients of $\mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}})$, and so, we find

$$\mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}) \cong E_\infty^{0,r} = E_2^{0,r} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0)$$

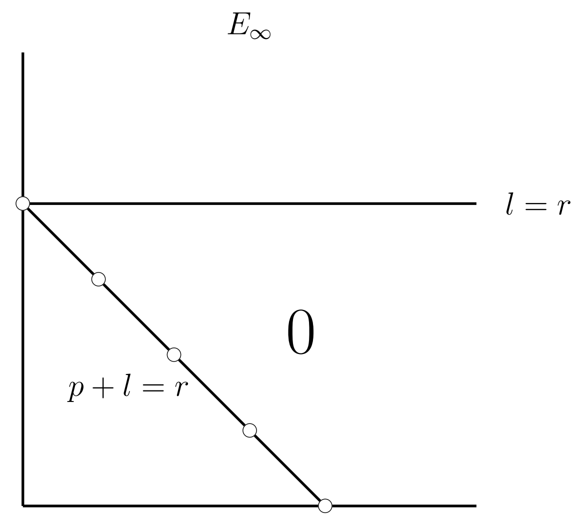

Figure 7.3: The E_∞ level of the spectral sequence

(by definition of ω_X^0), functorially in $\mathcal{F}$. We apply Serre Duality for $\mathbb{P} = \mathbb{P}_k^N$, it gives

$$\mathrm{Ext}^r_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}) \quad \text{is dual to} \quad H^{N-r}(\mathbb{P}, \mathcal{F}).$$

The latter group is just $H^d(X, \mathcal{F})$ because $\mathcal{F}$ has support in X, and $r = \mathrm{codim}(X)$; of course, $d = \dim(X)$. Consequently

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \quad \text{is dual to} \quad H^d(X, \mathcal{F}),$$

that is,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \cong H^d(X, \mathcal{F})^D.$$

Now, set $\mathcal{F} = \omega_X^0$ and take the identity map on the lefthand side to obtain the element $t \in H^d(X, \omega_X^0)^D$—the trace map. Therefore, indeed, (ω_X^0, t) is a dualizing sheaf. $\square$

Remark: Corollary 7.40 shows that the existence of ω_X^0 begins the duality (case $p = 0$) and gives the pairing

$$\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \otimes H^{n-p}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X^0) \xrightarrow{t} k,$$

which pairing is uniquely determined by the case $p = 0$. The problem is in extending the duality from $p = 0$ to the case when $p > 0$. We shall see below that this is directly related to the part of the second level of the spectral sequence

$$E_2^{p,l} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{O}_X, \omega_{\mathbb{P}_k^N})) \Longrightarrow \mathrm{Ext}^\bullet_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{F}, \omega_{\mathbb{P}_k^N}),$$

above the line $l = r$, shown shaded in Figure 7.4.

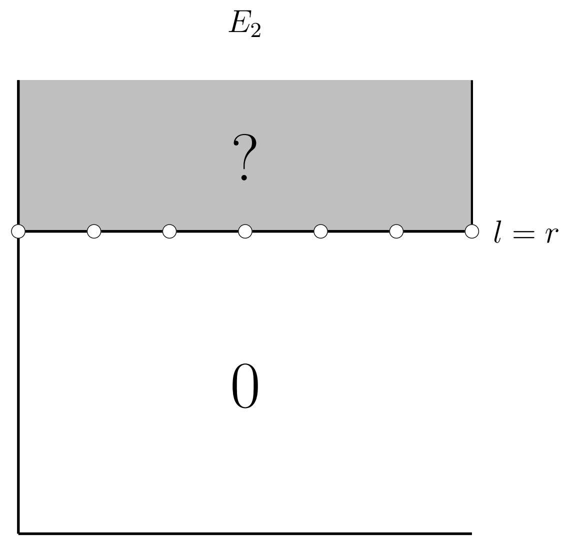

Figure 7.4: The E_2 level of the spectral sequence; obstruction to the duality

Along the line $l = r$, the groups are $E_2^{p,r}$, where

$$E_2^{p,r} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0),$$

and these are exactly the groups appearing in the duality pairing.

In all the following remarks, we will need to recall some facts about the notion of *depth* of a module. All these are quite standard commutative algebra (cf. Matsumura [40] Serre [52], Eisenbud[14]). The reader should skip these remarks now and proceed immediately to the Serre Duality theorem which follows, pausing only in the proof when the relevant fact is necessary. We have included more facts than actually needed; it seems reasonable to do this as they are part of a piece.

Remarks:

(1) If A is a ring and M is an A-module, then an *M-sequence* is just what we called a regular sequence in Definition 4.1. That is, a sequence of elements $a_1, \ldots, a_r$ of A so that a_i is not a zero divisor for the module $M/(a_1M + \cdots + a_{i-1}M)$. If all the a_i lie in an ideal, $\mathfrak{A}$, then we use the locution *M-sequence from* $\mathfrak{A}$. Because we have discussed M-sequences in Chapter 4 in connection with the Koszul complex, the following results will not be a surprise:

If A is noetherian and M is f.g. then the following are equivalent:

(a) There exists an M-sequence $(a_1, \ldots, a_r)$ from A.

(b) $\mathrm{Ext}^l_A(A/\mathfrak{A}, M) = (0)$ for $l < r$.

(c) $\mathrm{Ext}^l_A(N, M) = (0)$ for $l < r$ provided N is f.g. and $\mathrm{Supp}(N) \subseteq V(\mathfrak{A})$.

In the following remarks we always assume that M is f.g. and A is noetherian.

(2) The notion of a *maximal M-sequence from* $\mathfrak{A}$ should be clear, its cardinality (also called the length of this M-sequence) depends only on M and $\mathfrak{A}$. This length is the $\mathfrak{A}$-*depth* of M, denoted $\mathrm{depth}_{\mathfrak{A}}(M)$. We have

$$\mathrm{depth}_{\mathfrak{A}}(M) = n \quad \text{iff} \quad \mathrm{Ext}^l_A(A/\mathfrak{A}, M) = (0) \quad \text{for } l < r \text{ and } \quad \mathrm{Ext}^n_A(A/\mathfrak{A}, M) \neq (0).$$

(3) When A is local and $\mathfrak{A}$ is its maximal ideal, we merely write $\mathrm{depth}(M)$ instead of $\mathrm{depth}_{\mathfrak{A}}(M)$. We have

$$\mathrm{depth}(M_{\mathfrak{p}}) = (0) \quad \text{iff} \quad \mathfrak{p} \in \mathrm{Ass}(M) \quad \text{and} \quad \mathrm{depth}(M_{\mathfrak{p}}) \geq \mathrm{depth}_{\mathfrak{p}}(M).$$

In fact,

$$\mathrm{depth}_{\mathfrak{A}}(M) = \inf\{\mathrm{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in V(\mathfrak{A})\}.$$

(4) If A is local and $M \neq (0)$, then

$$\mathrm{depth}(M) \leq \dim(A/\mathfrak{p}) \quad \text{for every } \mathfrak{p} \in \mathrm{Ass}(M).$$

In particular

$$\mathrm{depth}(M) \leq \dim(M)$$

(where $\dim(M)$ is by definition the dimension of $A/\mathrm{Ann}(M)$). So the depth is ∞ iff $M = (0)$.

(5) Usually, the notion of M-sequence from $\mathfrak{A}$ depends on the order of the elements chosen but, if $\mathfrak{A} \subseteq \mathcal{J}(A)$ (= Jacobson radical of A), then an M-sequence from A is independent of the order of its elements.

(6) Because of (4), when A is local, special attention is paid to those M for which $\mathrm{depth}(M) = \dim(M)$. These are the *Cohen-Macaulay* modules (we also include $M = (0)$ as Cohen-Macaulay). Let us write C-M, instead of Cohen-Macaulay. A local ring is C-M if it is so as module over itself. When M is C-M over A, we have:

(a) $\mathrm{depth}(M) = \dim(A/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$; thus, *$M$ has no embedded primes* (this is the geometric meaning of Cohen-Macaulay).

(b) If $M \neq (0)$ and $a_1, \ldots, a_r$ is part of an M-sequence, then $M/(a_1M + \cdots + a_{i-1}M)$ is again C-M and

$$\dim(M/(a_1M + \cdots + a_{r-1}M)) = \dim(M) - r.$$

(The local geometric content of this statement should be clear.)

(c) For all $\mathfrak{p} \in \mathrm{Spec}\, A$, the module $M_{\mathfrak{p}}$ is C-M over $A_{\mathfrak{p}}$.

(d) If, for any ideal, $\mathfrak{A}$, we define

$$\mathrm{ht}(\mathfrak{A}) = \inf\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in V(\mathfrak{A})\},$$

then when A is C-M

$$\mathrm{ht}(\mathfrak{A}) + \dim(A/\mathfrak{A}) = \dim(A).$$

(7) In any noetherian ring, consider ideals, $\mathfrak{A}$, generated by r elements and having $\mathrm{ht}(\mathfrak{A}) = r$. Call such an ideal *unmixed* provided $A/\mathfrak{A}$ has no embedded primes. This is a directly geometric notion about $V(\mathfrak{A}) = \mathrm{Spec}(A/\mathfrak{A})$, as a subscheme of $\mathrm{Spec}\, A$—it is the origin of this whole collection of ideas and was initiated by F.S. Macaulay (about 1915). We say that *unmixedness holds in* A iff all $\mathfrak{A}$ (r generators, $\mathrm{ht}(\mathfrak{A}) = r$) are unmixed. This is a local property: Unmixedness holds in A iff unmixedness holds in $A_{\mathfrak{p}}$ (for all $\mathfrak{p}$) iff unmixedness holds in $A_{\mathfrak{m}}$ (for all maximal $\mathfrak{m}$).

And now, we have: The following are equivalent:

(a) Unmixedness holds for A.

(b) $A_{\mathfrak{p}}$ is C-M for all $\mathfrak{p} \in \operatorname{Spec} A$.

(c) $A_{\mathfrak{m}}$ is C-M for all $\mathfrak{m} \in \operatorname{Max} A$.

(8) If we agree to call a noetherian ring a *Cohen-Macaulay* ring whenever unmixedness holds in it (by (7), this agrees with the use of C-M in the local case), then Macaulay's theorem is the following:

Theorem 7.42 *(Macaulay) If A is a C-M ring then so is $A[X_1, \ldots, X_n]$. In particular, the rings $k[X_1, \ldots, X_n]$ (k a field) and $\mathbb{Z}[X_1, \ldots, X_n]$ are C-M.*

(9) If A is noetherian local and M is a f.g. A-module, suppose that $\mathrm{projdim}(M) < \infty$ (recall $\mathrm{projdim}(M) \leq \alpha$ iff $\mathrm{Ext}^l_A(M, -) = (0)$ when $l > \alpha$). Then,

$$\mathrm{projdim}(M) + \mathrm{depth}(M) = \mathrm{depth}(A).$$

This equality is due to Auslander and Buchsbaum [3]. When A is C-M, remark (6) shows that we have the equality

$$\mathrm{projdim}(M) + \mathrm{depth}(M) = \dim(A).$$

(10) Every regular ring is C-M.

(11) Serre [52] called attention to the following conditions on noetherian rings, A:

$$(S_r): \quad (\forall \mathfrak{p} \in \operatorname{Spec} A)(\mathrm{depth}(A_{\mathfrak{p}} \geq \min\{r, \mathrm{ht}(\mathfrak{p})\}).$$

A ring, A, is C-M iff it has condition (S_r) for every r.

Theorem 7.43 *Let X be a projective scheme over an algebraically closed field k. Write $d = \dim(X)$, suppose $\mathcal{O}_X(1)$ is a very ample sheaf, and denote by ω_X^0 the dualizing sheaf for X. Then, the following are equivalent:*

(1) X is Cohen-Macaulay and equidimensional (all components have the same dimension). (Being Cohen-Macaulay means that all its local rings are Cohen-Macaulay.)

(2) In the spectral sequence, $E_2^{p,l} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})) \Longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}})$, the terms $E_2^{p,l}$ vanish for $l > r$ and all $p \geq 0$.

(3) (Serre Duality) For all coherent sheaves, $\mathcal{F}$, on X, the pairings

$$\mathrm{Ext}^l(\mathcal{F}, \omega_X^0) \otimes_k H^{d-l}(X, \mathcal{F}) \longrightarrow H^d(X, \omega_X^0) \xrightarrow{t} k$$

are perfect duality pairings of finite dimensional vector spaces over k for all $l \geq 0$.

(4) For all locally free $\mathcal{F}$ on X and all $q >> 0$,

$$H^l(X, \mathcal{F}(-q)) = (0) \quad \textit{if } l < d.$$

Proof. (1) $\Longrightarrow$ (2). Choose any closed point, x, of X. By our assumption, the local ring $\mathcal{O}_{X,x}$ is d-dimensional and C-M; hence, $\mathrm{depth}(\mathcal{O}_{X,x}) = d$. Now, $\mathcal{O}_{\mathbb{P},x}$ acts on $\mathcal{O}_{X,x}$ through the surjection $\mathcal{O}_{\mathbb{P},x} \longrightarrow \mathcal{O}_{X,s}$; so, the depth of $\mathcal{O}_{X,x}$ as $\mathcal{O}_{\mathbb{P},x}$-module is again d. By Remark (9) above,

$$\mathrm{projdim}(\mathcal{O}_{X,x}) = \mathrm{depth}(\mathcal{O}_{\mathbb{P},x}) - d = N - d = r,$$

because $\mathbb{P}$ is smooth so Remark (10) says $\mathbb{P}$ is C-M and therefore $\mathrm{depth}(\mathcal{O}_{\mathbb{P},x}) = \dim(\mathcal{O}_{\mathbb{P},x}) = N$. From the definition of projdim, we find that

$$\mathrm{Ext}^l_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P}}) = (0) \quad \text{if } l > r.$$

But,

$$\mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})_x = \mathrm{Ext}^l_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P},x}),$$

and we find that $\mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) = (0)$ if $l > r$; this proves (2).

(2) $\Longrightarrow$ (3). Assume $E_2^{p,l} = (0)$ for $l > r$, so that actually, $E_2^{p,l} = (0)$, for all $l \neq r$. Then, the spectral sequence degenerates and we obtain

$$E_2^{p,r} \cong \mathrm{Ext}^{p+r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}).$$

By Serre Duality for $\mathbb{P}$, we get that $\mathrm{Ext}^{p+r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}})$ is dual to $H^{N-(p+r)}(\mathbb{P}, \mathcal{F}) = H^{d-p}(\mathbb{P}, \mathcal{F})$. But, $\mathcal{F}$ has support in X, so the last group is actually $H^{d-p}(X, \mathcal{F})$—which gives the duality for X.

[Actually, all these isomorphisms are natural in the following sense: The diagram

$$\begin{tikzcd}
\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \otimes H^{d-p}(X, \mathcal{F}) \arrow[r] \arrow[d] & H^d(X, \omega_X^0) \arrow[r, "t"] \arrow[d] & k \arrow[d, equal] \\
\mathrm{Ext}^{p+r}_{\mathcal{O}_X}(\mathcal{F}, \omega_{\mathbb{P}}) \otimes H^{d-p}(\mathbb{P}, \mathcal{F}) \arrow[r] & H^N(\mathbb{P}, \omega_{\mathbb{P}}) \arrow[r, "t"] & k
\end{tikzcd} \tag{$\dagger$}$$

commutes. Here the two left arrows are isomorphisms and the rightmost arrow comes about as follows: Consider the sheaf $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{G})$, where $\mathcal{G}$ is a sheaf on $\mathbb{P}$. From the exact sequence

$$0 \longrightarrow \mathfrak{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we obtain the inclusion $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{G}) \hookrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \mathcal{G}) = \mathcal{G}$ and hence, we have the inclusion

$$H^0(\mathbb{P}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{G})) \hookrightarrow H^0(\mathbb{P}, \mathcal{G}).$$

Both side are left-exact functors of $\mathcal{G}$, so, when we take derived functors we obtain the map

$$R^\bullet\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}, \mathcal{G}) \longrightarrow R^\bullet H^0(\mathbb{P}, \mathcal{G}) = H^\bullet(\mathbb{P}, \mathcal{G}). \tag{$*$}$$

On the lefthand side we have just $\mathrm{Ext}^\bullet_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{G})$, and also the spectral sequence

$$H^p(\mathbb{P}, \mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{G})) \Longrightarrow \mathrm{Ext}^\bullet(\mathcal{O}_X, \mathcal{G}). \tag{$\dagger\dagger$}$$

Now, set $\mathcal{G} = \omega_{\mathbb{P}}$, then $\mathrm{Ext}^\bullet(\mathcal{O}_X, \mathcal{G}) = (0)$ if $l < r$; hence, $E_\infty^{p,l} = (0)$ for $l < r$ in spectral sequence ($\dagger\dagger$). It follows that $E_\infty^{p,r}$ is the subobject (in the filtration) of $\mathrm{Ext}^{p+r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})$.

However, the maps d_2 taking $E_2^{p,r}$ to $E_2^{p+2,r-1}$ are all zero, and the same for the higher levels of the spectral sequence. We get a surjective map $E_2^{p,r} \longrightarrow E_\infty^{p,r}$, and coupled with the inclusion $E_\infty^{p,r} \longrightarrow \mathrm{Ext}^{p+r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})$, we obtain the map

$$E_2^{p,r} \longrightarrow \mathrm{Ext}^{p+r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \longrightarrow H^{p+r}(\mathbb{P}, \omega_{\mathbb{P}}),$$

where the righthand arrow comes from $(*)$. Take $p = d = \dim(X)$ and obtain

$$H^d(X, \omega_X^0) = H^d(\mathbb{P}, \omega_X^0) \longrightarrow H^N(\mathbb{P}, \omega_{\mathbb{P}}).\Big]$$

(3) $\Longrightarrow$ (4). Take a locally-free sheaf, $\mathcal{F}$, on X and apply duality to the sheaf $\mathcal{F}(-q)$. We obtain

$$H^l(X, \mathcal{F}(-q)) \quad \text{is dual to} \quad \mathrm{Ext}^{d-l}_{\mathcal{O}_X}(\mathcal{F}(-q), \omega_X^0)$$

and the latter group is just $\mathrm{Ext}^{d-l}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0(q))$, because $\mathcal{F}$ is locally free. However, $\mathrm{Ext}^t_{\mathcal{O}_X}(\mathcal{O}_X, -)$ is just $H^l(X, -)$. Therefore, we deduce $H^l(X, \mathcal{F}(-q))$ is dual to $H^{d-l}(X, (\mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0)(q))$. Since $l < d$, the Serre vanishing theorem shows that the latter group vanishes if $q >> 0$, i.e. (4).

(4) $\Longrightarrow$ (1). In the statement of (4) choose $\mathcal{F}$ to be $\mathcal{O}_X$. By (4), we obtain

$$H^l(X, \mathcal{O}_X(-q)) = (0) \quad \text{if } l < d \text{ and } q >> 0.$$

But, $\mathcal{O}_X(-q)$ has support in X and so, our group is just $H^l(\mathbb{P}^N, \mathcal{O}_X(-q))$, and the latter is dual to $\mathrm{Ext}^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X(-q), \omega_{\mathbb{P}})$, by Serre Duality for $\mathbb{P}^N$. Therefore, our cohomology group is dual to $\mathrm{Ext}^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(q))$. We know for $q >> 0$ that this last group is just $H^0(\mathbb{P}^N, \mathcal{E}xt^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(q)))$ (see the claim in Theorem 7.41). Now,

$$\mathcal{E}xt^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(q)) = \mathcal{E}xt^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})(q)$$

and if $q >> 0$, this last group is generated by its sections. By (4), these global sections vanish ($l < d$); and so, the sheaf $\mathcal{E}xt^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}})(q) = (0)$. Consequently, the sheaf $\mathcal{E}xt^{N-l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) = (0)$, and all its stalks must vanish. Therefore, we find

$$\mathrm{Ext}^{N-l}_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P},x}) = (0) \quad \text{if } l < d$$

and x is any closed point of X. But, $\omega_{\mathbb{P},x} = \mathcal{O}_{\mathbb{P},x}$ because $\omega_{\mathbb{P}}$ is a line bundle, and now one sees simply that if an Ext-group over $\mathcal{O}_{X,x}$ vanishes when the righthand argument is $\mathcal{O}_{X,x}$ it must vanish for all righthand arguments. Therefore, $\mathrm{projdim}(\mathcal{O}_{X,x}) \leq N - d$, as $\mathcal{O}_{\mathbb{P},x}$-module. By Remark (9) again, we find that $\mathrm{depth}(\mathcal{O}_{X,x}) \geq d$. Yet, $\mathrm{depth}(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x})$, and for some x, we have $\dim(\mathcal{O}_{X,x}) = d$. Therefore, for all closed points $x \in X$, we have

$$d \geq \dim(\mathcal{O}_{X,x}) \geq \mathrm{depth}(\mathcal{O}_{X,x}) \geq d,$$

and so, X is equidimensional and Cohen-Macaulay. $\square$

Corollary 7.44 *(Enriques-Severi-Zariski Lemma) Let X be a normal, projective variety over an algebraically closed field, k, and assume that* $\dim(X) \geq 2$*. Then, for all locally-free $\mathcal{F}$ on X and all $q >> 0$,*

$$H^1(X, \mathcal{F}(-q)) = (0).$$

Proof. Serre showed in [52]? that normality of a ring, A, is equivalent to condition (S_2) of Remark (11) and condition (R_1): For each prime, $\mathfrak{p}$, of height 1, the ring $A_{\mathfrak{p}}$ is regular. If we choose x to be a closed point of X, then as $\dim(X) \geq 2$, condition (S_2) implies that $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$. Now, $\mathcal{F}$ is locally free; so, $\mathcal{F}_x = \mathcal{O}_{X,x}^t$, for some t. Thus, $\mathrm{depth}(\mathcal{F}_x) \geq 2$. By Remark (9), $\mathrm{projdim}(\mathcal{F}_x)$ as $\mathcal{O}_{\mathbb{P},x}$-module satisfies

$$\mathrm{projdim}(\mathcal{F}_x) \leq N - 2.$$

Therefore, as in the proof of (1) $\Longrightarrow$ (2), we get

$$\mathcal{E}xt^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, -) = (0), \quad \text{if } l \geq N - 1.$$

However, by Serre Duality for $\mathbb{P}$, the vector space $\mathrm{Ext}^l_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}})$ is dual to $H^{N-l}(\mathbb{P}, \mathcal{F})$; so, for any q the vector space $\mathrm{Ext}^{N-1}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}(-q), \omega_{\mathbb{P}})$ is dual to $H^1(\mathbb{P}, \mathcal{F}(-q))$. Now,

$$H^1(\mathbb{P}, \mathcal{F}(-q)) = H^1(X, \mathcal{F}(-q)),$$

and

$$\mathrm{Ext}^{N-1}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}(-q), \omega_{\mathbb{P}}) = \mathrm{Ext}^{N-1}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}(q)).$$

We are reduced to proving that $\mathrm{Ext}^{N-1}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}(q)) = (0)$ if $q >> 0$; yet we know that for all $q >> 0$ the latter space is isomorphic to $H^0(\mathbb{P}^N, \mathcal{E}xt^{N-1}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}(q)))$. The coefficient sheaf of this H^0 vanishes and we are done. $\square$

Corollary 7.45 *Let X be an integral, normal, projective scheme of dimension $n = \dim(X) \geq 2$ over an algebraically closed field k, and let Y be a closed subset of X. Assume that Y is the support of a divisor D, where D is effective and ample on X. Then, Y is connected. Hence, in Bertini's theorem, the hyperplane sections are connected when* $\dim(X) \geq 2$*.*

Proof. As

$$\mathrm{Supp}|mD| = \mathrm{Supp}|D|,$$

since D is ample, we may assume that D is very ample. Therefore, $D = \mathcal{O}_X(1)$ for some embedding $X \hookrightarrow \mathbb{P}^M_k$. Let Y_p be the scheme given by pD. Then, $|Y_p| = Y$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{Y_p} \longrightarrow 0.$$

Taking cohomology, we get

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(Y, \mathcal{O}_{Y_p}) \longrightarrow H^1(X, \mathcal{O}_X(-p)).$$

By Lemma 7.44, we have
$$H^1(X, \mathcal{O}_X(-p)) = (0)$$
for $p >> 0$. Thus,
$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(Y, \mathcal{O}_{Y_p})$$
is surjective. But
$$H^0(X, \mathcal{O}_X) = k,$$
and $k \subseteq H(Y, \mathcal{O}_{Y_p})$, which implies that
$$H^0(Y, \mathcal{O}_{Y_p}) = k,$$
and Y is connected. $\square$

Remark: There are two ways to proceed from our present position. The first involves more generality and is indicated by the spectral sequence used to prove the Serre Duality. One wants a duality statement valid for proper morphisms or perhaps finite type morphisms, and so on. Necessarily, these are more abstract and less informative (c.f. Hartshorne [32]). We choose the second way: Less generality and more precision.

Instead of a Cohen-Macaulay subscheme of $\mathbb{P}^N_k$, let us look at the more special case of a local complete intersection. In this case, we can be quite precise about the nature of the dualizing sheaf ω^0_X:

Theorem 7.46 *If X is a closed subscheme of $\mathbb{P}^N_k$ which is a local complete intersection of codimension r, then*
$$\omega^0_X = \omega_{\mathbb{P}^N_k} \otimes \bigwedge^r \mathcal{N}_{X \hookrightarrow \mathbb{P}^N_k}.$$
So, ω^0_X is a line bundle on X and if X is nonsingular it is ω_X, the canonical line bundle on X.

Proof. We can cover $\mathbb{P}^N_k = \mathbb{P}$ by affine opens and on these opens X is actually a complete intersection. Call such an open, U; then, on U the ideal sheaf, $\mathfrak{I}$, of X is generated by some global sections, $f_1, \ldots, f_r$, in $\Gamma(U, \mathcal{O}_{\mathbb{P}} \restriction U)$. As $\mathbb{P}$ is regular (hence C-M), the elements $f_1, \ldots, f_r$ give a regular sequence in $\Gamma(U, \mathcal{O}_U)$. But then, the Koszul complex (Chapter 4) gives a resolution of $\Gamma(U, \mathcal{O}_U)/(f_1, \ldots, f_r)$:
$$K_\bullet(\overrightarrow{f}) \longrightarrow \Gamma(U, \mathcal{O}_U) \longrightarrow \Gamma(U, \mathcal{O}_U)/(f_1, \ldots, f_r) \longrightarrow 0, \qquad (K)$$
here, we have separated out $K_0(\overrightarrow{f}) = \Gamma(U, \mathcal{O}_U)$, for clarity. By taking sheaves, we obtain the corresponding resolution of $\mathcal{O}_X$:
$$\widetilde{K}_\bullet(\overrightarrow{f}) \longrightarrow \mathcal{O}_{\mathbb{P}} \restriction U \longrightarrow \mathcal{O}_X \restriction U \longrightarrow 0. \qquad (\widetilde{K})$$

To compute $\mathcal{E}xt^{\bullet}_{\mathcal{O}_{\mathbb{P}}}$, we use $(\widetilde{K})$ after we apply to it $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}$. By Proposition 4.3, we find

$$\mathcal{E}xt^{l}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) = (0), \quad \text{if } l < r \text{ and}$$

$$\omega_X^0 \restriction U = \mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \cong \omega_{\mathbb{P}}/(f_1, \ldots, f_r)\omega_{\mathbb{P}} \cong \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \restriction U.$$

The last isomorphism depends on the choice of the basis $f_1, \ldots, f_r$ of the ideal sheaf $\mathfrak{I}$. If $g_1, \ldots, g_r$ is another basis for $\mathfrak{I}$, say $g_j = \sum_{i=1}^r c_{ij} f_i$, then the exterior powers of the matrix (c_{ij}) gives an isomorphism of the complex $K_\bullet(\overrightarrow{f})$ to $K_\bullet(\overrightarrow{g})$. On $K_r(\overrightarrow{f})$, which computes $\mathcal{E}xt^r$, we have $\det(c_{ij})$ as the multiplier in the isomorphism. That is, the diagram

$$\begin{tikzcd}
\mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \restriction U \arrow[r, "\sim"] \arrow[d, equal] & \omega_{\mathbb{P}}/(f_1, \ldots, f_r)\omega_{\mathbb{P}} \arrow[r, "\sim"] \arrow[d, "\det(c_{ij})"] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \restriction U \arrow[d, "\det(c_{ij})"] \\
\mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \restriction U \arrow[r, "\sim"] & \omega_{\mathbb{P}}/(g_1, \ldots, g_r)\omega_{\mathbb{P}} \arrow[r, "\sim"] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \restriction U
\end{tikzcd}$$

commutes.

Now, $\mathfrak{I}/\mathfrak{I}^2$ is free over U with basis $f_1, \ldots, f_r$ (or basis $g_1, \ldots, g_r$) and so, $\bigwedge^r \mathfrak{I}/\mathfrak{I}^2$ is free of rank 1 with basis $f_1 \wedge \cdots \wedge f_r$ (resp. $g_1 \wedge \cdots \wedge g_r$). The isomorphism from $f_1 \wedge \cdots \wedge f_r$ to $g_1 \wedge \cdots \wedge g_r$ is just multiplication by $\det(c_{ij})$. Therefore, by tensoring with $(\bigwedge^r \mathfrak{I}/\mathfrak{I}^2)^D$, we obtain the commutative diagram:

$$\begin{tikzcd}
\mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \restriction U \arrow[r, "\sim f'\mathrm{s}"] \arrow[d, equal] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \restriction U \arrow[r, "\sim f'\mathrm{s}"] \arrow[d, "\det(c_{ij})"] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \otimes (\bigwedge^r \mathfrak{I}/\mathfrak{I}^2)^D \arrow[d, equal] \\
\mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \restriction U \arrow[r, "\sim g'\mathrm{s}"] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \restriction U \arrow[r, "\sim g'\mathrm{s}"] & \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_X \otimes (\bigwedge^r \mathfrak{I}/\mathfrak{I}^2)^D .
\end{tikzcd}$$

Thus, the isomorphisms on the righthand side patch over the covering by the U's and we get

$$\omega_X^0 = \mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \xrightarrow{\sim} \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \bigwedge^r \mathcal{N}_{X \hookrightarrow \mathbb{P}^N},$$

because $\mathcal{N}_{X \hookrightarrow \mathbb{P}^N}$ is $(\mathfrak{I}/\mathfrak{I}^2)^D$ (cf. Chapter 6, Section 6.2). The last statement is an immediate consequence of the adjunction formula (Proposition 6.14). $\square$

Notice that when X is nonsingular this gives the nonobvious isomorphism

$$H^d(X, \omega_X) \xrightarrow{t} k.$$

Even for X a curve, it is not an obvious map (even though it is true that $H^1(X, \omega_X)$ is one-dimensional when X is a curve).

Consequently, by specializing to a case where computations are available we have made the Serre Duality a bit more explicit. We can make it more explicit yet if we specialize in a slightly different direction: Namely assume of our sheaf $\mathcal{F}$ that it is *locally-free* on X. In this case, we have

$$\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) = H^l(X, \mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0).$$

To see this, observe that $\mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0$ is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0)$. Now, since $\mathcal{F}$ is locally free

$$\mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0 = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0).$$

Hence,

$$H^l(X, \mathcal{F}^D \otimes \omega_X^0) \cong H^l(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0)),$$

and the latter group is just $\mathrm{Ext}^l_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0)$.

Putting this together with the abstract statement of Serre Duality, we deduce the special case originally proved by Serre:

Corollary 7.47 *Under the hypotheses of the duality theorem (Theorem 7.43), if $\mathcal{F}$ is locally-free, the duality pairing is just*

$$H^l(X, \mathcal{F}) \otimes H^{d-l}(X, \mathcal{F}^D \otimes_{\mathcal{O}_X} \omega_X^0) \longrightarrow H^d(X, \omega_X^0) \overset{t}{\cong} k.$$

An even more interesting special case is the case when we restrict both X to be nonsingular and $\mathcal{F}$ to be a line bundle $\mathcal{O}_X(E)$, where E is a Cartier divisor on X. Write $\mathcal{K}_X$ for the canonical divisor on X, so that $\omega_X^0 = \omega_X = \mathcal{O}_X(\mathcal{K}_X)$. Then, Serre Duality becomes:

Corollary 7.48 *If X is a nonsingular projective scheme over the algebraicaly closed field, k, and if E is any Cartier divisor on X, the Serre pairing*

$$H^l(X, \mathcal{O}_X(E)) \otimes H^{d-l}(X, \mathcal{O}_X(\mathcal{K}_X - E)) \longrightarrow H^d(X, \mathcal{O}_X(\mathcal{K}_X)) \overset{t}{\cong} k$$

is a perfect pairing of finite dimensional vector spaces.

Serre Duality theorem leads immediately to a proof of the Riemann-Roch theorem for curves, a theorem we already used in Section 7.4. If $\mathcal{F}$ is a coherent sheaf on the projective d-dimensional scheme, X, we define

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{d} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

The function χ is an Euler function on $\mathrm{Coh}(X)$, that is, for any exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

we have $\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$. The case of most importance for us now is when $d = 1$, that is, X is a curve and X is nonsingular. Then, $\chi(X, \mathcal{F})$ is just

$$\dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}).$$

Theorem 7.49 *(Riemann-Roch For Line Bundles on X) If X is a smooth, proper curve over an algebraically closed field, k, then for all line bundles $\mathcal{O}_X(D)$ on X:*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(\mathcal{K}_X - D)) = \deg D + 1 - g.$$

Here, $\mathcal{K}_X$ is the canonical divisor on X and g is the genus of X.

Proof. Every proper curve is projective and so Serre Duality applies. Observe that by Serre Duality, the lefthand side of Riemann-Roch is just the Euler function $\chi(X, \mathcal{O}_X(D))$. Pick a closed point, P, of X and consider either one of the exact sequences

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0 \qquad (A)$$
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(P) \longrightarrow \mathcal{O}_P \longrightarrow 0. \qquad (B)$$

Note that $\mathcal{O}_P$ is a skyscraper sheaf supported at P, i.e., the stalk of $\mathcal{O}_P$ is zero outside P and at P it is $\kappa(P) = k$.

Given D, if P appears in D with positive multiplicity use (A) and if P appears with negative multiplicity use (B); we get

$$0 \longrightarrow \mathcal{O}_X(D - P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_P \longrightarrow 0 \qquad (A')$$
$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow \mathcal{O}_P \longrightarrow 0. \qquad (B')$$

For the function χ (omitting the argument X for simplicity of notation), exact sequence (A') gives

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_P) + \chi(\mathcal{O}_X(D - P)) = 1 + \chi(\mathcal{O}_X(D - P)),$$

while for (B'), we get

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D + P)) - 1.$$

In either case, $\mathcal{O}_X(D - P)$ or $\mathcal{O}_X(D + P)$ contains P with smaller absolute value of its multiplicity and so we can use induction on the sum of the absolute values of the multiplicities in D and obtain:

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X).$$

Now,

$$\begin{aligned} \chi(\mathcal{O}_X) &= \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) \\ &= \dim H^0(X, \mathcal{O}_X) - \dim H^0(X, \mathcal{K}_X) = 1 - g. \end{aligned}$$

$\square$

There is also a version of Riemann-Roch for vector bundles on a curve:

Theorem 7.50 *(Riemann-Roch For Vector Bundles on X) If X is a smooth, proper curve over an algebraically closed field, k, and if E is a vector bundle on X, then*

$$\dim H^0(X, \mathcal{O}_X(E)) - \dim H^0(X, \mathcal{O}_X(\omega_X \otimes E^D)) = \deg\Big(\bigwedge^{\bullet} E\Big) - (\operatorname{rk} E)(1 - g).$$

Here, $\bigwedge^{\bullet} E$ is the highest wedge of E, a line bundle.

Proof. Again, X being proper and a curve is projective; so, we may use Serre Duality. We use induction on $\mathrm{rk}\, E$, the case $\mathrm{rk}\, E = 1$ being Riemann-Roch for line bundles above. By the Atiyah-Serre Theorem (Theorem 5.22), there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \widetilde{E} \longrightarrow 0,$$

and $\mathrm{rk}\, \widetilde{E} = \mathrm{rk}\, E - 1$. Further, $\bigwedge^\bullet E = \bigwedge^\bullet \widetilde{E}$, hence, by induction as

$$\chi(\mathcal{O}_X(E)) = \chi(\mathcal{O}_X(\widetilde{E})) + \chi(\mathcal{O}_X),$$

we find

$$1 - g + \deg\Big(\bigwedge^{\bullet} \widetilde{E}\Big) + (\mathrm{rk}\, \widetilde{E})(1-g) = \chi(\mathcal{O}_X(E)).$$

And so, Serre Duality on the righthand side and simple addition on the lefthand side finish the proof. $\square$

** Remember: Exercise on RR on surfaces **.

If we use Serre Duality for some special line bundles, we obtain interesting results for a nonsingular projective variety of dimension d. As usual, we write $\Omega_{X/k}$ for the rank d bundle of one-forms on X, and write $\Omega^p_{X/k}$ for $\bigwedge^p \Omega_{X/k}$. This latter is the bundle of p-forms on X; here, $0 \le p \le d$. The natural pairing

$$\Omega^{d-p}_{X/k} \otimes \Omega^p_{X/k} \longrightarrow \Omega^{d-p}_{X/k} \bigwedge \Omega^p_{X/k} = \omega_X$$

gives the duality

$$\Omega^{d-p}_{X/k} \cong (\Omega^p_{X/k})^D \otimes \omega_X.$$

(This is a simple exercise and will be left to the reader.) We can apply Serre Duality to deduce that

$$H^{d-q}(X, \Omega^{d-p}_{X/k}) \quad \text{is dual to} \quad H^q(X, \Omega^p_{\Omega/k}).$$

Traditionally, the dimension of $H^q(X, \Omega^p_{\Omega/k})$ is denoted $h^{p,q}$ and is called the p,q-*Hodge number* of X. What we have shown is that $h^{p,q} = h^{d-p,d-q}$ on a d-dimensional nonsingular projective variety, X. The numbers $h^{p,q}$ are important invariants of X; they help to classify these vartieties.

** Exercise 1: cohomology class of a subvariety and mention the Hodge conjecture. **

** Exercise 2: Computation of $h^{p,q}$ on $\mathbb{P}^d$ and perhaps other varieties. **

As we have remarked, while $H^d(X, \Omega_X)$ is one-dimensional (X is nonsingular and d-dimensional) as yet we do not know how to compute the all important trace map from it to k, *even for curves*. All we know is that the trace map exists. But for curves, there is another approach which gives an explicit computation of the trace map. From now on, write X for a proper, smooth curve, Ω_X for its sheaf of one-forms, Ω_K for the module of differentials of K/k (where $K = \mathcal{M}\mathrm{er}(X)) =$ stalk of Ω_X at the generic point. Finally, if P is a closed point of X, write Ω_P for the stalk of Ω_X at P. Recall that at P we have a valuation of K: ord_P, namely, the order of zero or pole of a function (from K) at P. The computation of the trace map will be done in terms of the notion of residue at P, this is a classical notion from complex analysis, but we can abstract it as follows:

Theorem 7.51 *(Existence and Uniqueness of* Res_P*) If X is as above and P is any closed point of X then there exists a unique k-linear map,* $\mathrm{Res}_P\colon \Omega_K \to k$*, having the properties:*

(a) If $\xi \in \Omega_P$, then $\mathrm{Res}_P(\xi) = 0$.

(b) If $f \in K^*$, then $\mathrm{Res}_P(f^r df) = 0$ *if $r \neq -1$.*

(c) $\mathrm{Res}_P\left(\dfrac{df}{f}\right) = \mathrm{ord}_P(f)$.

Proof. (In characteristic zero.) First, we will show that (a), (b) and (c) determine Res_P and so, uniqueness will follow. In this part of the proof no use of characteristic zero will be necessary.

Choose P and let t be a local uniformizing parameter in $\mathcal{O}_P$. Then, dt generates Ω_K as a K-vector space; so, any element of Ω_K has the form $f dt$, for some $f \in K$. As $\mathcal{O}_P$ is a valuation ring of K, the function f can be written

$$f = \sum_{i=-N}^{-1} c_i t^i + g, \quad \text{for some } N < \infty.$$

Here, the $c_i \in k$ and $g \in \mathcal{O}_P$. Thus,

$$f dt = \sum_{i=-N}^{-2} c_i t^i dt + c_{-1}\frac{dt}{t} + g dt.$$

Now, $g dt \in \Omega_P$, so (a), (b) and (c) and linearity imply immediately that

$$\mathrm{Res}_P(f dt) = c_{-1}. \tag{$*$}$$

This proves uniqueness, once Res_P exists with properties (a), (b) and (c). Existence is more problematical. If one takes, as is natural, property ($*$) as the definition, one must show independence of t. This is not easy in characteristic $p > 0$ (see Serre [51]), but in characteristic 0 (which we are assuming) we can use analysis. Namely, by the Lefchetz principle we may asume $k = \mathbb{C}$. Then, if z is another uniformizer there exist locally-defined holomorphic functions

$$\begin{aligned} z &= \alpha(t) = at + O(t^2), \quad a \in \mathbb{C}^* \\ t &= \beta(z) = \frac{1}{a}z + O(z^2), \end{aligned}$$

relating t and z. We notice that by $(*)$ and the Cauchy residue theorem, the residue as defined by $(*)$ is just
$$\frac{1}{2\pi i}\int_\gamma f dt,$$
where γ is a small, simple closed curve about P in the local patch on X. If we change variables, $t = \beta(z)$, the curve γ shifts yet the value of the integral remains the same as the integral is independent of the curve up to its homology in the punctured disk about P, and this is determined by the winding number of γ about P. However, this winding number is the same for $\alpha(\gamma)$ because $\alpha(t)$ has a simple zero at P. Thus, the definition given by $(*)$ is in fact independent of the uniformizing parameter. $\square$

** Exercise on the existence of the residue a la Tate. **

The residue is a local invariant as we have defined it and we have all these local invariants, one for each point $P \in X$. Given a differential $\xi \in \Omega_K$, it has but finitely many poles and so, $\mathrm{Res}_P(\xi)$ is almost always zero. Therefore, the sum $\sum_{P\in X}\mathrm{Res}_P(\xi)$ makes sense. The local residues are not independent because of the main fact:

Theorem 7.52 *(Global Residue Theorem) If X is a smooth, proper curve over k (k algebraically closed), then*
$$\sum_{P\in X}\mathrm{Res}_P(\xi) = 0.$$

Proof. (In characteristic zero). We apply Stokes theorem to the Riemann surface X. Observe that the differential ξ is holomorphic outside a finite number of points, $P_1, \ldots, P_n$. Draw about each of these points a small circle. The complement, C, of the union of the disks defined by these circles has their union as a boundary and by Stokes
$$0 = \int_C d\xi = \sum_{j=1}^n \int_{\gamma_{P_j}} \xi = \sum_{j=1}^n \mathrm{Res}_{P_j}(\xi). \qquad \square$$

What is the connection of this material with the trace map? Recall Chapter 5 that since X is irreducible the sheaf $K(X)$ consisting of the meromorphic functions is the constant sheaf on X and so it is flasque. Also, the sheaf $\mathcal{P}_X = K(X)/\mathcal{O}_X$ is flasque; so,
$$0 \longrightarrow \mathcal{O}_X \longrightarrow K(X) \longrightarrow \mathcal{P}_X \longrightarrow 0$$
is a flasque resolution of $\mathcal{O}_X$. Now,
$$\mathcal{P}_X = \coprod_{P\in X}(i_P)_*(K(X)/\mathcal{O}_P),$$
as remarked in Chapter 5 (** Steve, where in Chapter 5? **) (here, i_P is the closed immersion of $\{P\} \hookrightarrow X$). Tensor the flasque resolution of $\mathcal{O}_X$ by Ω_X—we get a flasque resolution of Ω_X:
$$0 \longrightarrow \Omega_X \longrightarrow K(X)\otimes_{\mathcal{O}_X}\Omega_X \longrightarrow \coprod_{P\in X}(i_P)_*(\Omega_K/\Omega_P) \longrightarrow 0.$$

We can apply cohomology to this sequence and obtain a piece of the long cohomology exact sequence:

$$0 \longrightarrow H^0(X, \Omega_X) \longrightarrow \Omega_K \longrightarrow \coprod_{P \in X} (\Omega_K/\Omega_P) \longrightarrow H^1(X, \Omega_X) \longrightarrow 0.$$

Now define a map from $\coprod_{P \in X}(\Omega_K/\Omega_P)$ to k *via* $(\xi_P) \mapsto \sum_P \mathrm{Res}_P(\xi_P)$.

By the global residue theorem, this map is 0 on Ω_K and therefore it descends to a map

$$H^1(X, \Omega_X) \longrightarrow k.$$

Since $\dim X = 1$, we find $\Omega_X = \omega_X$ and the map above is the trace map.

Serre duality is also valid for compact, complex manifolds and holomorphic bundles on them (as shown by Serre himself using an analytic proof). In the case that k is an algebraically-closed field of characteristic zero one can prove further theorems, not provable in characteristic $p > 0$, and the proofs of these theorems use analysis and perhaps some differential geometry.

Typical of the above theorems is the circle of ideas concerning the Kodaira vanishing theorem. In discussing this theorem, we'll assume $k = \mathbb{C}$ (as we may by the Lefchetz principle) and furthermore we will not give proofs—they depend both on analysis and differential geometry. Indeed, for the Kodaira theorem, M. Raynaud gave a counterexample in characteristic $p > 0$. However, the statements are clear and connect with what we have done above.

In the following, assume that X is a compact, complex, analytic *manifold*. The first famous theorem is due to Hodge and it says that the cohomology with complex coefficients of such an X is the coproduct of the cohomology of the holomorphic differential forms:

Theorem 7.53 *(Hodge Decomposition) Let X be a compact, complex manifold, then for all $n \geq 0$*

$$H^n(X, \mathbb{C}) = \coprod_{p+q=n} H^q(X, \Omega_X^p).$$

A differential form of type (p, q) is one which in local coordinates everywhere has the form

$$\alpha = \sum_{(r),(s)} \alpha_{(r),(s)}(z, \overline{z})\, dz_{r_1} \wedge \cdots \wedge dz_{r_p} \wedge d\overline{z}_{s_1} \wedge \cdots \wedge d\overline{z}_{s_q}.$$

When α is a $(1,1)$-form, the coefficients α_{rs} form a matrix and we say that α is a *positive* $(1,1)$*-form* provided the matrix $-i\alpha_{rs}$ is a positive definite Hermitian symmetric matrix. (i.e, $\alpha_{rs} = i\beta_{rs}$, where β_{rs} is positive definite Hermitian symmetric). Let's write $\alpha > 0$ for this. If E is a holomorphic line bundle on X, then $c_1(E)$ lies in $H^2(X, \mathbb{C})$ and by Hodge $c_1(E)$ has a decomposition according to the coproduct

$$H^2(X, \mathbb{C}) = H^{2,0} \coprod H^{1,1} \coprod H^{0,2}.$$

Call the bundle, E, *positive* iff $c_1(E) \in H^{1,1}$ and is represented there by a positive $(1,1)$-form. (Recall the Dolbeault theorem:

$$0 \longrightarrow \Omega_X^p(E) \longrightarrow D_X^{p,0}(E) \xrightarrow{\overline{\partial}} D_X^{p,1}(E) \xrightarrow{\overline{\partial}} D_X^{p,2}(E) \xrightarrow{\overline{\partial}} \cdots$$

is a resolution of $\Omega_X^p(E)$. Here, $D_X^{p,q}(E)$ is $D_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$ and $D_X^{p,q}$ is the $\mathcal{O}_X$-module of C^∞-(p,q) forms on X.) Also, call E *negative* when E^D is positive.

Examples.

(1) The hyperplane bundle, $\mathcal{O}_{\mathbb{P}^N}(1)$, on $\mathbb{P}^N_{\mathbb{C}}$ is positive (DX).

(2) The tensor product of two positive bundles is positive; hence, $\mathcal{O}_{\mathbb{P}^N}(d)$ is positive if $d > 0$.

(3) If a tensor power of a bundle is positive, then the bundle itself is positive.

(4) An ample bundle on X is positive.

With the notion of positive bundle we can state Nakano's generalization of Kodaira's vanishing theorem (Nakano [**?**] (1955), Kodaira [**?**] (1953)).

Theorem 7.54 *(Kodaira/Nakano Vanishing Theorem) Suppose X is a compact, complex n-dimensional manifold and E is a holomorphic line bundle on X.*

(1) If $E \otimes \bigwedge^n T(X)$ is positive, then

$$H^p(X, \mathcal{O}_X(E)) = (0), \quad \textit{for } p > 0 \textit{ and}$$

(2) If E is negative then

$$H^q(X, \Omega_X^p(E)) = (0), \quad \textit{when } p + q < n.$$

Corollary 7.55 *(Original Kodaira Vanishing Theorem) Hypotheses on X and E as above, then (1) as above and*

(2) If E is negative, then

$$H^q(X, \mathcal{O}_X(E)) = (0), \quad \textit{if } q < n.$$

Note that the corollary (part (2)) is just the case $p = 0$ of the theorem.

To connect Kodaira's theorem with projective geometry let's suppose that X is a closed, smooth subvariety of $\mathbb{P}^n_{\mathbb{C}}$. Write L for the line bundle $E \otimes \bigwedge^n T(X)$ and assume that L is ample (hence positive by our remark above). Then, $E = L \otimes \omega_X$ and we get the form of Kodaira's theorem for smooth projective varieties:

Corollary 7.56 *If X is a smooth projective variety and L is an ample line bundle on X, then*

(1) $H^p(X, \mathcal{O}_X(L \otimes \omega_X)) = (0)$, if $p > 0$ and

(2) $H^p(X, \mathcal{O}_X(L^D)) = (0)$, if $p < n$.

Notice that (1) and (2) of Kodaira's theorem are equivalent by Serre Duality. If we apply Serre Duality to statement (2) of Nakano's vanishing theorem and use the fact that $\Omega^p_X(E)^D \otimes \omega_X$ is isomorphic to $\Omega^{n-p}_X(E^D)$, then we find that (2) becomes

$$H^r(X, \Omega^s_X(E^D)) = (0), \quad \text{if } E^D \text{ is positive and } r + s > n.$$

The use of Kodaira's theorem is in answering the question of when a compact, complex manifold is actually a projective variety. To understand this, observe that on a complex manifold there are lots of C^∞ Hermitian metrics. When $g_{\alpha\beta}$ is such a metric, we can make the associated $(1,1)$-form:

$$[g_{\alpha\beta}] = \sum_{\alpha,\beta} g_{\alpha,\beta}(z, \overline{z})\, dz_\alpha \wedge d\overline{z}_\beta.$$

The metric, $g_{\alpha\beta}$, is called a *Kähler metric* iff $[g_{\alpha\beta}]$ is a closed $(1,1)$-form and then X is a *Kähler manifold* iff it admits a Kähler metric. So, Kähler manifolds are special kinds of complex manifolds.

Examples.

(1) $\mathbb{P}^N_{\mathbb{C}}$ admits the Fubini-Study metric which is Kähler.

(2) Any smooth complex projective variety is thus Kähler using the restriction of the Fubini-Study metric.

Now, $H^{1,1} \subseteq H^2(X, \mathbb{C})$ and Hodge [**?**] called attention to the fact that if X is projective, algebraic, then its Kähler form comes from $H^2(X, \mathbb{Z})$ (with obvious $\frac{1}{2\pi i}$ factors). In his honnor a Kähler manifold whose Kähler metric comes from $H^2(X, \mathbb{Z})$ is called a *Hodge manifold*. Hodge conjectured that every Hodge manifold was, in fact, projective algebraic. Using his vanishing theorem, Kodaira proved that Hodge manifolds admit a closed holomorphic embedding into projective space and hence, by Chow's theorem [**?**] or by Serre's GAGA [48], every Hodge manifold *is* projective algebraic.

Since positive line bundles, coherent sheaves, etc. make sense for compact, complex manifolds one can ask if Kodaira's vanishing theorem can be generalized. Recall that Nakano's generalization concerned differential forms with coefficients negative bundles. There have been many generalizations of Kodaira's theorem, perhaps the following is representative:

Theorem 7.57 *(Grauert's Vanishing Theorem) If X is a compact, complex holomorphic manifold, E a positive holomorphic line bundle on X and $\mathcal{F}$ is a coherent (analytic) sheaf on X, then there exists $\mu_0(\mathcal{F})$ so that if $\mu \geq \mu_0$ we have*

$$H^p(X, \mathcal{O}_X(E)^{\otimes \mu} \otimes \mathcal{F}) = (0), \quad p > 0.$$

(That is, positive implies ample.)

In preparation for Chern classes, we need to define the projective bundle $\mathbb{P}(\mathcal{E})$ associated with a locally free sheaf $\mathcal{E}$ of rank r. We assume that we are in the following situation that we denote by (H).

We have a locally noetherian scheme X and $\mathcal{S}$ is a graded $\mathcal{O}_X$-algebra ($\mathcal{S} = \bigcup_{d \geq 0} \mathcal{S}_d$), so that

(1) $\mathcal{S}_0 = \mathcal{O}_X$.

(2) $\mathcal{S}$ is generated as an algebra over $\mathcal{S}_0 = \mathcal{O}_X$ by $\mathcal{S}_1$.

(3) $\mathcal{S}_1$ is coherent as $\mathcal{O}_X$-module.

Note that we basically have a "sheafified" ggr. Let U be some affine open in X. Then, we have

$$\mathcal{S} \restriction U = \widetilde{S_U},$$

where S_U is some A_U-algebra where $A_U = \Gamma(U, \mathcal{O}_U)$, and S_U is graded. We make $\mathrm{Proj}(S_U)$. Observe that if $f \in A_U$, then

$$\mathrm{Proj}(S_U)_{(f)} = \mathrm{Proj}(S_{U_f}) = \pi^{-1}(U_f),$$

where $\pi\colon \mathrm{Proj}(S_U) \to U$. This implies that these schemes patch, and we get $\mathrm{Proj}(\mathcal{S})$, a scheme over X.

Remarks:

(1)
$$(\mathrm{Proj}(\mathcal{S})) \restriction \pi^{-1}(U) = \mathrm{Proj}(S_U).$$

(2) $\mathrm{Proj}(\mathcal{S})$ is proper over X and locally projective. We will write

$$\mathrm{Proj}(S_U) \hookrightarrow \mathbb{P}^{n_U}_U.$$

The scheme $\mathrm{Proj}(\mathcal{S})$ is not necessarily projective over X.

(3) $Y = \mathrm{Proj}(\mathcal{S})$ always has an $\mathcal{O}_Y(1)$: Just patch the $\mathcal{O}_{\mathrm{Proj}(S_U)}(1)$ together.

(4) Assume that $\mathcal{S} = \mathcal{O}_X[T_0, \ldots, T_n]$. Then,

$$\mathrm{Proj}(\mathcal{S}) = \mathbb{P}^n_X.$$

(5) Take $\mathcal{E}$, a locally free sheaf of rank r on X. Make $\mathcal{E}^D$ and $\mathrm{Sym}(\mathcal{E}^D)$. The situation (H) holds, and we set

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^D)),$$

the *projective bundle over* X *determined by* $\mathcal{E}$. Note that the reason for using $\mathcal{E}^D$ instead of $\mathcal{E}$ is that given a morphism between $\mathcal{E}_1$ and $\mathcal{E}_2$, the arrows $\mathbb{P}(\mathcal{E}_1) \restriction U \longrightarrow \mathbb{P}(\mathcal{E}_2) \restriction U$ go in the right direction.

The maps α and β from Serre's Theorem 7.4 are relativized as follows. Letting $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ be the projection morphism,

$$\beta\colon \mathcal{S} = \mathrm{Sym}(\mathcal{E}^D) \longrightarrow \coprod_l \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l))$$

is an isomorphism. This implies the following facts.

(1)

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)) = (0)$$

if $l < 0$.

(2a)

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}) = \mathcal{O}_X.$$

(The fibres of π are connected).

(2b)

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{E}^D.$$

(2c)

$$\pi^*\mathcal{E}^D \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$$

is surjective (this is the same as saying that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by sections).

(3)

$$(R^p\pi_*)(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)) = (0)$$

if $0 < p < r$, where $r = rg(\mathcal{E})$, and all l.

(4)

$$\omega_{\mathbb{P}(\mathcal{E})} = \left(\pi^* \bigwedge^r \mathcal{E}^D\right)(-r).$$

(5) $\mathbb{P}(\mathcal{E})$ represents the functor

$$\mathrm{Hom}_X(T, \mathbb{P}(\mathcal{E})) = \{(\mathcal{L}, \psi) \mid \mathcal{L} \in \mathrm{Pic}(T),\ \pi_T^* \mathcal{E}^D \stackrel{\psi}{\longrightarrow} \mathcal{L} \longrightarrow 0\}.$$

The reason for (4) is that, as usual, with the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0,$$

we have the Euler bundle sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}(\mathcal{E})} \longrightarrow \pi^* \mathcal{E}^D(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow 0.$$

7.7 Blowing Up

The notion of blowing-up is the main means for producing non-flat modifications of a scheme in a natural way. As we shall see, the problem of reducing or resolving singularities of varieties (reduced, irreducible schemes over a field) is couched in terms of forming a new variety from the old by a sequence of appropriate blowings-up.

In Chapter 2 we introduced the notion of blowing-up a point on a variety. Here, we'll begin with the most general notion and show that it agrees with the simplest case which we have already defined.

Let X be a scheme and $\mathfrak{J}$ be a QC $\mathcal{O}_X$-ideal. We introduce the graded $\mathcal{O}_X$-algebra

$$\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) = \coprod_{n \geq 0} \mathfrak{J}^n T^n,$$

where $\mathfrak{J}^0$ stands for $\mathcal{O}_X$ and T is an indeterminate (which serves to keep track of degrees). Of course, each $\mathfrak{J}^n$ is QC and $\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J})$ is a QC good, graded $\mathcal{O}_X$-algebra. If X is locally noetherian and $\mathfrak{J}$ is cohherent, then all the $\mathfrak{J}^n$ will be coherent. If $\mathfrak{J}$ is just f.g. as $\mathcal{O}_X$-module then, all the $\mathfrak{J}^n$ are also f.g. In any case, we can form

$$\mathcal{B}_{\mathfrak{J}}(X) = \mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}))$$

and we call $\mathcal{B}_{\mathfrak{J}}(X)$ the *blowing-up of X along* $\mathfrak{J}$. Since $\mathfrak{J}$ is a QC-ideal, it corresponds to a closed subscheme, Y, of X. We also denote $\mathcal{B}_{\mathfrak{J}}(X)$ by $\mathcal{B}_Y(X)$ and say that $\mathcal{B}_Y(X)$ is the *blowing-up of X along Y* or *with center Y*. On $\mathcal{B}_Y(X)$ we have the natural invertible sheaf $\mathcal{O}_{\mathcal{B}}(1)$.

If U is an affine open in X, then $\Gamma(U, \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \restriction U)$ is a graded $\Gamma(U, \mathcal{O}_X \restriction U)$-algebra, and $\mathcal{B}_Y(X)$ is glued together from the schemes $\mathrm{Proj}(\Gamma(U, \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \restriction U))$. Each of the latter has a surjective morphism to U, so we get the natural, surjective structure morphism

$$\pi \colon \mathcal{B}_Y(X) \to X.$$

As for the sheaf, $\mathcal{O}_{\mathcal{B}}(1)$, on $\pi^{-1}(U)$ it is given by the graded $\Gamma(U, \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \restriction U)$-module

$$\Gamma(U, \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \restriction U)(1) = \coprod_{n\geq 0} \Gamma(U, \mathfrak{J}^{n+1}T^{n+1} \restriction U).$$

The morphism $\mathcal{B}_Y(X) \longrightarrow X$ is not projective unless $\mathfrak{J}$ is a f.g. $\mathcal{O}_X$-ideal, for part of the definition of projective morphism is that it be a finite-type morphism.

Say our open, U, is the complement of Y, where Y is defined by $\mathfrak{J}$. As $\mathfrak{J} \restriction U$ is just $\mathcal{O}_X \restriction U$, we find that

$$\pi^{-1}(U) = \mathrm{Proj}\Big(\coprod_{n\geq 0} \mathcal{O}_U T^n\Big) \xrightarrow{\sim} U.$$

Hence, outside of Y, the blow-up morphism $\mathcal{B}_Y(X) \longrightarrow X$ is an isomorphism. This is what we had in mind when speaking of a "modification" of X by the process of blowing-up; we change X only over Y.

Example 1. As mentioned above, in Chapter 2 we considered the blow-up of a point of $\mathbb{A}^n_k$. Let's consider the above definition in this case. Of course, we take our point, P, to be the origin; so, J—the ideal defining P—is just $(X_1, \ldots, X_n)$ in the ring $A = k[X_1, \ldots, X_n]$. Now,

$$\mathrm{Pow}_A(J) = \coprod_{r\geq 0} (X_1, \ldots, X_n)^r T^r = \coprod_{r\geq 0} (X_1 T, \ldots, X_n T)^r.$$

We have a map

$$A[Y_1, \ldots, Y_n] \longrightarrow \mathrm{Pow}_A(J),$$

by sending Y_l to $X_l T$. This is a surjection. What is the kernel? Because we have a map of graded rings the kernel is generated by homogeneous forms in the Y's. When one writes down a form of degree z in the Y's with coefficients polynomials in the X's, one finds that it goes to zero when and only when our expression contains monomials of degree z in the X's times monomials of degree z in the Y's which agree under the switch $X_i \longleftrightarrow Y_i$ and appear with opposite signs. For example,

$$R(X, Y) = X_1^5 X_2 X_3^2 Y_1^3 Y_2^5 - X_1^3 X_2^5 Y_1^5 Y_2 Y_3^2.$$

Let us continue with this example which renders the general case clear. We factor:

$$\begin{aligned} R(X, Y) &= X_1^3 X_2 Y_1^3 Y_2 (X_1^2 X_3^2 Y_2^4 - X_2^4 Y_1^2 Y_3^2) \\ &= X_1^3 X_2 Y_1^3 Y_2 (X_1 X_3 Y_2^2 + X_2^2 Y_1 Y_3)(X_1 X_3 Y_2^2 - X_2^2 Y_1 Y_3). \end{aligned}$$

So, we have only to examine $X_1 X_3 Y_2^2 - X_2^2 Y_1 Y_3$. But, the latter is

$$X_2 Y_1 (X_3 Y_2 - Y_3 X_2) + X_2 Y_3 (Y_1 X_2 - X_1 Y_2).$$

A simple generalization shows our kernel is generated by the set

$$\{X_i Y_j - X_j Y_i \mid 1 \leq i, j \leq n\}.$$

This shows that $\mathcal{B}_P\mathbb{A}^n$ is just the subscheme of

$$\mathbb{P}_A^{n-1} = \mathrm{Proj}(A[Y_1, \ldots, Y_n])$$

defined by the Y-homogeneous equations

$$X_iY_j - X_jY_i = 0, \quad \text{for} \quad 1 \leq i, j \leq n,$$

exactly as defined in Chapter 2.

Notice that $E = \pi^{-1}(Y)$ is $\mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$, and since $\mathcal{O}_Y$ is $\mathcal{O}_X \restriction \mathfrak{J}$, we find

$$E = \pi^{-1}(Y) = \mathrm{Proj}\Big(\coprod_{n\geq 0} \mathfrak{J}^n/\mathfrak{J}^{n+1}\Big).$$

But, $\mathrm{Sym}_{\mathcal{O}_Y}(\mathfrak{J}/\mathfrak{J}^2)$ is $\mathfrak{J}^n/\mathfrak{J}^{n+1}$; hence,

$$E = \pi^{-1}(Y) = \mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Y}(\mathfrak{J}/\mathfrak{J}^2)).$$

We call E the *exceptional locus* of the blow-up.

If $\varphi\colon Z \to W$ is a morphism of scheme, then we have $|\varphi|\colon |Z| \to |W|$, the corresponding morphism of the underlying topological spaces. If $\mathfrak{J}$ is an ideal of $\mathcal{O}_W$, then $|\varphi|^{-1}(\mathfrak{J})$ is an $|\varphi|^{-1}\mathcal{O}_W$-ideal. We know there is a ring map $|\varphi|^{-1}\mathcal{O}_W \longrightarrow \mathcal{O}_Z$, so the ideal generated by the image of $|\varphi|^{-1}(\mathfrak{J})$ in $\mathcal{O}_Z$ is an $\mathcal{O}_Z$-ideal; it is $\mathcal{O}_Z \cdot |\varphi|^{-1}(\mathfrak{J})$ and it is also the image of $\varphi^*(\mathfrak{J})$ *via* the natural map

$$\varphi^*(\mathfrak{J}) = |\varphi|^{-1}(\mathfrak{J}) \otimes_{|\varphi|^{-1}\mathcal{O}_W} \mathcal{O}_Z \longrightarrow \mathcal{O}_Z.$$

Let's denote $\mathcal{O}_Z \cdot |\varphi|^{-1}(\mathfrak{J})$ by $\varphi^\bullet(\mathfrak{J})$ and refer to it as the *inverse image ideal of* $\mathfrak{J}$ *in* $\mathcal{O}_Z$. (Because tensor product is only right-exact, $\varphi^\bullet(\mathfrak{J})$ may very well be different from $\varphi^*(\mathfrak{J})$.)

Now, for the morphism $\pi\colon \mathcal{B}_Y(X) \to X$, the ideal $\pi^\bullet(\mathfrak{J})$ is just

$$\mathfrak{J} \cdot \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) = \mathfrak{J} \cdot \coprod_{n\geq 0} \mathfrak{J}^n = \coprod_{n\geq 0} \mathfrak{J}^{n+1}.$$

However, we've already noted that the later module is $\mathcal{O}_{\mathcal{B}}(1)$. Therefore, $\varphi^\bullet(\mathfrak{J})$ is the invertible ideal $\mathcal{O}_{\mathcal{B}}(1)$ on $\mathcal{B}_Y(X)$. These remarks prove

Proposition 7.58 *If X is a scheme and Y is a closed subscheme with corresponding QC ideal $\mathfrak{J}$, then*

(1) If $\mathfrak{J}$ is f.g. the scheme $\mathcal{B} = \mathcal{B}_Y(X)$ is projective over X.

(2) In general, the inverse image ideal, $\pi^\bullet(\mathfrak{J})$, is the invertible $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{O}_{\mathcal{B}}(1)$; and, in fact, the inverse image ideal $\pi^\bullet(\mathfrak{J}^n)$ is $\mathcal{O}_{\mathcal{B}}(n)$, for all $n \geq 0$.

(3) *The scheme $E = \pi^{-1}(Y)$ equals* $\mathrm{Proj}(\mathrm{Sym}(\mathfrak{J}/\mathfrak{J}^2))$*. (Recall that $\mathfrak{J}/\mathfrak{J}^2$ is the conormal sheaf of Y in X.)*

(4) *The scheme $E = \pi^{-1}(Y)$ is defined, as subscheme of $\mathcal{B}_Y(X)$, by the invertible ideal $\mathcal{O}_{\mathcal{B}}(1)$. That is, E is a Cartier divisor on $\mathcal{B}$.*

(5) *If U denotes the complement of Y in X, then the morphism*

$$\pi\colon \pi^{-1}(U) \to U$$

is an isomorphism.

Proof. All we need to prove is (4). Now, E is defined by $\pi^{\bullet}(\mathfrak{J})$ according to (3) and, by (2), this is just $\mathcal{O}_{\mathcal{B}}(1)$. $\square$

One can still be a little more precise about the basic injection $\mathcal{O}_{\mathcal{B}}(1) \xrightarrow{\sim} \pi^{\bullet}(\mathfrak{J}) \hookrightarrow \mathcal{O}_{\mathcal{B}}$. Namely, we have the inclusions $\mathfrak{J}^{n+1} \hookrightarrow \mathfrak{J}^n$ and there results a degree 0, injective, $\mathcal{O}_{\mathcal{B}}$-module map

$$\mathrm{Pow}^+_{\mathcal{O}_X}(\mathfrak{J})(l+1) \hookrightarrow \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J})(l), \quad \text{all } l \in \mathbb{Z}.$$

But, $\mathrm{Pow}^+_{\mathcal{O}_X}(\mathfrak{J})(l+1)$ and $\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J})(l+1)$ are (TN)-isomorphic, therefore we get the sheaf injection

$$\sigma_l\colon \mathcal{O}_{\mathcal{B}}(l+1) \to \mathcal{O}_{\mathcal{B}}(l).$$

When $l = 0$, this is our basic injection

$$\sigma_0\colon \mathcal{O}_{\mathcal{B}}(1) \to \mathcal{O}_{\mathcal{B}}.$$

whose image is just the ideal $\pi^{\bullet}(\mathfrak{J})$.

When $l = -1$, however, we get

$$s = \sigma_{-1}\colon \mathcal{O}_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}}(-1).$$

Of course, s corresponds to a section of $\mathcal{O}_{\mathcal{B}}(-1)$ over $\mathcal{B}$, we continue to refer to this section by the letter s and call it the *canonical section* of $\mathcal{O}_{\mathcal{B}}(-1)$. By some standard commutative diagram which will be left to the reader, it is not hard to see that σ_l is given in terms of s as follows:

We know $\mathcal{O}_{\mathcal{B}}(l)$ is $\mathcal{O}_{\mathcal{B}}(1)^{\otimes l}$, hence we get the map

$$\mathcal{O}_{\mathcal{B}}(l+1) = \mathcal{O}_{\mathcal{B}}(l+1) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}} \xrightarrow{1\otimes s} \mathcal{O}_{\mathcal{B}}(l+1) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(-1) = \mathcal{O}_{\mathcal{B}}(l)$$

which is just σ_l. The repeated composition

$$\mathcal{O}_{\mathcal{B}}(l) \longrightarrow \mathcal{O}_{\mathcal{B}}(l-1) \longrightarrow \mathcal{O}_{\mathcal{B}}(l-2) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathcal{B}}(1) \longrightarrow \mathcal{O}_{\mathcal{B}}$$

is then just the map $1 \otimes s^{\otimes l}$ (here, $1 =$ identity map on $\mathcal{O}_{\mathcal{B}}(l)$).

In terms of the canonical section, s, we can describe the space underlying E:

Proposition 7.59 *For the blow-up $\pi\colon \mathcal{B}_Y(X) \to X$, if $E = \pi^{-1}(Y)$ designates the exceptional locus, then $|E|$ is just the set of zeros of the canonical section, s. So, s defines the Cartier divisor, E.*

Proof. We can check this pointwise, so choose $x \in \mathcal{B}_Y(X)$. Write ξ_x for a choice of generator of $\mathcal{O}_{\mathcal{B}}(1)_x$, then we know that $\xi_x \otimes s_x$ generates the stalk of $\pi^\bullet(\mathfrak{J})$ at x. Thus, $\xi_x \otimes s_x$ is a unit (i.e., $x \notin |E|$) iff $s_X \notin \mathfrak{m}_x \mathcal{O}_{\mathcal{B}}(-1)_x$ iff $s(x) \neq 0$. $\square$

Suppose the $\mathcal{O}_Y$-module, $\mathfrak{J}/\mathfrak{J}^2$ is locally free on Y. Then, we know that its dual is the normal bundle to Y in X. We also know that the condition that $\mathfrak{J}/\mathfrak{J}^2$ be locally free will be satisfied when Y is smooth. In the locally free case, by (3) of Proposition 7.58, our exceptional locus is exactly $\mathbb{P}(\mathcal{N}^D_{Y \hookrightarrow X})$; so, the blow-up along Y is obtained by excising Y from X and inserting in its place the projectivized conormal bundle to Y in X. Now, we've defined projective space in terms of hyperplanes in an affine space. The dual projective space is defined in terms of the lines in the affine. Consequently, our $\mathbb{P}(\mathcal{N}^D_{Y \hookrightarrow X})$ is just the bundle whose fibre at each point of Y is the space of lines in the fibre of the normal bundle to Y in X at the corresponding point of Y. This bundle is what most authors define as the projectivized normal bundle to Y in X. The glueing of X with Y removed to the insertion of the projectivized normal bundle is clear: Follw along a line normal to Y and connect to the point of $\mathbb{P}(\mathcal{N}^D_{Y \hookrightarrow X})$ which corresponds.

Suppose that Z is a closed subscheme of X and Y is a closed subscheme of Z. Then, we seem to have two notions of the blow-up of Z along Y:

(1) $\mathcal{B}_Y(Z)$ in our sense as $\mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_Z}(\mathfrak{J})$, where $\mathfrak{J}$ defines Y in Z and

(2) Blow-up X along Y to get $\mathcal{B}_Y(X)$, consider $\pi^{-1}(Z - Y)$ and take its closure in $\mathcal{B}_Y(X)$. (This was the way $\mathcal{B}_P(Z)$ was defined in Chapter 2, Section 2.8, where Z was there a closed subvariety of $\mathbb{P}^n_k$ and P was a point of Z.) This second process gives what is usually called the *strict transform* (or *proper transform*) of Z under the blow-up of Y in X.

Do the two processes agree? Of course, the answer had better be "yes." And in fact, the blow-up enjoys a lifting property which will imply the affirmative result.

Proposition 7.60 *(Lifting property of blowing-up) Suppose X is a scheme, $\mathfrak{J}$ is a QC, f.g., $\mathcal{O}_X$-ideal and $\mathcal{B}_{\mathfrak{J}}(X)$ is the blowing-up of X along $\mathfrak{J}$. Given a morphism $\varphi\colon Z \to X$ assume that $\varphi^\bullet(\mathfrak{J})$ is an invertible ideal of $\mathcal{O}_Z$. Then, there exists a* **unique** *morphism $\psi\colon Z \to \mathcal{B}_{\mathfrak{J}}(X)$ lifting φ in the sense that the diagram below*

$$\begin{tikzcd} & \mathcal{B}_{\mathfrak{J}}(X) \arrow[d, "\pi"] \\ Z \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & X \end{tikzcd} \tag{†}$$

commutes.

Proof. Cover X by affines. If we can prove the result here, the uniqueness of our lifts shows they glue together to give the global morphism. Therefore, we may and do assume $X = \mathrm{Spec}\, A$, $\mathfrak{J} = \widetilde{J}$, where J is a f.g. A-ideal and then, $\mathcal{B} = \mathcal{B}_{\mathfrak{J}}(X)$ is just $\mathrm{Proj}(\mathrm{Pow}_A(J))$. Write $\alpha_0, \ldots, \alpha_r$ for the generators of J, then of course there is a surjection of graded algebras

$$\theta\colon A[T_0, \ldots, T_r] \to \mathrm{Pow}_A(J) = \coprod_{n\geq 0} J^n T^n$$

via $T_j \mapsto \alpha_j T$. From this, we get the closed immersion $\mathcal{B} \hookrightarrow \mathbb{P}^r_A$. Note that the kernel of θ is generated by all forms $F \in A[T_0, \ldots, T_r]$ so that $F(\alpha_0, \ldots, \alpha_r) = 0$.

For the morphism $Z \longrightarrow Z$ write $\mathcal{L}$ instead of $\varphi^\bullet(\mathfrak{J})$, then the images, s_j, of the α_j's—considered as global sections of $\mathcal{O}_Z$—are global sections of $\mathcal{L}$ that generate $\mathcal{L}$ everywhere on Z. Consequently, by the characterization of morphisms to $\mathbb{P}^n_A$ (Corollary 7.12), there exists a *unique* morphism

$$\widetilde{\psi}\colon Z \to \mathbb{P}^n_A$$

so that $\big(\widetilde{\psi}\big)^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) = \mathcal{L}$ and $s_i = \big(\widetilde{\psi}\big)^{-1}(T_i)$, where T_i is considered as a global section of $\mathcal{O}_{\mathbb{P}^n_A}(1)$.

The morphism $\widetilde{\psi}$ will actually factor through the subscheme $\mathcal{B}$ of $\mathbb{P}^n_A$ provided all the forms of $\mathrm{Ker}\,\theta$ (the ideal defining $\mathcal{B}$ in $\mathbb{P}^n_A$) vanish on $s_0, \ldots, s_r$. if $F \in \mathrm{Ker}\,\theta$ and has degree d, then $F(s_0, \ldots, s_r) \in \Gamma(Z, \mathcal{L}^{\otimes d})$ is the image of $F(\alpha_0, \ldots, \alpha_r)$. Yet, we've already remarked that $F(\alpha_0, \ldots, \alpha_r) = 0$; so, we get our diagram (†), as required.

We need to prove the uniqueness of the lifted morphism ψ (perhaps it does not come from $\widetilde{\psi}$). If ψ exists, then $\varphi^\bullet(J) = \psi^\bullet(\pi^\bullet(J))$, where $\pi^\bullet(J)$ is an ideal of $\mathcal{O}_{\mathcal{B}}$. Since $\pi^\bullet(J)$ is $\mathcal{O}_{\mathcal{B}}(1)$, we get

$$\mathcal{L} = \varphi^\bullet(J) = \psi^\bullet(\mathcal{O}_{\mathcal{B}}(1)).$$

Now, $\psi^*(\mathcal{O}_{\mathcal{B}}(1)) \longrightarrow \psi^\bullet(\mathcal{O}_{\mathcal{B}}(1)) = \mathcal{L}$ is a surjective map and (by Nakayama) surjective maps of locally free equal (finite) rank sheaves on LRS's are isomorphisms, we see that $\psi^*(\mathcal{O}_{\mathcal{B}}(1)) = \mathcal{L}$. The commutativity of our diagram shows that the sections $s_0, \ldots, s_r$ are the pullbacks of $\alpha_0 T, \ldots, \alpha_r T$, and these are just the pullbacks of $T_0, \ldots, T_r$ (as sections of $\mathcal{O}_{\mathbb{P}^n_A}(1)$). Therefore, our morphism ψ really does come from a morphism $\widetilde{\psi}$, and so uniqueness of the $\widetilde{\psi}$ yields the uniqueness for ψ. $\square$

Based on our proposition, we can answer the question about the differing notions of blowing-up.

Corollary 7.61 *If $\theta\colon Z \to X$ is a morphism and $\mathfrak{J}$ is a f.g. QC $\mathcal{O}_X$-ideal and we set $\mathfrak{I} = \theta^\bullet(\mathfrak{J})$, then there exists a unique morphism, $\widetilde{\theta}\colon \mathcal{B}_{\mathfrak{I}}(Z) \to \mathcal{B}_{\mathfrak{J}}(X)$, so that the diagram*

$$\begin{tikzcd}
\mathcal{B}_{\mathfrak{I}}(Z) \arrow[r, "\widetilde{\theta}"] \arrow[d, "\pi_Z"'] & \mathcal{B}_{\mathfrak{J}}(X) \arrow[d, "\pi_X"] \\
Z \arrow[r, "\theta"'] & X
\end{tikzcd}$$

commutes. When θ is a closed immersion, so is $\widetilde{\theta}$.

Proof. The existence and uniqueness of $\widetilde{\theta}$ follows directly from the proposition. All we need to prove is the last statement. We have $\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J})$, $\mathrm{Pow}_{\mathcal{O}_Z}(\mathfrak{I})$ and further $\mathfrak{I} = \theta^\bullet(\mathfrak{J})$. But, Z is a closed subscheme of X so there is a surjection $\mathcal{O}_X \longrightarrow \mathcal{O}_Z$. It follows immediately that $\mathfrak{J}$ maps onto $\mathfrak{I}$ under this surjection; thus, we get the surjective map of graded rings $\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J}) \longrightarrow \mathrm{Pow}_{\mathcal{O}_Z}(\mathfrak{I})$. Of course, this shows $\widetilde{\theta}$ is a closed immersion. $\square$

In the situation described by the closed immersion $Y \hookrightarrow Z \hookrightarrow X$, the blow-ups being separated morphisms, it follows immediately that the strict transform of Z in $\mathcal{B}_Y(X)$ is just $\mathcal{B}_Y(Z)$ (as a subscheme of $\mathcal{B}_Y(X)$). This answers our question and it shows that "embedded blow-up" is actually intrinsic (being just the abstract blow-up).

Example 2. Here, we will apply the blowing-up procedure to a singular surface in $\mathbb{A}^3$. The equation we'll consider is
$$f(x, y, z) = x^2 - z^3(z - y^2).$$
It is easy to see that the y-axis, given by $x = z = 0$ is contained in our surface. The partial derivatives of f w.r.t. x, y, z are
$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2yz^3, \quad \frac{\partial f}{\partial z} = -4z^3 + 3z^2y^2.$$
Hence, the singular locus (which is the set where f and its partial derivatives simultaneously vanish) is exactly $x = z = 0$. A picture of the real points of this surface is shown in Figure 7.5.

Figure 7.5: The surface $\mathcal{X}\colon f(x, y, z) = 0$.

We blow up the singular locus inside our surface $\mathcal{X}$. By our corollary, it is the same to blow-up the line $x = z = 0$ in $\mathbb{A}^3$ and to take the strict transform of $\mathcal{X}$ in this blow-up. Since the ideal definining the line is given by two generators, we need two variables u, v and form $\mathbb{P}^1_{\mathbb{A}^3} = \mathrm{Proj}(k[x, y, z][u, v])$ and consider the locus given by the orginal equation $f(x, y, z) = 0$ and the new relation $xv - uz = 0$. There are two patches to the $\mathbb{P}^1$ given by $u \neq 0$, respectively $v \neq 0$. If $u \neq 0$, set $\xi = \frac{v}{u}$, then $z = x\xi$. We substitute this in $f(x, y, z)$ and obtain the equation
$$x^2(1 - x\xi^3(x - y^2)) = 0. \tag{$*$}$$
Excising the singular locus from the surface means looking at the points of the surface where either x or z is $\neq 0$. Since $z = x\xi$, we must assume $x \neq 0$. From $(*)$, we get
$$1 - x\xi^3(x - y^2) = 0.$$

Now, for the closure of the smoth part of the surface in this coordinate patch, we let x and z got to 0. Our above equation leads to the contradiction $1 = 0$; hence, there are no points of the strict transform in this patch. In the second patch (where $v \neq 0$), we set $\eta = \frac{u}{v}$. Then, $x = \eta z$ and once again we obtain an equation

$$\eta^2 z^2 - z^3(z - y^2) = 0.$$

For the regular locus, x and z cannot simultaneously vanish, so $z \neq 0$. We obtain

$$\eta^2 - z(z - y^2) = 0.$$

For the closure, let z go to 0, then $\eta^2 = 0$. Hence, our strict transform cuts the exceptional locus in one point but tangent there. Now, $\eta = \frac{u}{v}$ and $v \neq 0$; hence we have only to deal with the affine patch $v = 1$. So, the proper transform of our surface is given by

$$u^2 - z(z - y^2) = 0. \tag{$**$}$$

We can relabel this equatin by setting $u = x$. A sketch of the blow-up situation is given in Figure 7.6.

Figure 7.6: The surface $\mathcal{X}$ after blowing-up the line $x = z = 0$.

We now have the new equation

$$g(x, y, z) = x^2 - z(z - y^2).$$

Once again, a check of the partial derivatives shows the only singularity is at the origin $x = y = z = 0$.

We now blow up the origin in $\mathbb{A}^3$. The ideal of the origin is given by three generators: x, y, z. And so, we need homogeneous coordinates u, v, w and we form $\mathbb{P}^2_{\mathbb{A}^3}$ and examine the subvariety whose equations are:

$$\begin{aligned} g(x, y, x) &= 0 \\ xv - yu &= 0 \\ xw - zu &= 0 \\ yw - zv &= 0. \end{aligned}$$

Here, there are three patches: $u \neq 0$; $v \neq 0$; $w \neq 0$. Let us look at the patch where $u \neq 0$. Set $\xi = \frac{v}{u}$ and $\eta = \frac{w}{u}$ so that $y = x\xi$ and $z = x\eta$. Upon substitution into $g = 0$ we obtain

$$x^2(1 - \eta^2 + \eta\xi^2) = 0.$$

Outside the singular locus, one of x, y, z must not be zero. If $x = 0$, so are y and z; so, $x \neq 0$. We cancel in the equation above and then to find the closure let x, y, z go to 0. This gives the equation

$$h(\eta, \xi) = 1 - \eta^2 + \eta\xi^2.$$

By ckecking its partial derivatives, we find that the surface $h = 0$ has no singularities. The (u, v, w)-form of $h = 0$ is

$$u^2 - w^2 + wv^2 = 0. \qquad (***)$$

There remain the patches $v \neq 0$ and $w \neq 0$. However, all we need to examine are the pieces of those patches where $u = 0$. So, assume $v \neq 0$ and $u = 0$. Write $\alpha = \frac{u}{v} = 0$ and $\beta = \frac{w}{v}$, then $x = \alpha y = 0$ and $z = \beta y$. To be off the singular locus we need to have $y \neq 0$. Upon substitution in our equation $g(x, y, z) = 0$, we obtain the equation

$$y^2(-\beta^2 + \beta y) = 0.$$

We may cancel y and let y and z go to 0. This shows that $\beta^2 = 0$ and gives one further point of tangency to the exceptional locus. On the affine patche where $v \neq 0$ we may take $v = 1$ and then our equation is

$$u^2 - w^2 + w = 0.$$

This is nonsingular.

There remains just the point where $u = v = 0$ and $w \neq 0$. An easy check shows that the strict transform does not go through this point.

Consequently for the singular surface sketched above, two blow-ups suffice to resolve singularities of our surface. Notice that had the origin been less singular than it actually was, one blow-up would have sufficed.

Example 3. We can use the blowing-up process to construct an example of a smooth, proper 3-fold which is *not* projective. The example we choose is due to H. Hironaka in his thesis [34]. We start with a nonsingular projective 3-fold over $\mathbb{C}$ and with two nonsingular curves, γ and $\widetilde{\gamma}$, on X so that γ and $\widetilde{\gamma}$ meet transversally in exactly two points P and Q. There are many ways to find such an X and curves $\gamma, \widetilde{\gamma}$. For example, we could take $X = \mathbb{P}^3_{\mathbb{C}}$ and γ a line and $\widetilde{\gamma}$ a conic intersecting γ. More generally, take any two curves in $\mathbb{P}^3_{\mathbb{C}}$ which intersect transversally at points $R_1, \ldots, R_q$ and blow up $R_2, \ldots, R_{q-1}$. Then, $X = \mathcal{B}_{R_2,\ldots,R_{q-1}}\mathbb{P}^3_{\mathbb{C}}$ and $\gamma, \widetilde{\gamma} =$ the strict transforms our two curves will do.

In any case, take such an X and $\gamma, \widetilde{\gamma}$. Consider $X - \{Q\}$ and blow up γ in $X - \{Q\}$ to get $\mathcal{B}_\gamma(X - Q)$. Let $\widetilde{\gamma}(P)$ be the strict transform of $\widetilde{\gamma}$ on $\mathcal{B}_\gamma(X - Q)$. Now blow up $\mathcal{B}_\gamma(X - Q)$ along $\widetilde{\gamma}(P)$ and get $\mathcal{B}_{\widetilde{\gamma}(P)}(\mathcal{B}_\gamma(X - Q))$. Repeat the same process replacing Q by P and interchanging the roles of γ and $\widetilde{\gamma}$. We obtain $\mathcal{B}_{\gamma(Q)}(\mathcal{B}_{\widetilde{\gamma}}(X - P))$. A picture of these various blow-ups in shown in Figure 7.7.

Figure 7.7: Hironaka's example of a 3-fold which is not projective

Over $X - P - Q$, by the lifting property, the two blow-ups in either order are isomorphic; so, we can glue together the two parts:

$$\mathcal{B}_{\widetilde{\gamma}(P)}(\mathcal{B}_{\gamma}(X-Q)) \quad \text{and} \quad \mathcal{B}_{\gamma(Q)}(\mathcal{B}_{\widetilde{\gamma}}(X-P))$$

along their overlap. This gives a new variety $\mathcal{X}$ over $\mathbb{C}$. Also, α (resp. β) is the generic point of γ (resp. $\widetilde{\gamma}$) and l_α (resp. l_β) is the generic fibre in the projectivized normal bundle sewn in by blowing up. Now, on $\mathcal{B}_{\gamma}(X-Q)$, l_α is algebraically equivalent to l_P (write: $l_\alpha \approx l_P$). Therefore, on $\mathcal{B}_{\widetilde{\gamma}(P)}(\mathcal{B}_{\gamma}(X-Q))$, we have $l_\alpha \approx l_P + \lambda_P$, while $l_\beta \approx \lambda_P$. Similarly, on $\mathcal{B}_{\widetilde{\gamma}}(X-P)$, we have $l_\beta \approx l_Q$ and on $\mathcal{B}_{\gamma(Q)}(\mathcal{B}_{\widetilde{\gamma}}(X-P))$, we have $l_\beta \approx l_Q + \lambda_Q$ while $l_\alpha \approx \lambda_Q$. On the glued variety, $\mathcal{X}$, we get

$$\lambda_Q \approx l_\alpha \approx l_P + \lambda_P \quad \text{and} \quad \lambda_P \approx l_\beta \approx l_Q + \lambda_Q.$$

Eliminate λ_P from these two algebraic equivalences and obtain

$$\lambda_Q \approx l_P + l_Q + \lambda_Q,$$

hence

$$l_P + l_Q \approx 0.$$

Were $\mathcal{X}$ projective, the algebraic curves (both are $\mathbb{P}^1$'s) would each have a degree and degrees add under addition of cycles and are preserved by algebraic equivalence (c.f. Section 7.4 and the discussion in the next section on flat families). Since the degree of l_P and l_Q is positive, the algebraic equivalence $l_P + l_Q \approx 0$ would be impossible in this case. Hence. $\mathcal{X}$ is not projective. However, $\mathcal{X}$ is proper because properness can be checked locally on the base and over the opens $X - Q$ or $X - P$ our $\mathcal{X}$ is projective hence proper.

It turns out that blowing-up a QC ideal is a very general process as we are going to see below. Because of this, to use blowing-up in an efficient manner we must restrict to specialized centers for blowing-up and to less than general schemes. We shall now restrict ourselves to irreducible varieties by which we always mean reduced, irreducible, separated, finite-type schemes over a field k, and we'll assume k is algebraically closed. Of course, we must now check that $\mathcal{B}_Y(X)$ is again a variety.

Proposition 7.62 *If X is a variety over k and $\mathfrak{I}$ is a coherent $\mathcal{O}_X$-ideal then $\mathcal{B}_{\mathfrak{I}}(X)$ is again a variety. Moreover, the map $\pi\colon \mathcal{B}_{\mathfrak{I}}(X) \to X$ is a birational, surjective, proper morphism and if X is quasi-projective or projective then so is $\mathcal{B}_{\mathfrak{I}}(X)$ and π is a projective morphism.*

Proof. $\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{I})$ is a sheaf of integral domains, so $\mathcal{B}_{\mathfrak{I}}(X)$ is reduced and irreducible. We know that $\pi\colon \mathcal{B}_{\mathfrak{I}}(X) \to X$ is proper (properness is local on the base and over an affine, $\mathcal{B}_{\mathfrak{I}}(X)$

is just an ordinary projective scheme so that it is proper, by Theorem 7.17 or Theorem 2.36.) Therefore, π is a separated and finite-type morphism and so $\mathcal{B}_{\mathfrak{J}}(X)$ is indeed a variety.

Let Y be the closed subscheme of X defined by $\mathfrak{J}$ then $Y \neq X$ and so the nonempty open $U = X - Y$ is dense. Now, $\pi\colon \pi^{-1}(U) \to U$ is an isomorphism and $\pi^{-1}(U)$ is also dense in $\mathcal{B}_{\mathfrak{J}}(X)$; so, π is a birational morphism. We already know that π is surjective.

Suppose that X is either projective or quasi-projective. Then, there exists an ample invertible sheaf, $\mathcal{L}$, on X so that $\mathfrak{J} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections if $n >> 0$. Now, X is noetherian so only finitely many sections are necessary. Consequently, we obtain the surjection

$$\mathcal{O}_X[T_0, \ldots, T_N] \longrightarrow \mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J} \otimes \mathcal{L}^{\otimes n}).$$

This gives the closed immersion

$$\mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J} \otimes \mathcal{L}^{\otimes n})) \longrightarrow \mathbb{P}^N_X.$$

But, $\mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J} \otimes \mathcal{L}^{\otimes n}))$ is isomorphic to $\mathrm{Proj}(\mathrm{Pow}_{\mathcal{O}_X}(\mathfrak{J})) = \mathcal{B}_{\mathfrak{J}}(X)$, and so, $\mathcal{B}_{\mathfrak{J}}(X)$ is a closed subscheme of $\mathbb{P}^N_X$. (We already know $\mathcal{B}_{\mathfrak{J}}(X)$ embedded in $\mathbb{P}(\mathcal{E})$, but here, we have proved the stronger assertion that it sits inside ordinary projective space over X.) Since π is a projective morphism, the scheme $\mathcal{B}_{\mathfrak{J}}(X)$ is quasi-projective or projective according as X is so. $\square$

Now we face the proof that blowing-up a coherent ideal is a very general process . This is the following

Theorem 7.63 *Suppose X is a quasi-projective variety and $\theta\colon Y \to X$ is a birational* **projective** *morphism where Y is another variety. Then, there exists a coherent ideal $\mathfrak{J} \subseteq \mathcal{O}_X$ and an X-isomorphism $\mathcal{B}_{\mathfrak{J}}(X) \xrightarrow{\sim} Y$.*

Proof. The morphism θ is projective and we are in the noetherian situation so there is a closed immersion $i\colon Y \to \mathbb{P}^r_X$, so that

$$\begin{tikzcd} & \mathbb{P}^r_X \arrow[d] \\ Y \arrow[ur, "i"] \arrow[r, "\theta"'] & X \end{tikzcd}$$

commutes. Write $i^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{L}$, an invertible sheaf on Y. Then, $\theta_*(\mathcal{L})$ is a coherent $\mathcal{O}_X$-module and we can form

$$\mathcal{S} = \mathcal{O}_X \amalg \coprod_{d \geq 1} \theta_*(\mathcal{L})^d T^d.$$

This $\mathcal{S}$ is a QC $\mathcal{O}_X$-algebra, but it may not be a ggr because $\mathcal{S}_1 = \theta_*(\mathcal{L})$ may not generate it.

To remedy this, we use the q-uple embedding, that is, we form

$$\mathcal{S}^{(q)} = \coprod_{d \geq 0} \mathcal{S}_{dq}.$$

We'll show that if $q >> 0$, then $\mathcal{S}^{(q)}$ is generated by its $\mathcal{S}_1^{(q)} = \mathcal{S}_q$. If we show this for each affine of an affine open cover of X, then, X being quasi-compact, it will be true for X. Therefore, we may assume X is affine, say $X = \operatorname{Spec} A$. Note that A is a f.g. k-algebra. Now, recall that if S is a ggr and S_1 is a f.g. A-module, then for any f.g., graded S-module, M, we have a (TN)-isomorphim

$$M \longrightarrow (M^\sharp)^\flat = \coprod_d \Gamma(\operatorname{Proj} S, M^\sharp(d)).$$

Further, if $W \hookrightarrow \mathbb{P}^r_A$ is $\operatorname{Proj}(A[T_0, \ldots, T_r]/I)$, it follows that

$$A[T_0, \ldots, T_r]/I \longrightarrow ((A[T_0, \ldots, T_r]/I)^\sharp)^\flat = \coprod_d \Gamma(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S}(d))$$

is a (TN)-isomorphism. (c.f. Theorem 7.4.) Our Y is given as $\operatorname{Proj}(A[T_0, \ldots, T_r]/I)$ corresponding to its closed immersion $Y \hookrightarrow \mathbb{P}^r_A$ and so, the algebra $S = A \amalg \coprod_{d\geq 1} \theta_*(\mathcal{L})^d T^d$ is (TN)-isomorphic to $A[T_0, \ldots, T_r]/I$ and the latter is an sggr.

When we couple the closed immersions $Y \hookrightarrow \mathbb{P}^r_A$ with the q-uple embedding (for some $q >> 0$) $\mathbb{P}^r_A \hookrightarrow \mathbb{P}^N_A$, we replace $\mathcal{S}$ by $\mathcal{S}^{(q)}$ and then the latter is a ggr as we've just seen. Therefore, we now have $Y \cong \operatorname{Proj} \mathcal{S}$ and $\mathcal{S}$ is a ggr as $\mathcal{O}_X$-algebra.

Were $\theta_*(\mathcal{L})$ an ideal, $\mathfrak{J}$, of $\mathcal{O}_X$, then $\mathcal{S}$ would be $\operatorname{Pow}_{\mathcal{O}_X}(\mathfrak{J})$ and we'd be done. Here, we'll see that "fractional ideals"' enter the picture. Recall that Y is reduced and irreducible so its sheaf of meromorphic functions, $\mathcal{M}\mathrm{er}(Y)$, is *constant*. Further, we can find an embedding $\mathcal{L} \longrightarrow \mathcal{M}\mathrm{er}(Y)$, so $\mathcal{L}$ is a subsheaf of $\mathcal{M}\mathrm{er}(Y)$ (c.f. Proposition 5.30.) Now, θ_* is left-exact, hence $\theta_*(\mathcal{L}) \hookrightarrow \theta_*(\mathcal{M}\mathrm{er}(Y))$. However, θ is *birational* therefore, $\theta_*(\mathcal{M}\mathrm{er}(Y)) = \mathcal{M}\mathrm{er}(X)$; and so, $\theta_*(\mathcal{L}) \hookrightarrow \mathcal{M}\mathrm{er}(X)$. We want to show that $\theta_*(\mathcal{L})$ has "bounded denominators."

Consider the "ideal of denominators" $(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X)$, where

$$(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X) = \{\xi \in \mathcal{O}_X \mid \xi\theta_*(\mathcal{L}) \subseteq \mathcal{O}_X\}.$$

(The above definition makes sense locally on affine patches and defines an $\mathcal{O}_X$-ideal.) I claim $(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X)$ is coherent. Of course, this is a local question, and on an affine, $\theta_*(\mathcal{L})$ corresponds to a f.g. A-submodule of $\mathcal{M}\mathrm{er}(X)$. So, we can take common denominators for its generators and get $(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X)$ is f.g. on affine patches. As A is noetherian, the coherence of $(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X)$ follows.

Now, X is assumed *quasi-projective*, so there exists an ample invertible sheaf, $\mathcal{M}$, on X. Therefore, $(\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X) \otimes \mathcal{M}^{\otimes q}$ is generated by its (finitely many) global sections if $q >> 0$. In particular, there exits a nonzero section

$$\mathcal{O}_X \longrightarrow (\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X) \otimes \mathcal{M}^{\otimes q}, \quad \text{if } q >> 0;$$

hence, a nonzero map

$$\mathcal{M}^{-\otimes q} \longrightarrow (\theta_*(\mathcal{L}) \longrightarrow \mathcal{O}_X)$$

(bounded denominators!). But then, by definition,

$$\mathcal{M}^{-\otimes q} \cdot \theta_*(\mathcal{L}) \hookrightarrow \mathcal{O}_X,$$

so, $\mathfrak{J} = \mathcal{M}^{-\otimes q}\, \theta_*(\mathcal{L})$ is a coherent $\mathcal{O}_X$-ideal.

Finally, we will prove that $Y \cong \mathcal{B}_{\mathfrak{J}}(X)$. Since $Y = \mathrm{Proj}\,\mathcal{S}$, we find, of course, that

$$Y = \mathrm{Proj}\left(\coprod_{d\geq 0}(\mathcal{M}^{-\otimes dq} \otimes \mathcal{S}_d)\right),$$

so we have to show that $\mathfrak{J}^d = \mathcal{M}^{-\otimes dq} \otimes \mathcal{S}_d$, for all $d \geq 1$. Of course, $\theta_*(\mathcal{L}^{\otimes d}) \subseteq \mathcal{M}\mathrm{er}(X)$ (just as with $d = 1$, $\theta_*(\mathcal{L}) \subseteq \mathcal{M}\mathrm{er}(X)$) and $\mathcal{M}$ is invertible (so flat) hence $\mathcal{M}^{-\otimes dq} \otimes \mathcal{S}_d = \mathcal{M}^{-dq}\, \mathcal{S}_d$. Now, $\mathcal{S}$ is a ggr and $\mathfrak{J} = \mathcal{M}^{-q}\, \theta_*(\mathcal{L}) = \mathcal{M}^{-q}\, \mathcal{S}_1$, so we find the natural surjection

$$\mathfrak{J}^d \longrightarrow \mathcal{M}^{-qd}\, \mathcal{S}_d, \quad d \geq 1.$$

However, both are subschemes of $\mathcal{M}\mathrm{er}(X)$ and the diagram

$$\begin{tikzcd} & \mathcal{M}\mathrm{er}(X) & \\ \mathfrak{J}^d \arrow[ur] \arrow[rr] & & \mathcal{M}^{-dq}\, \mathcal{S}_d \arrow[ul] \end{tikzcd}$$

commutes. We see that the lower arrow is injective and we are done. $\square$

Remark: Our theorem shows that blowing-up a coherent ideal is a *very* general process. It gives us *all* birational morphisms from a quasi-projective variety to another. Obviously, if we wish to understand birational projective morphisms as the result of a sequence of blowings-up, we will need to choose the centers of these blowings-up to be as simple as we possibly can make them.

One of the ways we can understand the blowing-up is to view it as the graph of a birational map. Recall that a rational map from one projective variety, X, to another, Y, is just an equivalence class of morphisms, $\varphi\colon U \to Y$, where U is a dense open of X and $\varphi\colon U \to Y$ and $\psi\colon V \to Y$ are equivalent iff $\varphi \restriction (U \cap V) = \psi \restriction (U \cap V)$. Obviously, there is a maximal open subvariety of X on which φ is defined and $X - \{\text{this open}\}$ is called the *indeterminacy locus* of the rational map φ.

Take a rational map $\varphi\colon X \dashrightarrow Y$, let U be its domain of definition, then in $U \prod Y$ we have the graph, $\Gamma_{\varphi\restriction U}$, of φ in our ususal sense. This is a closed subvariety of $U \prod Y$. Take the closure, Γ_φ, of $\Gamma_{\varphi\restriction U}$ in $X \prod Y$; this closure is called the *graph of the rational map, φ*. Observe that the graph of φ is a closed subvariety of $X \prod Y$. (It turns out that in characteristic 0 one can characterize rational maps as exactly those set maps from opens $U \subseteq X$ to Y for which the closure of the graph of the set map is a variety in $X \prod Y$. Unfortunately, this is not true in characteristic $p > 0$, as the Frobenius map $x \mapsto x^p$ from $\mathbb{A}^1$ to itself will show.) The second projection, pr_2, takes our graph Γ_φ to a closed subvariety in Y and this closed subvariey is always called the *image of φ*.

The image of φ is not the usual kind of image for it can happen that there is a point y in the image and no point x where φ is defined so that $\varphi(x) = y$.

We can also define the *inverse image of a subvariety, Z, of Y* by

$$\varphi^{-1}(Z) = pr_1(pr_2^{-1}(Z)).$$

In the other direction, we can take some subvariety, T, of X and form $pr_2(pr_1^{-1}(T))$. This is called the *image of T under φ* or the *total transform of T*. We can also restrict φ to T (meaning take the dense open U, intersect it with T, make the morphism taking $U \cap T$ to Y and close up its graph). Then, $\varphi \upharpoonright T$ is a rational map $T \dashrightarrow Y$, and so we can form its image. Unfortunately this image is not equal to the image of T, it is smaller, and is called the *proper transform of T*. The reader should take as rational map the inverse of a blow-up morphism and check these concepts for himself. Note that the maximal open of X on which our rational map, φ, is defined is just a variety birational to the original X; and so, to be a rational map just means that φ is a morphism on a variety birational to X. The concept that replaces surjectivity in this order of things is the notion of dominance, where we recall that $\varphi\colon X \dashrightarrow Y$ is *dominant* iff the image of φ (in the sense of rational maps) is all of Y. Of course all this means is that $\overline{\varphi(U)} = Y$.

Remark: In the complex case there is a famous theorem of Chow which says that a complex submanifold of $\mathbb{P}^r_{\mathbb{C}}$ is actually a projective algebraic variety. So, if $\varphi\colon X \to \mathbb{P}^1$ is a meromorphic function on our complex submanifold of $\mathbb{P}^r$ then, the graph of φ is a complex submanifold of $\mathbb{P}^1 \prod \mathbb{P}^r$. By Chow's theorem, this graph is algebraic and by the characterization of rational maps in characteristic zero as set maps whose graphs are algebraic subvarieties, we deduce that φ is a rational function.

Example 4. Quadric surfaces in $\mathbb{P}^3$.

Let $\mathcal{Q}$ be the quadric surface in $\mathbb{P}^3$ whose equation is $z_0z_3 - z_1z_2 = 0$. The point $P = (0\colon 0\colon 0\colon 1)$ is in $\mathcal{Q}$ and we can form the projection map from P defined on $\mathbb{P}^3 - \{P\}$ to $\mathbb{P}^2$, which in our chosen coordinates has the form:

$$\pi_P\colon (z_0\colon z_1\colon z_2\colon z_3) \mapsto (z_0\colon z_1\colon z_2).$$

Restrict π_P to $\mathcal{Q}$ and we obtain a rational map from $\mathcal{Q}$ to $\mathbb{P}^2$. Now, in general, projection from a point $P \in \mathbb{P}^n$ to $\mathbb{P}^{n-1}$ is a rational map from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ and let $\widetilde{\mathbb{P}^n}$ be the graph of this rational map in $\mathbb{P}^n \prod \mathbb{P}^{n-1}$. Then, the reader should check that the graph of the projection in this case together with its map $pr_1\colon \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ is exactly the blowing-up of $\mathbb{P}^n$ at the point P. In the case of a quadric when $n = 3$, the graph of the restriction of π_P to $\mathcal{Q}$ is precisely the blowing-up of $\mathcal{Q}$ at P, that is, the proper transfrom of $\mathcal{Q}$ in $\mathcal{B}_P(\mathbb{P}^3)$.

We can go a little further in our example by considering the second projection of $\Gamma_{\pi_P} \subseteq \mathcal{Q}\prod\mathbb{P}^2$ to $\mathbb{P}^2$. As a diagram of morphisms and rational maps the situation is

$$\begin{tikzcd}
 & \mathcal{Q}\prod\mathbb{P}^2 & \\
 & \Gamma_{\pi_P} \arrow[u] \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
\mathcal{Q} \arrow[rr, dashed, "\pi_P"'] & & \mathbb{P}^2
\end{tikzcd}$$

Let w_0, w_2, w_2 be homogeneous coordinates on $\mathbb{P}^2$, then Γ_{π_P} is the subvariety defined by the equations

$$\begin{aligned}
z_0z_3 &= z_1z_2\\
z_0w_1 &= z_1w_0\\
z_0w_2 &= z_2w_0\\
z_1w_2 &= z_2w_1.
\end{aligned}$$

Write E, as usual, for the exceptional locus of the blow-up at P, i.e., $E = pr_1^{-1}(P)$. We know E is a projective line. Now pr_2 maps E to $\mathbb{P}^2$, and we want to see what the image of E is. Cover $\mathbb{P}^2$ by the opens, $w_0 \neq 0$, $w_1 \neq 0$, $w_2 \neq 0$. On the first open where $w_0 \neq 0$, we let $\xi = w_1/w_0$, and $\eta = w_2/w_0$, then our equations give us

$$z_0\xi = z_1, \quad z_0\eta = z_2 \quad \text{and} \quad z_0z_3 - z_0^2\xi\eta = 0.$$

Off the exceptional locus, we must have $z_0 \neq 0$, else $z_0 = z_1 = z_2 = 0$ and we are at P. Thus, we find $z_3 = z_0\xi\eta$ and then to find the proper transform we let z_0 go to 0. We find $z_3 = 0$. Hence, on the patch $w_0 \neq 0$, the only possible point above P is $(0\colon 0\colon 0\colon 0)$, a non-point, that is, for $w_0 \neq 0$ there is *no* point of the proper transform.

Take $w_0 = 0$ but $w_1 \neq 0$. Let $u = w_0/w_1 = 0$, and $v = w_2/w_1$. Then, $z_0 = z_1u = 0$, and $z_2 = z_1v$. The last equation of Γ_{π_P} gives us

$$0 = z_1z_3u = z_1^2v.$$

Of course, outside E, the coordinate z_1 can't be zero, so we cancel it in the above and deduce that $z_1v = 0$. Let z_1 go to zero, then there is no restriction on v, so v is arbitrary.

Lastly, we take $w_2 \neq 0$ while $w_0 = w_1 = 0$ and find there is one extra point. Therefore, $pr_2(E)$ is exactly the line $w_0 = 0$.

Now look at the projection $pr_2\colon \Gamma_{\pi_P} \to \mathbb{P}^2$. Notice that if $w_0 \neq 0$ and we examine $pr_2^{-1}(w_0\colon w_1\colon w_2)$ we get the equations

$$z_0\xi = z_1, \quad z_0\eta = z_2, \quad z_0z_3 - z_0^2\xi\eta = 0,$$

where $\xi = w_1/w_0$ and $\eta = w_2/w_0$. Now, $z_0 \neq 0$ because $z_0 = 0$ implies $z_1 = z_2 = 0$ and then our inverse image intersects E, and we know the intersection is empty when $w_0 \neq 0$. Thus, we can cancel z_0 in the last equation and get the equation

$$z_3 = z_0 \xi \eta.$$

It follows immediately that above the point $(1\colon \xi\colon \eta)$ there is exactly one point

$$((z_0\colon z_0\xi\colon z_0\eta\colon z_0\xi\eta); (1\colon \xi\colon \eta)).$$

Now examine the case that $w_0 = 0$, and look at the point $(0\colon 1\colon 0)$, i.e., $w_1 \neq 0$ and $w_2 = 0$. Our equation gives us $z_0 = z_2 = 0$ and z_1 and z_3 are arbitrary and therefore pr_2 is not one-to-one. Over the other point $(0\colon 0\colon 1)$, we find $z_0 = z_1 = 0$ and again, the map pr_2 is not one-to-one. If $w_1 \neq 0$ and $w_2 \neq 0$, then $z_0 = 0$ and $z_1 = (w_1/w_2)z_2$, therefore z_1 and z_2 vanish or don't vanish simultaneously. If both are 0 only one point lies above, namely

$$((0\colon 0\colon 1); (0\colon w_1\colon w_2)).$$

If neither vanishes, then $z_1 = (w_1/w_2)z_2$ and $z_0z_3 = z_1z_2 = 0$, yet $z_1 \neq 0$ and $z_2 \neq 0$, a contradiction. So, above $(0\colon w_1\colon w_2)$, we have exactly one point. In conclusion, the map pr_2 is not one-to-one exactly over the points $Q = (0\colon 1\colon 0)$ and $R = (0\colon 0\colon 1)$ and the inverse images are lines. Observe that Γ_{π_P} is an algebraic variety mapping to $\mathbb{P}^2$ and that at two points Q and R, the inverse image under pr_2, is a codimension one subvariety. By the lifting property of blow-ups we obtain the isomorphism

$$\mathcal{B}_{Q,R}(\mathbb{P}^2) \xrightarrow{\sim} \Gamma_{\pi_P}.$$

So, the picture is exactly the one shown in Figure 7.8.

Figure 7.8: A rational map from $\mathcal{Q}$ to $\mathbb{P}^2$ and its graph

Here, L_Q and L_R are the images of E_Q and E_R under pr_1, while L_P is the image of E_P under pr_2. Thus, there is a birational map (not a morphism in any direction) between our quadric $\mathcal{Q}$ and $\mathbb{P}^2$. It blows up a point, P, of $\mathcal{Q}$ and blows down the two lines passing through P on $\mathcal{Q}$ and in the other direction it blows up two points, Q, R, of $\mathbb{P}^2$ and blows down the line joining them.

Suppose $X \dashrightarrow \mathbb{P}^n$ is a rational map. Can we find successive subvarieties, Y_i, in successive blow-ups, X_i, with $Y_0 = Y \subseteq X$ so that our rational map fits into a diagram

$$\begin{tikzcd}
X_t \arrow[d] \arrow[dddr, "\widetilde{\varphi}"] & \\
\vdots \arrow[d] & \\
X_1 \arrow[d] & \\
X \arrow[r, dashed, "\varphi"'] & \mathbb{P}^n
\end{tikzcd}$$

where $X_{i+1} = \mathcal{B}_{Y_i}(X_i)$ and $\widetilde{\varphi}\colon X_t \to \mathbb{P}^n$ is a *morphism*? Observe that in Example 4 we have exactly this situation. Here is a general theorem about this situation.

Theorem 7.64 *Suppose that $X \longrightarrow Y$ is a scheme over Y and $\mathcal{L}$ is a line bundle on X. Suppose further we are given sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ so that on an open $U \subseteq X$ they generate $\mathcal{L}$. Then, for the morphism*

$$\varphi_{\mathcal{L}}\colon U \to \mathbb{P}^n_Y$$

given by these sections, there exists a QC ideal, $\mathfrak{J}$, of $\mathcal{O}_X$ and there exists a morphism

$$\widetilde{\varphi}\colon \mathcal{B}_{\mathfrak{J}}(X) \to \mathbb{P}^n_Y$$

so that the diagram

$$\begin{tikzcd}
 & \mathcal{B}_{\mathfrak{J}}(X) \arrow[d, "\pi"] \arrow[ddr, "\widetilde{\varphi}"] & \\
 & X & \\
U \arrow[ur] \arrow[rr, "\varphi_{\mathcal{L}}"] & & \mathbb{P}^n_Y
\end{tikzcd}$$

commutes. In particular, when X is a variety ($Y = \operatorname{Spec} k$) and $\varphi_{\mathcal{L}}$ is a rational map we can use blowing-up to resolve the indeterminacy locus. The support of $\mathcal{O}_X/\mathfrak{J}$ is exactly $X - U$.

Proof. Of course we may assume that Y is affine as the question is local on Y. Write $\mathcal{F}$ for the subsheaf of $\mathcal{L}$ generated by $s_0, \ldots, s_n$. Cover X by opens X_α so that $\mathcal{L} \restriction X_\alpha$ is a free $\mathcal{O}_{X_\alpha}$-module. Then, $\psi_\alpha\colon \mathcal{L} \restriction X_\alpha \to \mathcal{O}_{X_\alpha}$ denotes our isomorphism and so, $\psi_\alpha(\mathcal{F} \restriction X_\alpha)$ is an ideal in $\mathcal{O}_{X_\alpha}$. On the overlaps $X_\alpha \cap X_\beta$, the isomorphisms ψ_α and ψ_β do not patch but are related one to the other by multiplication by g_α/g_β, which is a unit on $X_\alpha \cap X_\beta$. Hence, the ideals $\psi_\alpha(\mathcal{F} \restriction X_\alpha)$ and $\psi_\beta(\mathcal{F} \restriction X_\beta)$ are the same on $X_\alpha \cap X_\beta$ and therefore, we obtain a QC ideal, $\mathfrak{J}$, of $\mathcal{O}_X$ by glueing.

Since $\mathcal{F} = \mathcal{L}$ exactly on U we deduce that $\mathfrak{J} = \mathcal{O}_X$ exactly on U and therefore, $\operatorname{Supp} \mathcal{O}_X/\mathfrak{J} = X - U$.

Take $\mathcal{B}_{\mathfrak{J}}(X)$, we know that $\pi^{\bullet}(\mathfrak{J})$ is an invertible sheaf of ideals on $\mathcal{B}_{\mathfrak{J}}(X)$. Therefore, the global sections $\pi^*(s_0), \ldots, \pi^*(s_n)$ of $\pi^*(\mathcal{L})$ generate an invertible subsheaf, $\widetilde{\mathcal{L}}$, of $\pi^*(\mathcal{L})$. However, an invertible sheaf and generating sections are exactly what we need to give a morphism $\widetilde{\varphi}\colon \mathcal{B}_{\mathfrak{J}}(X) \to \mathbb{P}^n_Y$ and on $\pi^{-1}(U)$, since $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ are the same, our morphism is just $\varphi_{\mathcal{L}}$. $\square$

The reader should examine the simplest example: $\varphi\colon \mathbb{A}^2 \to \mathbb{P}^1$ given by

$$(x, y) \mapsto (x\colon y)$$

and go through the proof of the theorem to see that all we have done is to blow up the origin.

Staying in the classical case again, there is the notion of linear system and basepoint. We know that our line bundle $\mathcal{L}$ and our sections $s_0, \ldots, s_n$ give a linear system $\mathcal{D}$ on X, namely we look at all divisors linearly equivalent to the loci $\sum c_j s_j = 0$. The basepoint locus is given by $s_0 = s_1 = \cdots = s_n = 0$, call it Y and $U = X - Y$ is the locus where our linear system gives a morphism to projective space. The theorem says we can blow up the basepoint locus and obtain a new linear system on the blow-up without basepoints.

One of the main applications of blowing-up is to the question of *resolution of singularities*. This is the following problem: Given a variety X over a field k (we assume $\overline{k} = k$), find a proper birational *morphism* $\widetilde{X} \longrightarrow X$ so that $\widetilde{X}$ is nonsingular, i.e. a manifold. By Theorem 7.63, at least in the projective case, we know that $\widetilde{X} = \mathcal{B}_{\mathfrak{J}}(X)$ for some QC ideal $\mathfrak{J}$. We want some control over the morphism $\widetilde{X} \longrightarrow X$ and this means to make successive blowing-ups from X with known kinds of centers. There is also the question of *embedded resolution* of singularities. Here, $X \hookrightarrow W$, where W is already a nonsingular variety. The problem is to find a nonsingular variety $\widetilde{W}$ and a proper birational map $\pi\colon \widetilde{W} \to W$ so that

(1) The strict transform, $\widetilde{X}$, of X in $\widetilde{W}$ is nonsingular.

(2) $\pi^{-1}(X)$ is a divisor with normal crossings (i.e., if $Z_1, \ldots, Z_s$ are the irreducible components of $\pi^{-1}(X)$ meeting at some point P and if $f_1, \ldots, f_s$ are the local equations of these irreducible components, then the f's are part of a regular sequence in $\mathcal{O}_P$, which means of course, that $f_1, \ldots, f_s$ are linearly independent modulo $\mathfrak{m}_P^2$. The latter also means that the equation $f_1 f_2 \cdots f_s = 0$ describes $\pi^{-1}(X)$ at P.)

One can show that the morphism $\widetilde{X} \longrightarrow X$ induced by π is independent of the embedding $X \hookrightarrow W$.

The history of this problem and attempts at its solution is very long. For the case of curves both resolution and embedded resolution were settled by M. Noether and G. Halphen in the nineteenth century. In the early twentieth century, the Italian School by its synthetic method gave a proof for surfaces and a rigorous proof over $\mathbb{C}$ was first given by R.J. Walker in

1935 [?], while Zariski gave an algebraic proof valid in characteristic zero in 1939 [?]. Further, in 1944, Zariski settled embedded resolution for surfaces (char. 0) [?] and resolution for 3-folds, again in characteristic zero [?]. In his thesis in 1956, Abhyankar proved resolution for surfaces in characteristic $p > 0$ and in 1966 he gave a proof for 3-folds valid in characteristic $p > 5$. However, in 1964, Hironaka proved both resolution and embedded resolution in characteristic zero but for *all* dimensions. One the consequences of Hironaka's theorem is the following result:

Theorem 7.65 *Suppose X and X' are projective varieties in characteristic zero and both are nonsingular. If $\varphi\colon X \dashrightarrow X'$ is a rational map, then there exists a morphism $\widetilde{X} \longrightarrow X$ which is birational and proper and given by a finite succession of blowings-up with nonsingular varieties as centers and there exists a* **morphism** *$\widetilde{\varphi}\colon \widetilde{X} \to X'$ so that the diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[d] \arrow[dr, "\widetilde{\varphi}"] & \\ X \arrow[r, dashed, "\varphi"'] & X' \end{tikzcd}$$

commutes.

It is this theorem which is most often used in applications of Hironaka's resolution of singularities to problems in algebraic geometry, partial differential equations, etc.

If one will allow a change of function fields by finite extension, i.e., not insist on birationality in the theorem above, then the morphism $\widetilde{X} \longrightarrow X$ is called an *alteration* and De Jong has proved the above theorem for alterations in *any* characteristic [?]. De Jong's theorem suffices for almost all the applications in which Hironaka's theorem was used even though it is weaker. But, it holds in all characteristics.

Hironaka proposed an interesting combinatorial game in connection with the problem of resolving singularities. Write, as usual, $\mathbb{N}$ for the natural numbers $(0, 1, 2, \ldots)$. The game is as follows: One starts with a finite set of points, A, in $\mathbb{N}^n \subseteq \mathbb{R}^n_{\geq 0}$ and forms $B = \mathrm{conv}(A) + \mathbb{R}^n_{\geq 0}$, where B is simply the positive convex hull of A (see Figure 7.9).

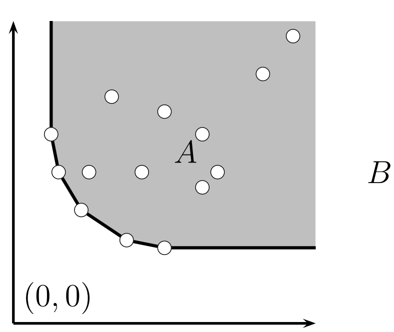

Figure 7.9: Positive convex hull B of A in $\mathbb{N}^n$

There are two players $\mathcal{P}_1$ and $\mathcal{P}_2$ and they make their moves by the following process: $\mathcal{P}_1$ chooses a nonempty subset, J, of $\{1, \ldots, n\}$ and $\mathcal{P}_2$ responds by choosing some $j \in J$. Then, the set A is changed by taking the j-component of each of its points and replacing that component with the sum of the components of it indexed by the chosen J.
$(P = (\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{j-1}, \sum_{k\in J} \alpha_k, \alpha_{j+1}, \ldots, \alpha_n).)$ We obtain a new set A' and we form again $B' = \mathrm{conv}(A') + \mathbb{R}^n_{\geq 0}$. And now, players $\mathcal{P}_1$ and $\mathcal{P}_2$ make their moves with the new set A'. And so the game goes. How is the winner determined? Simply this: player $\mathcal{P}_1$ wins after finitely moves if the positive convex hull B has become an orthant, i.e., $B = v + \mathbb{R}^n_{\geq 0}$, for some vector v. If this never occurs, $\mathcal{P}_2$ wins. Hironaka's problem is to show that $\mathcal{P}_1$ has a winning strategy no matter how $\mathcal{P}_2$ chooses his moves.

The case $n = 2$ is not only instructive, it is easy. For then, there are exactly three choices for $\mathcal{P}_1$: $J = \{1\}$, $J = \{2\}$, or $J = \{1, 2\}$. If $\mathcal{P}_1$ picks either of the first two, of course he forces $\mathcal{P}_2$'s move and the new "board" A' is identical to A. So, $\mathcal{P}_1$'s only choice is $J = \{1, 2\}$. But then, the change of board is just effected by a shear either horizontally (or vertically) and the shearing factor is larger the larger the y (resp. x) coordinate. So, it is clear that $\mathcal{P}_1$ will win.

What has this to do with singularities? Simply this: The points of A are the exponents of the monomials in our polynomial equations defining the possibly singular variety. Hironaka's game corresponds to blowing-up, *e.g.*, take the equation defining the cuspidal cubic $y^2 - x^3 = 0$. Either perform the blowing-up $z = x/y$ or $w = y/x$, then in the first case $zy = x$ so the equation becomes $y^2(1 - yz^3) = 0$. The points of the initial board are $(0, 2), (3, 0)$, while the points of the second board are $(0, 2), (3, 3)$. Notice that $(3, 3)$ is exactly the transform of $(3, 0)$ under the upward shear corresponding to $\mathcal{P}_2$'s choice of $j = 2$. Had $\mathcal{P}_2$ chosen $j = 1$, we would have had the blow-up $w = y/x$.

While Hironaka's game is trivial if $n = 2$, it is already nontrivial if $n = 3$.

Finally, here is another example of blowing-up to resolve singularities.

Example 5. Consider the surface $x^2 = z^2(z - y^2)$ whose singular locus is given by $x = z = 0$. A picture of this surface, $\mathcal{S}$, appears in Figure 7.10.

Figure 7.10: The surface $x^2 = z^2(z - y^2)$

Our surface $\mathcal{S}$ is embedded in $\mathbb{A}^3$ and we blow up the singular locus $x = z = 0$ and then take the proper transform of $\mathcal{S}$. The equation of the blow-up of $\mathbb{A}^3$ is

$$xw_2 - zw_1 = 0$$

in $\mathbb{A}^3 \prod \mathbb{P}^1$. We have two patches, $w_1 \neq 0$ and $w_2 \neq 0$ and we let $\xi = w_2/w_1$ on the first patch and $\eta = w_1/w_2$ on the second. Then, off the singular locus of $\mathcal{S}$ and on the first patch we have

$$x^2 = x^2\xi^2(x\xi - y^2).$$

Since x cannot be 0, we can cancel x^2 and get the equation of the proper transform

$$1 = \xi^2(x\xi - y^2).$$

Note that as x goes to 0 we find $1 = -\xi^2 y^2$ and so, if $y \neq 0$, the proper transform meets the exceptional locus in the two points

$$\left((0, y, 0); \left(1\colon\ \pm i\frac{1}{y}\right)\right).$$

When $y = 0$, the proper transform does not meet this part of the exceptional locus. On the second patch where $w_2 \neq 0$ and off the singular locus of $\mathcal{S}$, we find the equation

$$z^2\eta^2 = z^2(z - y^2).$$

Of course, $z \neq 0$, so we can cancel and obtain the equation of the proper transform in the second patch

$$\eta^2 = z - y^2.$$

When z goes to 0, we get $\eta^2 = -y^2$ and so, if $y \neq 0$ the proper transform meets the exceptional locus in the two points

$$((0, y, 0), (\pm iy\colon 1)).$$

However, when $y = 0$, our proper transform meets the exceptional locus tangentially in the point

$$((0, 0, 0); (0\colon 1)).$$

The two parts of the proper transform are nonsingular hypersurfaces.

Chapter 8

Proper Schemes and Morphisms

In the last chapter we made a fairly extensive study of what is perhaps the most important class of morphisms in algebraic geometry–projective morphisms. In the applications it turns out that one is faced frequently with morphisms which are not projective and yet are well-behaved in their topological properties. These morphisms are the analogs in algebraic geometry of the maps of topological spaces which are "relatively compact"–that is, the maps so that the inverse image of a compact set is compact. Notive that the base space need not be compact (a standard example is the open unit disk or the punctured unit disk) but, the fibres are always compact. These are the morphisms wich are called *proper morphisms*. The reader might review both Theorem 2.36 of Chapter 2, Section 2.5 and Theorem 7.17 of Chapter 7, Section 7.3; these theorems assert that projective morphisms are proper morphisms.

8.1 Proper Morphisms

We begin by recalling the formal definition of a proper morphism from Section 7.3.

Definition 7.3 If X is a scheme over S, then the morphism, $X \longrightarrow S$, is a *proper morphism* (we also say X *proper over* S) iff

(1) X is separated over S.

(2) $X \longrightarrow X$ is a finite-type morphism.

(3) The map $X \longrightarrow X$ is *universally closed*, that is, for every T over S, the morphism $pr_2 \colon X \prod_S T \to T$ is a closed map.

Remark: It is not clear at the outset that there exist non-projective proper morphisms. In fact, for a curve over a field, properness and projectivity coalesce. If X is a surface over a field and X is smooth, then again, properness and projectivity coalesce (at least in characteristic zero).

The following examples show that indeed there exist non-projective yet proper morphisms. Moreover, they occur by deforming projective morphisms; so, it is clear that they are a part of the natural landscape and not some isolated pathology.

** Examples to be supplied **

Having seem the existence of non-projective yet proper morphisms, there is an obvious question of how close they are. This question was considered by W.L. Chow and he proved the following theorem:

Theorem 8.1 *(Chow's Lemma) If S is a scheme and X is a separated finite-type S-scheme, assume one of:*

(α) S is noetherian or

(β) S is quasi-compact and X has only finitely many irreducible components.

Then,

(1) There exists a quasi-projective S-scheme, X', and an S-morphism, $f\colon X' \to X$, which is both surjective and projective.

(2) One can choose X' and $f\colon X' \to X$ so that there is an open, U, of X for which $f^{-1}(U)$ is dense in X' and f is an isomorphism of $f^{-1}(U)$ and U.

(3) If X is irreducible or reduced, X' may be chosen with the same property

$$\begin{tikzcd}
 & f^{-1}(U) \arrow[r] \arrow[dl, "f\restriction U"'] & X' \arrow[r] \arrow[dl, "f"'] \arrow[dd] \arrow[dr] & \mathbb{P}_X(\mathcal{F}) \arrow[dll] \\
U \arrow[r] & X \arrow[dr] & & \mathbb{P}_S(\mathcal{E}) \arrow[dl] \\
 & & X &
\end{tikzcd}$$

where $X' \longrightarrow \mathbb{P}_X(\mathcal{F})$ is a closed immersion and $X' \longrightarrow \mathbb{P}_S(\mathcal{E})$ is an immersion.

Proof. The main case is when X is an irreducible scheme. For, suppose the result is known in this case. If hypothesis (α) holds, X itself is noetherian and so has only finitely many components; and if (β) holds we have assumed only finitely many components. Give each reducible component, X_j, its reduced structure and consider the scheme, X'_j, assumed to exist in the irreducible case. Write $X' = \coprod_j X'_j$ and let f be the morphism $f\colon X' \to X$ induced by the morphisms $X'_j \longrightarrow X_j \longrightarrow X$. The scheme X' is projective over X because

$$\mathrm{Proj}_X(\mathcal{S}_1 \coprod \cdots \coprod \mathcal{S}_r) = \coprod_{i=1}^{r} \mathrm{Proj}_X(\mathcal{S}_i).$$

Of course, by the theorem in the irreducible case, each X'_j many be chosen to be reduced and so, X' is reduced, which would prove (3) in the general case. As for (2), consider the open U_i in X_i and write

$$\widetilde{U}_i = U_i \cap \Big(\bigcup_{j \neq i} X_j\Big)^c = U_i \cap \Big(\bigcap_{j \neq i} X_j^c\Big).$$

Take U to be the union of these $\widetilde{U}_i$. The removal of $\left(\bigcup_{j\neq i} X_j\right) \cap X_i$ from U_i to form $\widetilde{U}_i$ is necessitated because X' is the disjoint union of the X'_i. A picture of this situation is shown in Figure 8.1.

Figure 8.1: Construction of X' in Chow's lemma

It follows (by a formal argument or just by looking at the picture) that $f^{-1}(U)$ and U are isomorphic *via* f. Hence, the irreducibility statement of (3) is also proved. Finally, (1) holds as each X'_j is certainly X-projective, so therefore is the finite disjoint union, X'. Surjectivity is built in.

We are now reduced to the case X is irreducible. Consider the finite-type morphism $\pi\colon X \to S$. By definition, we can cover S by finitely many affine opens, S_α, so that each $X_\alpha = \pi^{-1}(S_\alpha)$ is itself covered by finitely many affine opens, X_α^β. Moreover, each $\Gamma(X_\alpha^\beta, \mathcal{O}_{X_\alpha^\beta})$ is a finitely generated $\Gamma(S_\alpha, \mathcal{O}_{S_\alpha})$-algebra. Thus, $X_\alpha^\beta \longrightarrow S_\alpha$ is a quasi-projective morphism, and as $S_\alpha \longrightarrow S$ is an open immersion, the composition, $\pi \upharpoonright X_\alpha^\beta\colon X_\alpha^\beta \to S$ is quasi-projective. It follows that for each α and β there is an open immersion, $\varphi_\alpha^\beta\colon X_\alpha^\beta \to P_\alpha^\beta$, where P_α^β is a projective S-scheme. Write $U = \bigcap_{\alpha,\beta} X_\alpha^\beta$; each X_α^β is open, hence dense in X (remember, X is irreducible), therefore, U is open and dense in X. But then, we have the morphism

$$\varphi\colon U \longrightarrow P = \prod_{\alpha,\beta} P_\alpha^\beta$$

induced by the φ_α^β; that is, the diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & P \arrow[d, "pr_\alpha^\beta"] \\ X_\alpha^\beta \arrow[r] & P_\alpha^\beta \end{tikzcd} \tag{$\dagger$}$$

commute. Now, we have two morphisms, $U \longrightarrow X$ and $U \longrightarrow P$; so, we get the immersion $\psi\colon U \to X \prod_S P$. If hypothesis (α) holds, then $X \prod_S P$ is certainly noetherian, while if (β) holds it is quasi-compact, In either case, the closure of the scheme induced on the subspace, $\psi(U)$, by $X \prod_S P$ exists; this is our scheme X'.

We have the morphism

$$U \overset{\psi'}{\hookrightarrow} X' \overset{g}{\hookrightarrow} X \prod_S P,$$

in which ψ' is an open immersion and g is a closed immersion. Now, define f to be the composed morphism

$$f\colon X' \overset{g}{\hookrightarrow} X \prod_S P \xrightarrow{pr_1} X$$

and θ to be the composed morphism

$$\theta\colon X' \overset{g}{\hookrightarrow} X \prod_S P \xrightarrow{pr_2} X.$$

Of course, as $X \prod_S P$ is projective over X, our morphism $f\colon X' \to X$ is projective.

There are several things to prove:

(a) $f \restriction f^{-1}(U)\colon f^{-1}(U) \longrightarrow U$ is an isomorphism and f itself is surjective.

(b) θ is an immersion (so that X' is indeed quasi-projective over S).

(c) If X is reduced, so is X'. (Note that irreducibility of X' follows as $f^{-1}(U)$ is irreducible being isomorphic to U by (a).)

(a) Look a the diagram

$$\begin{tikzcd} U \arrow[r, "\psi'"] \arrow[dr] & X' \arrow[r, "g"] \arrow[d, "f"] & X \prod_S P, \arrow[dl, "pr_1"] \\ & X & \end{tikzcd}$$

it commutes, so $f(X')$ contains the dense open set, U, of X. However, f is the composition of closed morphisms, so it is closed. Hence, $f(X') = X$.

To see that f is birational (i.e., $f^{-1}(U) \cong U$ *via* f), write $U' = g^{-1}(U \prod_S P)$ and note that the scheme structure on it is induced by X'. However, in the standard way, the immersion $\psi\colon U \to X \prod_S P$ factors as

$$U \xrightarrow{\Gamma} U \prod_S P \longrightarrow X \prod_S P,$$

where Γ is the graph morphism of $\varphi\colon U \to P$. The scheme P is separared, so Γ is a closed immersion. Now, as remarked, the scheme whose closure is X' has as closure in $U \prod_S P$ the scheme U', but Γ is a closed immersion, so this scheme is already closed in U'; hence, it is U'. Since $\psi\colon U \to X \prod_S P$ is an immersion we find that $f \restriction U'$ and $\psi'\colon U \to U' \subseteq X'$ are inverse isomorphisms. (Since $U' = f^{-1}(U)$ and X' is the closure of U', we see that u' is dense in X' and X' is indeed irreducible.)

(b) We now must show that

$$\theta\colon X' \overset{g}{\hookrightarrow} X \prod_S P \xrightarrow{pr_2} X$$

is an S-immersion. Of course, this is a local question on X' and will be achieved by finding a convenient open cover of X' where immersion can be proved.

We have the open immersions $\varphi_\alpha^\beta\colon X_\alpha^\beta \to P_\alpha^\beta$, so let Y_α^β be $\varphi_\alpha^\beta(X_\alpha^\beta)$, each Y_α^β is then an open of the corresponding P_α^β. Then, there are the projections $pr_\alpha^\beta\colon P \to P_\alpha^\beta$, so let $W_\alpha^\beta = (pr_\alpha^\beta)^{-1}(Y_\alpha^\beta)$. The W_α^β are opens in P. Finally, set

$$\begin{aligned} (X_\alpha^\beta)' &= f^{-1}(X_\alpha^\beta) \\ (X_\alpha^\beta)'' &= \theta^{-1}(W_\alpha^\beta). \end{aligned}$$

Both $(X_\alpha^\beta)'$ and $(X_\alpha^\beta)''$ are families of opens in X'; the convenient one will turmn out to be $(X_\alpha^\beta)''$. As f is surjective, it is clear that the $(X_\alpha^\beta)'$ form an open cover of X'. Suppose we can show that $(X_\alpha^\beta)' \subseteq (X_\alpha^\beta)''$ for all α and β, then the latter will be an open cover of X', too. *Assume for the moment the statement* $(X_\alpha^\beta)' \subseteq (X_\alpha^\beta)''$.

The W_α^β are an open cover of $\theta(X')$, and θ will therefore be an immersion if each $\theta \restriction (X_\alpha^\beta)''$ is an immersion into W_α^β. To prove this local immersion property, consider the morphism

$$w_\alpha^\beta\colon W_\alpha^\beta \xrightarrow{pr_\alpha^\beta} Y_\alpha^\beta \xrightarrow{(\varphi_\alpha^\beta)^{-1}} X_\alpha^\beta \hookrightarrow X,$$

and use the hypothesis that X is separated over S to deduce that the graph morphism, Γ_α^β, of w_α^β is a closed immersion

$$\Gamma_\alpha^\beta\colon W_\alpha^\beta \longrightarrow W_\alpha^\beta \prod_S X.$$

Write T_α^β for the image of W_α^β in $W_\alpha^\beta \prod_S X$, of course, T_α^β is closed in the latter scheme. I claim there is a morphism

$$z_\alpha^\beta\colon U' \to W_\alpha^\beta,$$

so that the diagram

$$\begin{tikzcd} U' \arrow[rr, "z_\alpha^\beta"] \arrow[dr, "v_\alpha^\beta"'] & & W_\alpha^\beta \arrow[dl, "\Gamma_\alpha^\beta"] \\ & X \prod_S W_\alpha^\beta & \end{tikzcd} \tag{††}$$

commutes, where v_α^β is the canonical injection. To see this, by definition of the product, we need to prove that the two arrows

$$U' \xrightarrow{v_\alpha^\beta} X \prod_S W_\alpha^\beta \xrightarrow{pr_1} X \quad \text{and} \quad U' \xrightarrow{v_\alpha^\beta} X \prod_S W_\alpha^\beta \xrightarrow{pr_2} W_\alpha^\beta \xrightarrow{w_\alpha^\beta} X$$

are the same map. However, this is clear from the commutativity of the diagram

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d] & (X_\alpha^\beta)' \arrow[r] & W_\alpha^\beta \subseteq P \arrow[d, "pr_\alpha^\beta"] \\ U \arrow[r, hook] & X_\alpha^\beta \arrow[r] & Y_\alpha^\beta \subseteq P_\alpha^\beta, \end{tikzcd} \tag{$*$}$$

which in turn is commutative because of the commutativity of (†). So, we do indeed get the morphism $z_\alpha^\beta \colon U' \to W_\alpha^\beta$, factoring the canonical injection *via* the graph, as shown in (††).

From diagram (††), we see that T_α^β contains the scheme U', and so–as X' is the closure of U'–the scheme T_α^β also contains the scheme induced by X' on its open $(X_\alpha^\beta)''$. But, the second projection

$$X \prod_S W_\alpha^\beta \longrightarrow W_\alpha^\beta$$

is an isomorphism of T_α^β and W_α^β; therefore, we finally find the immersion of $(X_\alpha^\beta)''$ into W_α^β, *via* θ.

To finish this part of the proof it remains to show the assumed assertion: $(X_\alpha^\beta)' \subseteq (X_\alpha^\beta)''$.

Now, $g^{-1}(X_\alpha^\beta \prod_S P)$ is the scheme that X' induces on $(X_\alpha^\beta)'$. So, it is the closure of U' in $(X_\alpha^\beta)'$. Look at the diagram exactly analogous to $(*)$ but with $(X_\alpha^\beta)'$ replacing $(X_\alpha^\beta)''$. The righthand inner square of this diagram:

$$\begin{tikzcd} (X_\alpha^\beta)' \arrow[r, "\theta"] \arrow[d, "f"'] & P \arrow[d, "pr_\alpha^\beta"] \\ U \arrow[r, "\varphi_\alpha^\beta"] & P_\alpha^\beta, \end{tikzcd} \tag{$**$}$$

will commute (by separation of P and P_α^β over S) provided the outer square analog of $(*)$:

$$\begin{tikzcd} U' \arrow[r, "\theta"] \arrow[d, "f"'] & P \arrow[d, "pr_\alpha^\beta"] \\ U \arrow[r, "\varphi_\alpha^\beta"] & P_\alpha^\beta, \end{tikzcd}$$

also commutes. But, the latter is just (†), which commutes. But, (∗∗) implies immediately that $(X_\alpha^\beta)' \subseteq (X_\alpha^\beta)''$ by their definitions, as contended.

(c) We need only prove that if X is reduced, so is X'. But, if X is reduced, so is U; therefore, so is U' as it is isomorphic to U. Then, X', as the closure of U', is again reduced. □

Corollary 8.2 *(Chow's Lemma, proper case) If S is a scheme and X is proper over S and if either*

(α) S is noetherian or

(β) S is quasi-compact and X has only finitely many irreducible components,

then we can find a projective S-scheme, X', and a surjective morphism, $f\colon X' \to X$, so that there is an open, U, in X whose inverse image, $f^{-1}(U)$, is dense in X' and f is an isomorphism of $f^{-1}(U)$ and U. We can choose X' irreducible if X is irreducible and X' reduced is X is reduced. In case X is irreducible the surjective morphism $f\colon X' \to X$ is birational.

Proof. This is just our theorem but with the extra assertion that X' is *projective over S*. However, the composed morphism $X' \longrightarrow X \longrightarrow S$ makes X' a proper scheme over S because the first morphism ($X' \longrightarrow X$) is projective hence proper and the second morphism ($X \longrightarrow S$) is assumed proper. Then, X' is both proper and quasi-projective (by Theorem 8.1); so, it is projective. □

Remark: If $X' \longrightarrow X$ is surjective and if X' is itself S-projective, then X will be proper provided it is separated and finite-type over S. (This statement is essentially the converse of Chow's lemma in the proper case.)

8.2 Finiteness Theorems for Proper Morphisms; Applications

In Chapter 4, Section 4.3, after we proved that for reasonable morphisms the higher-direct images of QC sheaves were themselves QC, we mentioned that for coherent sheaves the situation was more difficult. For projective morphisms, the Serre finiteness theorem (Chapter 7, Section 7.5). Here, we face the general case of a proper morphism. Of course, an obvious idea is to somehow use Chow's lemma (our Theorem 8.1) to get the Serre theorem to apply. But, it is not at all obvious how to do this. The essential trick is due to Grothendieck and is contained in the next theorem. But first, we need some terminology.

If $\mathcal{A}$ is an abelian category and $\widetilde{\mathcal{A}}$ is a subclass of $\mathcal{O}\mathrm{b}(\mathcal{A})$, we say that $\mathcal{A}$ is *thick* when it has the following property: Given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of $\mathcal{A}$ if any two of the three objects A, B, C are in $\widetilde{\mathcal{A}}$, then so is the third.

For example, in Appendix A, we showed that the subclass of *coherent* $\mathcal{O}_X$-modules in the abelian category of all $\mathcal{O}_X$-modules is thick. Let us also call $\widetilde{\mathcal{A}}$ *very thick* whenever it is both thick and satisfies:

If A is a cofactor of an object in $\widetilde{\mathcal{A}}$ (i.e., $A \coprod B$ is in $\widetilde{\mathcal{A}}$) then A itself in in $\widetilde{\mathcal{A}}$.

In addition we need another property of a subclass, $\widetilde{\mathcal{K}}$, of the abelian category, $\mathrm{Coh}(\mathcal{O}_X)$, of coherent $\mathcal{O}_X$-modules, this time connected with a closed subset, $\widetilde{X}$, of X. This is:

$\widetilde{\mathcal{K}}$ is *strongly* $\widetilde{X}$*-dense in* $\mathrm{Coh}(\mathcal{O}_X)$ iff for every irreducible closed subset, Y, of $\widetilde{X}$, with generic point y, there exists a coherent $\mathcal{O}_X$-module, $\mathcal{G}$, in $\widetilde{\mathcal{K}}$ whose *stalk*, $\mathcal{G}_y$, at y, is a one-dimensional $\kappa(y)$-space. The subclass $\widetilde{\mathcal{K}}$ is $\widetilde{X}$*-dense in* $\mathrm{Coh}(\mathcal{O}_X)$ iff for all irreducible closed $Y \subseteq \widetilde{X}$ there is a coherent $\mathcal{O}_X$-module, $\mathcal{G}$, whose support is Y and which belongs to $\widetilde{\mathcal{K}}$.

Remarks:

(1) The terminology "thick" is not what one finds in EGA–there one finds (unfortunately) the overworked word "exact." However, it is clear why $\widetilde{\mathcal{A}}$ should be called thick.

(2) As for the concept of denseness, first observe that because it is the *stalk* of $\mathcal{G}$ at y (*not* the fibre of $\mathcal{G}$ at y) which is a one-dimensional $\kappa(y)$-space, the support of such a $\mathcal{G}$ is exactly Y. (It is contained in Y as one should be able to see immediately, and it is all of Y being closed and containing the generic point.) So, strong $\widetilde{X}$-denseness implies $\widetilde{X}$-denseness. Moreover, $\widetilde{X}$-denseness merely says that

$$\widetilde{\mathcal{K}} \cap (\text{part of } \mathrm{Coh}(\mathcal{O}_X) \text{ with support} = Y) \neq \emptyset$$

for every irreducible $Y \subseteq \widetilde{X}$.

Having explained the terminology we can prove

Theorem 8.3 *(Unscrewing Lemma) If X is a noetherian scheme and $\widetilde{X}$ is a closed subset of X, and if we are given a subclass, $\widetilde{\mathcal{K}}$, of* $\mathrm{Coh}(\mathcal{O}_X)$ *so that either*

(I) $\widetilde{\mathcal{K}}$ is thick in $\mathrm{Coh}(\mathcal{O}_X)$ *and strongly $\widetilde{X}$-dense there, or*

(II) $\widetilde{\mathcal{K}}$ is very thick in $\mathrm{Coh}(\mathcal{O}_X)$ *and $\widetilde{X}$-dense there,*

holds, then every coherent $\mathcal{O}_X$-module with support in $\widetilde{X}$ is already in $\widetilde{\mathcal{K}}$. In particular, when $\widetilde{X} = X$, we obtain $\widetilde{\mathcal{K}} =$ $\mathrm{Coh}(\mathcal{O}_X)$.

Proof. We use noetherian induction; that is, if we write $P(Z)$ for the statement: A coherent $\mathcal{O}_X$-module with support in Z is already in $\widetilde{\mathcal{K}}$, we must prove

whenever Y is closed in $\widetilde{X}$ and if for all Y' closed and strictly contained in Y we know $P(Y')$ is true, then $P(Y)$ is true.

So, take Y closed in $\widetilde{X}$, assume for each closed $Y' < Y$ every sheaf, $\mathcal{M}$, with support contained in Y' is already in $\widetilde{\mathcal{K}}$, and take a sheaf $\mathcal{F}$ whose support is in Y, we must show $\mathcal{F} \in \widetilde{\mathcal{K}}$. Give Y its reduced structure as scheme so that Y is defined by a coherent ideal. $\mathfrak{I}$, of $\mathcal{O}_X$. Now, X is noetherian, $\mathcal{F}$ is coherent and the support of $\mathcal{F}$ is in Y, so there is an integer $n > 0$ with $\mathcal{G}^n\mathcal{F} = (0)$. For any integer, k, with $1 \leq k \leq n$, we have the exact sequence

$$0 \longrightarrow \mathfrak{I}^{k-1}\mathcal{F}/\mathfrak{I}^k\mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{I}^k\mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{I}^{k-1}\mathcal{F} \longrightarrow 0.$$

By ordinary induction on k we must prove the sheaf $\mathfrak{I}^{k-1}\mathcal{F}/\mathfrak{I}^k\mathcal{F}$ is in $\widetilde{\mathcal{K}}$ and then $\mathcal{F}$ will be in $\widetilde{\mathcal{K}}$ because $\widetilde{\mathcal{K}}$ is thick. The latter sheaf is killed by $\mathfrak{I}$, therefore we may and do assume $\mathfrak{I}\mathcal{F} = (0)$. This means that $\mathcal{F} = i_*(i^*(\mathcal{F}))$, where $i\colon Y \to X$ is the canonical closed immersion. There are two cases:

Case A: Y is reducible. We have $Y = Y' \cup Y''$ with closed subsets $Y' < Y$ and $Y'' < Y$. Again, give Y' and Y'' their reduced scheme structures, defined by the coherent ideals, $\mathfrak{I}'$ and $\mathfrak{I}''$ of $\mathcal{O}_X$. If we set $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_X/\mathfrak{I}'$ and $\mathcal{F}'' = \mathcal{F} \otimes \mathcal{O}_X/\mathfrak{I}''$, we get the homomorphisms $\mathcal{F} \longrightarrow \mathcal{F}'$ and $\mathcal{F} \longrightarrow \mathcal{F}''$. Therefore, we deduce a map $\theta\colon \mathcal{F} \to \mathcal{F}' \coprod \mathcal{F}''$ and the question is local where everything is affine. So, if $z \notin Y' \cap Y''$, either $\mathfrak{I}'_z = \mathcal{O}_{X,z}$ or $\mathfrak{I}''_z = \mathcal{O}_{X,z}$. In the either case, the map θ_z is bijective. Hence, Ker θ and Coker θ have their supports in $Y' \cap Y''$. But Ker θ and Coker θ are coherent sheaves, and so by our assumption lie in $\widetilde{\mathcal{K}}$. Then, the exact sequence

$$0 \longrightarrow \mathrm{Im}\ \theta \longrightarrow \mathcal{F}' \coprod \mathcal{F}'' \longrightarrow \mathrm{Coker}\ \theta \longrightarrow 0$$

shows that Im θ is in $\widetilde{\mathcal{K}}$ because, $\mathcal{F}'$ and $\mathcal{F}''$ are there (assumption) and $\widetilde{\mathcal{K}}$ is thick. Now, the exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \theta \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im}\ \theta \longrightarrow 0$$

proves that $\mathcal{F}$ is in $\widetilde{\mathcal{K}}$, and case A is proved. Notice that the density of $\widetilde{\mathcal{K}}$ was not used in this part of the proof.

Case B: Y is irreducible—hence integral. Here, we will need some form of the $\widetilde{X}$-density of $\widetilde{\mathcal{K}}$.

First assume (I): $\widetilde{\mathcal{K}}$ is thick and strongly $\widetilde{X}$-dense. For y, the generic point of Y we have $\mathcal{O}_{Y,y} = \kappa(y)$ and $\mathcal{F}_y = (i^*(\mathcal{F}))_y$ is a finite-dimensional $\kappa(y)$-space because $i^*(\mathcal{F})$ is coherent. As $\widetilde{\mathcal{K}}$ is strongly $\widetilde{X}$-dense, there is a coherent $\mathcal{O}_X$-module, $\mathcal{G}$, with

(α) $\mathcal{G} \in \widetilde{\mathcal{K}}$.

(β) $\mathcal{G}_y$ is a one-dimensional $\kappa(y)$-space.

So, there is some $\kappa(y)$-isomorphism $\mathcal{G}_y^m \cong \mathcal{F}_y$. But, both $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-coherent, so the iomorphism come from an isomorphism

$$\mathcal{G}^m \restriction W \cong \mathcal{F} \restriction W$$

for some open neighborhood, W, of y *in* X (coherence implies finite presentation, cf. Appendix A, Corollary A.20).

Write $\mathcal{H}$ for the graph of this last isomorphism; it is a coherent $\mathcal{O}_W$-submodule of $(\mathcal{G}^m \prod \mathcal{F}) \restriction W$ isomorphic to both $\mathcal{G}^m \restriction W$ and $\mathcal{F} \restriction W$. Because $\mathcal{G}_y$ is a finite-dimensional $\kappa(y)$-space, the sheaf $\mathcal{G}^m$ has support exactly Y (as already mentioned) and so the sheaf $\mathcal{G}^m \prod \mathcal{F}$ has support eaxctly Y. Consequently, there is a coherent $\mathcal{O}_X$-module, $\widetilde{\mathcal{H}}$, contained in $\mathcal{G}^m \prod \mathcal{F}$ so that

(i) $\widetilde{\mathcal{H}} \restriction (X - Y) = (0)$ and

(ii) $\widetilde{\mathcal{H}} \restriction W = \mathcal{H}$.

Look at the two projections $\mathcal{G}^m \prod \mathcal{F} \longrightarrow \mathcal{G}^m$ and $\mathcal{G}^m \prod \mathcal{F} \longrightarrow \mathcal{F}$, and restrict them to the submodule $\widetilde{\mathcal{H}}$. We get the $\mathcal{O}_X$-module maps

$$\varphi\colon \widetilde{\mathcal{H}} \to \mathcal{G}^m \quad \text{and} \quad \psi\colon \widetilde{\mathcal{H}} \to \mathcal{F}.$$

On the open, W, these maps are isomorphisms, and on $X - Y$ they are also isomorphisms because both sides are (0). Thus, the kernel and cokernel of φ and ψ have their supports in $Y - W \cap Y$; and this a proper closed subset of Y. Our assumption shows that Ker φ, Coker φ, Ker ψ, Coker ψ all lie in $\widetilde{\mathcal{K}}$. Also $\mathcal{G} \in \widetilde{\mathcal{K}}$ and therefore, by thickness, $\mathcal{G}^m \in \widetilde{\mathcal{K}}$. We deduce as before from φ that $\widetilde{\mathcal{H}} \in \widetilde{\mathcal{K}}$. And now, we deduce again as before from ψ that $\mathcal{F} \in \widetilde{\mathcal{K}}$.

Now, assume (II). Here, the sheaf $\mathcal{G}$ has a stalk of dimension $n > 0$ at y (because the support of $\mathcal{G}$ is exactly Y). This means that for some m and q, we have an isomorphism

$$(\mathcal{G}_y)^m \cong (\mathcal{F}_y)^q.$$

We continue exactly the same argument as above and deduce that $\mathcal{F}^q = \underbrace{\mathcal{F} \coprod \cdots \coprod \mathcal{F}}_{q}$ lies in $\widetilde{\mathcal{K}}$. But now, $\widetilde{\mathcal{K}}$ is very thick, so $\mathcal{F} \in \widetilde{\mathcal{K}}$. $\square$

We can finally prove the finteness theorem for proper morphisms. In the proof we will use the following easy lemma:

Lemma 8.4 *Suppose*

$$\mathcal{F}_1 \xrightarrow{u} \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_4 \xrightarrow{v} \mathcal{F}_5$$

is an exact sequence of $\mathcal{O}_X$*-modules and* $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_5$ *are all coherent. Then,* $\mathcal{F}_3$ *is coherent.*

Proof. From our exact sequence, we get the short exact sequence

$$0 \longrightarrow \mathrm{Coker}\, u \longrightarrow \mathcal{F}_3 \longrightarrow \mathrm{Ker}\, v \longrightarrow 0. \qquad (*)$$

We also have the short exact sequences

$$0 \longrightarrow \mathrm{Im}\, u \longrightarrow \mathcal{F}_2 \longrightarrow \mathrm{Coker}\, u \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ker}\, v \longrightarrow \mathcal{F}_4 \longrightarrow \mathrm{Im}\, v \longrightarrow 0.$$

If we prove that both Coker u and Ker v are coherent, then $(*)$ implies $\mathcal{F}_3$ is coherent. From the last two exact sequences, we need only prove Im u and Im v are coherent. However, both are locally f.g. because they are images of coherent sheaves. But then, as subsheaves of coherent sheaves, being locally finitely generated they are themselves coherent. (cf. Remark (3) just after Definition A.6 in Appendix A). □

Theorem 8.5 *(Finiteness Theorem for Proper Morphisms) If Y is locally noetherian and $\pi\colon X \to Y$ is a proper morphism, then for each coherent $\mathcal{O}_X$-module, $\mathcal{F}$, the $\mathcal{O}_Y$-modules $R^q\pi_*(\mathcal{F})$ are all coherent ($q \geq 0$).*

Proof. The question is local on Y, so we may assume Y is noetherian. As π is proper, X is also noetherian. Now, let $\mathcal{K}$ be the subclass of $\mathcal{O}\mathrm{b}(\mathrm{Coh}(\mathcal{O}_X))$ consisting of those coherent sheaves $\mathcal{F}$ for which the conclusion of the theorem is true. Of course, $0 \in \mathcal{K}$.

I claim that $\mathcal{K}$ is very thick.

For, suppose that

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of coherent $\mathcal{O}_X$-modules then we get the piece of the long exact sequence of derived functors

$$R^{q-1}\pi_*\mathcal{F} \longrightarrow R^{q-1}\pi_*\mathcal{F}'' \longrightarrow R^q\pi_*\mathcal{F}' \longrightarrow R^q\pi_*\mathcal{F} \longrightarrow R^q\pi_*\mathcal{F}'' \longrightarrow R^{q+1}\pi_*\mathcal{F}' \longrightarrow R^{q+1}\pi_*\mathcal{F}.$$

(a) If $\mathcal{F}'$ and $\mathcal{F}''$ are in $\mathcal{K}$, we use

$$R^{q-1}\pi_*\mathcal{F}'' \longrightarrow R^q\pi_*\mathcal{F}' \longrightarrow R^q\pi_*\mathcal{F} \longrightarrow R^q\pi_*\mathcal{F}'' \longrightarrow R^{q+1}\pi_*\mathcal{F}'$$

and deduce from the lemma that $R^q\pi_*\mathcal{F}$ is coherent and so, $\mathcal{F} \in \mathcal{K}$.

(b) If $\mathcal{F}$ and $\mathcal{F}''$ are in $\mathcal{K}$, we use

$$R^{q-1}\pi_*\mathcal{F} \longrightarrow R^{q-1}\pi_*\mathcal{F}'' \longrightarrow R^q\pi_*\mathcal{F}' \longrightarrow R^q\pi_*\mathcal{F} \longrightarrow R^q\pi_*\mathcal{F}''$$

and proceed as in (a); we get $\mathcal{F}' \in \mathcal{K}$.

(c) If $\mathcal{F}'$ and $\mathcal{F}$ are in $\mathcal{K}$, we use

$$R^q\pi_*\mathcal{F}' \longrightarrow R^q\pi_*\mathcal{F} \longrightarrow R^q\pi_*\mathcal{F}'' \longrightarrow R^{q+1}\pi_*\mathcal{F}' \longrightarrow R^{q+1}\pi_*\mathcal{F}$$

and proceed as above.

Hence, $\mathcal{K}$ is thick. If $\mathcal{F} \in \mathcal{K}$ has the form $\mathcal{F}' \coprod \mathcal{F}''$, with $\mathcal{F}'$ coherent, then from the split exact sequence

$$0 \longrightarrow \mathcal{F}' \rightleftarrows \mathcal{F} \rightleftarrows \mathcal{F}'' \longrightarrow 0$$

we find that

$$R^q\pi_*\mathcal{F} = R^q\pi_*\mathcal{F}' \coprod R^q\pi_*\mathcal{F}''.$$

Now, $R^q\pi_*\mathcal{F}$ is coherent and hence, is locally f.g.; so, $R^q\pi_*\mathcal{F}'$ is locally f.g. However, it is quasi-coherent (Chapter 4, Theorem 4.18) and Y is noetherian. (Another way to see the same thing is that $R^q\pi_*\mathcal{F}'$ is a submodule, locally f.g., of the coherent $\mathcal{O}_X$-module $R^q\pi_*\mathcal{F}$. Thus, $R^q\pi_*\mathcal{F}'$ is coherent.) And, so $\mathcal{F}' \in \mathcal{K}$; this proves our claim that $\mathcal{K}$ is very thick.

We must now pove that $\mathcal{K}$ is X-dense in $\mathrm{Coh}(\mathcal{O}_X)$. Suppose we can show the following statement: *If $X \longrightarrow Y$ is proper (Y locally noetherian, of course) and X is irreducible, then there is an $\mathcal{F} \in \mathcal{K}$ so that $\mathcal{F}_x \neq (0)$, where x is generic in X.* Then we will be done as follows:

Choose any irreducible subscheme, Z, of X and let i be the closed immersion $Z \hookrightarrow X$. The composed morphism $Z \xrightarrow{i} X \xrightarrow{\pi} Y$ is proper. By our statement there exists a coherent $\mathcal{O}_Z$-module, $\mathcal{G}$, so that $\mathcal{G} \in \mathcal{K}_Z$ and $\mathcal{G}_z \neq (0)$, where z is the generic point of Z. This means that the support of $\mathcal{G}$ is equal to Z, and $R^q((\pi \circ i)_*\mathcal{G})$ is coherent. But, $i_*\mathcal{G}$ is a coherent $\mathcal{O}_X$-module (cf. Proposition 4.21, Chapter 4) with $(i_*\mathcal{G})_z \neq (0)$ and the spectral sequence of composed functors

$$R^p\pi_*(R^q i_*\mathcal{G}) \Longrightarrow R^\bullet(\pi \circ i)_*(\mathcal{G})$$

degenerates because i is an affine morphism (so, $R^q i_*\mathcal{G} = (0)$ when $q > 0$, cf. Corollary 4.12, Chapter 4). Therefore, we have the isomorphism

$$R^q\pi_*(i_*\mathcal{G}) \cong R^q(\pi \circ i)_*(\mathcal{G}),$$

and the righthand side is coherent. So, $i_*\mathcal{G}$ is the sheaf in $\mathcal{K}_X$ $(= \mathcal{K})$ we need in order to show that $\mathcal{K}$ is X-dense.

Finally, we are reduced to proving the italicized statement above. It is here that we use Chow's lemma. By it, there exists a morphism, $\varphi\colon X' \to X$, which is X-projective and X' is irreducible and, moreover, X' is projective over Y by Corollary 8.2. Since φ is projective, X' possesses an ample $\mathcal{O}_{X'}$-module, call it $\mathcal{L}$. (Of course, we write $\mathcal{O}_{X'}(n)$ to refer to twisting $\mathcal{O}_{X'}$ by $\mathcal{L}^{\otimes n}$.) Apply Serre's finiteness theorem (cf. Chapter 7, Theorem 7.36) to the morphism φ; this gives:

(i) $R^q\varphi_*\mathcal{O}_{X'}(n)$ is $\mathcal{O}_X$-coherent, for all $n \geq 0$ and all $q \geq 0$.

(ii) $R^q\varphi_*\mathcal{O}_{X'}(n) = (0)$, for every $q > 0$ provided only that $n \geq n_0$, for some fixed n_0.

(iii) The morphism $\varphi^*\varphi_*(\mathcal{O}_{X'}(n)) \longrightarrow \mathcal{O}_{X'}(n)$ is surjective if $n >> 0$.

We pick n big enough to satisfy (ii) and (iii) above. Write $\mathcal{F}$ for $\varphi_*\mathcal{O}_{X'}(n)$. From (iii), we find that $\mathcal{F}_x \neq (9)$, where x is generic for X. Now, we need to ptove $R^q\pi_*\mathcal{F}$ is coherent for all q (i.e., $\mathcal{F} \in \mathcal{K}$). But, $\pi \circ \varphi\colon X' \to Y$ is projective; hence, $R^q(\pi \circ \varphi)_*(\mathcal{O}_{X'}(n))$ is coherent for all n and q ((i) for the projective morphism $\pi \circ \varphi$). Use the spectral sequence of composed functors to obtain

$$R^p\pi_*(R^q\varphi_*(\mathcal{O}_{X'}(n))) \Longrightarrow R^\bullet(\pi \circ \varphi)_*(\mathcal{O}_{X'}(n)),$$

and observe that by (ii), $R^q\varphi_*(\mathcal{O}_{X'}(n)) = (0)$ for all $q > 0$. The spectral sequence therefore degenerates and we obtain the isomorphism

$$R^q\pi_*(\mathcal{F}) \cong R^q(\pi \circ \varphi)_*(\mathcal{O}_{X'}(n))$$

and the righthand side is coherent. $\square$

We single out two cases for special mention:

Corollary 8.6 *If Y is locally noetherian and $\pi\colon X \to Y$ is a proper morphism, then $\pi_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module whenever $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. (Case $q = 0$ of the theorem.)*

Corollary 8.7 *Suppose that $Y = \operatorname{Spec} A$ and A is a noetherian ring and that $\pi\colon X \to Y$ is a proper morphism. If $\mathcal{F}$ is any coherent $\mathcal{O}_X$-module, then the cohomology groups $H^q(X, \mathcal{F})$ are finitely generated A-modules for every $q \geq 0$.*

Proof. We know that $R^q\pi_*\mathcal{F}$ is $\widetilde{H^q(X, \mathcal{F})}$ (cf. Chapter 4, Corollary 4.19). As A is noetherian, the only way $R^q\pi_*\mathcal{F}$ will be coherent is for $H^q(X, \mathcal{F})$ to be finitely generated. $\square$

Remark: When $A = k$, a field, then the cohomology groups, $H^q(X, \mathcal{F})$ are finite dimensional vector spaces over k. This corresponds to the well-known topological statement that for compact topological spaces cohomology with coefficients in the base field is finite dimensional in all dimensions. It shows that proper is the correct analog of compactness in the norm topological case.

There is a slight generalization of the finiteness theorem that is very useful:

Corollary 8.8 *Assume that Y is locally noetherian, that $\pi\colon X \to Y$ is a finite-type morphism and that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module whose support is proper over Y. Then, $R^q\pi_*\mathcal{F}$ is coherent for all $q \geq 0$.*

Proof. The conclusion is again local on Y and so we may assume Y is noetherian and hence X is noetherian. Give $\mathrm{Supp}(\mathcal{F})$ its reduced induced scheme structure, call it Z, then $Z \overset{j}{\hookrightarrow} X \longrightarrow Y$ is a proper morphism. If $\mathfrak{I}$ is the ideal sheaf of $\mathcal{O}_X$ defining Z, then there is an integer N so that $\mathfrak{I}^N\mathcal{F} = (0)$. Just as in the proof of the untwisting lemma, we use induction on N to reduce to the case that $\mathfrak{I}\mathcal{F} = (0)$. This means that $\mathcal{F} = j_*\mathcal{G}$, where $\mathcal{G} = j^*\mathcal{F}$. Now, $\mathcal{G}$ is coherent on Z, so the spectral sequence of composed functors

$$R^p\pi_*(R^qj_*(\mathcal{G})) \Longrightarrow R^\bullet(\pi \circ j)_*\mathcal{G}$$

degenerates (because j is an affine morphism) and yields the isomorphism

$$R^q\pi_*\mathcal{F} = R^q\pi_*(j_*(\mathcal{G})) \cong R^q(\pi \circ j)_*\mathcal{G}.$$

But the righthand side is coherent since $\pi \circ j$ is proper as remarked above. $\square$

Chapter 9

Chern Classes and the Hirzebruch Riemann-Roch Theorem

9.1 Chern Classes

In order to minimize the amount of preliminaries, we assume that X is a nonsingular projective connected variety over $\mathbb{C}$. Let $n = \dim(X)$. We have the cohomology groups $H^r(X, \mathbb{Z})$, $0 \leq r \leq 2n$. They have no torsion, and thus are free, and they are dual to the homology groups $H_r(X, \mathbb{Z})$. Poincaré duality implies that

$$H^r(X, \mathbb{Z}) = (H^{2n-r}(X, \mathbb{Z}))^D.$$

Assume that $Y \subseteq X$ and that Y has codimension r as complex algebraic variety. Then, the homology class of Y is $2r$-codimensional, i.e., in $H_{2n-2r}(X, \mathbb{Z})$. By Poincaré duality, $H_{2n-2r}(X, \mathbb{Z})$ is isomorphic to $H^{2r}(X, \mathbb{Z})$. The intersection of Y and Z in X (we may have to move Y and Z to have a good intersection) corresponds to the cup product of cohomology classes.

Let $\mathcal{E}$ be a locally free sheaf of rank r on X. We want elements $c_j(\mathcal{E})$, where

$$c_j(\mathcal{E}) \in H^{2j}(X, \mathbb{Z}),$$

the *Chern classes* of $\mathcal{E}$. We define

$$c(\mathcal{E})(t) = 1 + c_1(\mathcal{E})t + \cdots + c_n(\mathcal{E})t^n$$

to be the *Chern polynomial* of $\mathcal{E}$.

The following conditions on Chern classes are required.

Definition 9.1 *Chern classes* satisfy the following conditions.

(CI) (Naturality) Let $\varphi\colon Y \to X$ be a morphism and $\mathcal{E}$ a locally free sheaf on X. Then,

$$c_l(\varphi^*\mathcal{E}) = \varphi^*(c_l(\mathcal{E}))$$

in $H^\bullet(Y, \mathbb{Z})$.

(CII) (Euler property) Let $A(X) = H^{\bullet}(X, \mathbb{Z})$, as graded ring. If

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is exact (where $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ are locally free sheaves), then

$$c(\mathcal{E})(t) = c(\mathcal{E}')(t)\, c(\mathcal{E}'')(t).$$

(CIII) (Normalization) Let $\mathcal{E} = \mathcal{L} = \mathcal{O}_X(D)$ for some divisor D on X ($D =$ its cohomology class in $H^2(X, \mathbb{Z})$). Then,

$$c_1(\mathcal{O}_X(D)) = D.$$

Given X and $\mathcal{E}$, take a test scheme T over X, with $\pi_T\colon T \to X$. We get $\pi_T^* \mathcal{E}^D$ on T. Look at *flags*

$$\pi_T^* \mathcal{E}^D = \mathcal{F}_r \supseteq \mathcal{F}_{r-1} \supseteq \cdots \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_0 = (0),$$

so that

(1) $\mathcal{F}_j$ is a locally free $\mathcal{O}_T$-module.

(2) $\mathcal{F}_j/\mathcal{F}_{j-1} = \mathcal{L}_j$ is invertible.

These are complete T-flags for $\pi_T^* \mathcal{E}^D$. We have the functor on X-schemes

$$T \mapsto \{\text{complete } T\text{-flags for } \pi_T^* \mathcal{E}^D\}.$$

This is representable, the object representing it is the *flag scheme of* $\mathcal{E}$ *over* X, denoted by $\mathbb{F}_X(\mathcal{E})$. Observe that

$$\mathbb{F}_X(\mathcal{E}) \xrightarrow{\Phi} \mathbb{P}(\mathcal{E}) \longrightarrow X.$$

The mapping Φ is obtained by sending the flag

$$\pi_T^* \mathcal{E}^D \supseteq \mathcal{F}_{r-1} \supseteq \cdots \supseteq \mathcal{F}_1 \supseteq (0)$$

to $\pi_T^* \mathcal{E}^D \supseteq \mathcal{F}_{r-1}$, and then, to the surjection

$$\pi_T^* \mathcal{E}^D \mapsto \pi_T^* \mathcal{E}^D/\mathcal{F}_{r-1}$$

(using the fact that $\mathbb{P}(\mathcal{E})$ represents the functor $\mathrm{Hom}_X(T, \mathbb{P}(\mathcal{E}))$.)

Remark: For every $p \in \mathbb{P}(\mathcal{E})$, the fibre at p is "$\mathbb{F}_{\mathbb{P}(\mathcal{E})}(\mathcal{F}_{r-1})$," a lower dimensional flag.

If we admit that $\mathbb{F}_X(\mathcal{E})$ exists as a scheme and that we have a morphism $\Theta\colon \mathbb{F}_X(\mathcal{E}) \to X$, then $\Theta^* \mathcal{E}^D$ has an $\mathbb{F}_X(\mathcal{E})$-flag (DX). Then,

$$\Theta^* \mathcal{E}^D \supseteq \mathcal{F}_{r-1} \supseteq \cdots \supseteq \mathcal{F}_1 \supseteq (0),$$

with invertible sheaves $\mathcal{L}_j = \mathcal{F}_j/\mathcal{F}_{j-1}$ on $\mathbb{F}_X(\mathcal{E})$, $j = 1, \ldots, r$. Apply this to $\mathcal{E}^D$. Over $\mathbb{F}_X(\mathcal{E}^D)$, we have the "splitting"

$$\Theta^*\mathcal{E} \supseteq \mathcal{F}_{r-1} \supseteq \cdots \supseteq \mathcal{F}_1 \supseteq (0),$$

with $\mathcal{L}_j$ on $\mathbb{F}_X(\mathcal{E}^D)$. However,

$$c(\Theta^*\mathcal{E})(t) = \prod_{j=1}^{r} c(\mathcal{L}_j)(t) = \prod_{j=1}^{r}(1 + D_j t),$$

where $D_j \in A(\mathbb{F}_X(\mathcal{E}^D))$, by (CII) and (CIII). If we have Chern classes on X, then (CI) implies that

$$c(\Theta^*\mathcal{E})(t) = \Theta^*(c(\mathcal{E})(t)),$$

and then,

$$\Theta^*(c(\mathcal{E})(t)) = \prod_{j=1}^{r}(1 + D_j t).$$

If the map $\Theta^*\colon A(X) \to A(\mathbb{F}_X(\mathcal{E}^D))$ is injective, two ways of defining Chern classes agree in $A(\mathbb{F}_X(\mathcal{E}^D))$, and hence on X.

Proposition 9.1 *The Chern polynomial $c(\mathcal{E})(t)$ is unique if $\Theta^*\colon A(X) \to A(\mathbb{F}_X(\mathcal{E}^D))$ is injective.*

The proof that $\Theta^*\colon A(X) \to A(\mathbb{F}_X(\mathcal{E}^D))$ is injective proceeds by induction and uses the fact the fibres of $\mathbb{F}_X(\mathcal{E}^D) \longrightarrow \mathbb{P}(\mathcal{E}^D)$ are projective bundles, similarly for $\mathbb{P}(\mathcal{E}^D) \longrightarrow X$, and reduce to the case $\mathbb{P}(\mathcal{E}^D) \longrightarrow X$ by a spectral argument due to Armand Borel.

We now turn to the existence of Chern classes. Given $\pi\colon \mathbb{P}(\mathcal{E}) \to X$, it turns out that $A(\mathbb{P}(\mathcal{E})) = H^\bullet(\mathbb{P}(\mathcal{E}), \mathbb{Z})$ is an $A(X) = H^\bullet(X, \mathbb{Z})$-algebra free as an $A(X)$-module, of rank $r = \mathrm{rk}(\mathcal{E})$, and it is generated by

$$1, H, H^2, \ldots, H^{r-1},$$

where H is the cohomology class of the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ ($H \in H^2(\mathbb{P}(\mathcal{E}), \mathbb{Z})$). The map $A(X) \longrightarrow A(\mathbb{P}(\mathcal{E}))$ is injective. Thus, H^r is a linear combination of $1, H, H^2, \ldots, H^{r-1}$ with coefficients in $H^\bullet(X, \mathbb{Z})$. We get

$$H^r + \alpha_1 H^{r-1} + \alpha_2 H^{r-2} + \cdots + \alpha_r = 0. \tag{$*$}$$

Observe that $\alpha_j \in H^{2j}(X, \mathbb{Z})$. We let

$$c_j(\mathcal{E}) = \alpha_j.$$

Since

$$\mathbb{P}(\varphi^*\mathcal{E}) = \varphi^*(\mathbb{P}(\mathcal{E})),$$

we get (CI). Assume that $\mathcal{E} = \mathcal{O}_X(D) \in \mathrm{Pic}(X)$. In this case, $\mathbb{P}(\mathcal{E}) = X$,

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}^D = \mathcal{O}_X(-D),$$

and thus, $H = -D$. Thus, $(*)$ implies (since $r = 1$) that

$$-D + \alpha_1 = 0,$$

that is, $\alpha_1 = D$. Thus,

$$c(\mathcal{O}_X(D))(t) = 1 + Dt,$$

and (CIII) holds. To prove (CII) requires more work.

Computations.

(1) The splitting principle suggests the introduction of the *Chern roots* of $\mathcal{E}$. Write

$$c(\mathcal{E})(t) = 1 + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r = \prod_{j=1}^{r}(1 + \gamma_j t),$$

where γ_j is the jth Chern root. Then,

$$c_j(\mathcal{E}) = \sigma_j(\gamma_1, \ldots, \gamma_r),$$

where σ_j is the jth symmetric function in r variables. Thus, we can compute for $\mathcal{E}$ as if it were a sum of line bundles whose Chern classes are $\gamma_1, \ldots, \gamma_r$. As an illustration, we get the following.

(a) $c(\mathcal{E} \otimes \mathcal{F})(t) = \prod_{i,j}(1 + (\gamma_i + \delta_j)t)$.

(b) $c(\mathcal{E} \coprod \mathcal{F})(t) = c(\mathcal{E})(t)\, c(\mathcal{F})(t) = \prod_i(1 + \gamma_i t) \prod_j(1 + \delta_j t)$.

(c) $c(\mathcal{E}^D)(t) = \prod_j(1 - \gamma_i t)$, and thus, $c_j(\mathcal{E}^D) = (-1)^j c_j(\mathcal{E})$.

(d) $c(\bigwedge^d \mathcal{E})(t) = \prod_{j_1 < \cdots < j_d}(1 + (\gamma_{j_1} + \cdots + \gamma_{j_d})t)$.

(e) $c(\mathcal{S}^d\mathcal{E})(t) = \prod_{m_1 + \cdots + m_r = d}(1 + (m_1\gamma_1 + \cdots + m_r\gamma_r)t)$.

(2) Given $\mathcal{E}$, twist by $\mathcal{O}_X$, to get $\mathcal{E}(n)$. If $n >> 0$, the sheaf $\mathcal{E}(n)$ has lots of sections, and it is generated by these sections. Pick $\sigma_1, \ldots, \sigma_r$, generic sections of $\mathcal{E}(n)$ (where $\mathrm{rk}(\mathcal{E}) = r$). If $q \leq r$, consider

$$\sigma_1 \wedge \cdots \wedge \sigma_{r-q+1},$$

a non-generic section of $\bigwedge^{r-q+1} \mathcal{E}$. The zero locus turns out to have codimension q and is the carrier of $c_q(\mathcal{E}(n))$. By (a),

$$c(\mathcal{E}(n))(t) = \prod_{j=1}^{r}(1 + (\gamma_j + nH)t),$$

where the γ_j's are the Chern roots of $\mathcal{E}$ (and H is the class of the hyperplane bundle, as before).

Example 9.1 Assume that $r = 2$, and that c_1, c_2 are known for $\mathcal{E}(n)$. We have

$$\begin{aligned} c(\mathcal{E}(n))(t) &= (1 + (\gamma_1 + nH)t)(1 + (\gamma_2 + nH)t) \\ &= 1 + (\gamma_1 + \gamma_2 + 2nH)t + (\gamma_1\gamma_2 + (\gamma_1 + \gamma_2)nH + n^2H^2)t^2 \\ &= 1 + (c_1(\mathcal{E}) + 2nH)t + (c_2(\mathcal{E}) + n(c_1(\mathcal{E}) \cdot H) + n^2(H \cdot H))t^2. \end{aligned}$$

This implies that

$$\begin{aligned} c_1(\mathcal{E}(n)) &= c_1(\mathcal{E}) + 2nH \\ c_2(\mathcal{E}(n)) &= c_2(\mathcal{E}) + n(c_1(\mathcal{E}) \cdot H) + n^2(H \cdot H). \end{aligned}$$

Thus, we can solve for $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$.

Let $p_1, \ldots, p_n$ be some indeterminates and look at

$$1 + p_1 z + p_2 z^2 + \cdots$$

Give p_j some degree d_j (generally, $d_j \to \infty$ as $j \to \infty$.) We consider functions K from power series to power series (with first term 1)

$$K\left(\sum_{j=0}^{\infty} p_j z^j\right) = \sum_{j=0}^{\infty} K_j(p_1, \ldots, p_j) z^j.$$

Such a function is called *multiplicative* and the family $\{K_j(p_1, \ldots, p_j)\}_{j=1}^{\infty}$ a *multiplicative sequence* if every identity

$$\sum_{j=0}^{\infty} p_j z^j = \left(\sum_{j=0}^{\infty} p'_j z^j\right)\left(\sum_{j=0}^{\infty} p''_j z^j\right)$$

is equivalent with an identity

$$\sum_{j=0}^{\infty} K_j(p\text{'s}) z^j = \left(\sum_{j=0}^{\infty} K_j(p'\text{'s}) z^j\right)\left(\sum_{j=0}^{\infty} K_j(p''\text{'s}) z^j\right).$$

We can construct such sequences. Observe that if we know

$$K(1 + z) = \sum_{j=0}^{\infty} K_j(1, 0, \ldots, 0) z^j$$

then we know K in general. Introduce the formal roots γ_j's. Look at

$$1 + p_1 z + \cdots + p_n z^n = (1 + \gamma_1 z)(1 + \gamma_2 z) \cdots (1 + \gamma_n z),$$

and apply K. We get

$$\sum_{j=0}^{n} K_j(p\text{'s})z^j = \prod_{j=1}^{n} K(1+\gamma_j z) = \prod_{j=1}^{n}\left(\sum_{l=0}^{\infty} K_l(\gamma_j, 0, \ldots, 0)z^l\right).$$

So, given a power series

$$Q(z) = 1 + h_1 z + h_2 z^2 + \cdots +,$$

consider K defined by

$$K(1+z) = Q(z).$$

For convenience, we now change notation: Let $z = t^2$ and make $c_1, \ldots, c_r$ via the equation

$$\sum_{j=0}^{\infty}(-1)^j p_j z^j = \left(\sum_{k=0}^{\infty}(-1)^k c_k t^k\right)\left(\sum_{l=0}^{\infty} c_l t^l\right).$$

Now, where we had $K_j(p\text{'s})$ we have $T_j(c\text{'s})$. The relationship is as follows. Given $Q(z)$, we make the $K_j(p\text{'s})$. Let

$$\widetilde{Q}(z) = Q(z^2).$$

Then, we can make from $\widetilde{Q}(z)$ the sequence $\widetilde{K}_l(c\text{'s})$, and we get

$$K_j(p\text{'s}) = \widetilde{K}_{2j}(c\text{'s}), \quad \widetilde{K}_{2j+1}(c\text{'s}) = 0.$$

We have the following two facts.

Proposition 9.2 *The following properties are equivalent for power series.*

(1) In $T_n(c_1, \ldots, c_n)$, substitute $c_j = \binom{n+1}{j}$. Then, $T(c_1, \ldots, c_n) = 1$.

(2) The coefficient of t^k in $Q(t)^{k+1}$ is 1.

(3) $T_1(c_1) = \frac{1}{2}c_1$ and the coefficient of c_1^k and the coefficient of c_k in $T_k(c_1, \ldots, c_k)$ are equal.

Proposition 9.3 *There exists a unique power series having the above properties, namely*

$$Q(t) = \frac{t}{1 - e^{-t}}.$$

Remark: If X is a nonsingular variety, the *Chern classes of X* are by definition the Chern classes of its tangent bundle. What are the Chern classes of $\mathbb{P}^n$? We have the exact sequence (Euler sequence)

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

We get

$$c(\Omega^1_{\mathbb{P}^n})(t) = (1 - Ht)^{n+1} \bmod H^{n+1},$$

that is,

$$\sum_{j=0}^{n} (-1)^j c_j(\mathbb{P}^n) t^j = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} H^j t^j,$$

and thus,

$$c_j(\mathbb{P}^n) = \binom{n+1}{j} H^j.$$

Now, introduce the Chern roots ($c_j = c_j(X)$),

$$\sum_{j=0}^{n} c_j t^j = \prod_{l=1}^{n} (1 + \gamma_l t),$$

and apply Q. We get

$$\sum_{j=0}^{n} T_j(c_1, \ldots, c_j) t^j = \prod_{l=1}^{n} \frac{\gamma_l t}{1 - e^{-\gamma_l t}}.$$

Definition 9.2 Given a nonsingular projective variety X of dimension n, the (unique) polynomial $\sum_{j=0}^{n} T_j(c_1, \ldots, c_j) t^j$ is the *(total) Todd polynomial* of X, denoted by $\mathrm{td}(X)(t)$. The coefficient $T_n(c_1, \ldots, c_n)$ is called the *Todd genus* of X, and is denoted by $\mathrm{td}(X)$.

Example 9.2 Consider a curve X, i.e, $\dim(X) = 1$. Then, $\gamma_1 = c_1$. We need the term of degree 1 in

$$\frac{c_1 t}{1 - e^{-c_1 t}}.$$

We have

$$\begin{aligned}
e^{-z} &= 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + O(z^5) \\
1 - e^{-z} &= z - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} + O(z^5) \\
\frac{1}{1 - e^{-z}} &= \frac{1}{z} + \alpha_0 z + \alpha_1 z^1 + \alpha_2 z^2 + \alpha_3 z^3 + O(z^4).
\end{aligned}$$

Thus, by multiplication, we get

$$1 = 1 + \left(-\frac{1}{2} + \alpha_0\right) z + \left(\frac{1}{6} - \frac{\alpha_0}{2} + \alpha_1\right) z^2 + O(z^3),$$

and

$$\alpha_0 = \frac{1}{2}, \quad \alpha_1 = \frac{1}{12}.$$

We get

$$\frac{c_1 t}{1 - e^{-c_1 t}} = c_1 t \left(\frac{1}{c_1 t} + \frac{1}{2} + \frac{c_1}{12} t \right) + O(t^3) = 1 + \frac{c_1}{2} t + O(t^2).$$

Thus, for curves, the Todd genus $\mathrm{td}(X)$ is given by

$$\mathrm{td}(X) = \frac{1}{2} c_1 = -\frac{1}{2} K,$$

where K is the class of the canonical bundle, $K \in H^2(X, \mathbb{Z})$. Under the isomorphism

$$H^2(X, \mathbb{Z}) \cong \mathbb{Z}$$

(evaluate on $[X] \in H^2(X, \mathbb{Z})$), we get

$$\mathrm{td}(X) = -\mathrm{deg}\left(\frac{1}{2} K \right) = -\frac{1}{2}(2g - 2) = 1 - g.$$

Now, assume that X is a surface, i.e., $\dim(X) = 2$. We have

$$1 + c_1 t + c_2 t^2 = (1 + \gamma_1 t)(1 + \gamma_2 t)$$

and

$$\mathrm{td}(X)(t) = \frac{\gamma_1 t}{(1 - e^{-\gamma_1 t})} \frac{\gamma_2 t}{(1 - e^{-\gamma_2 t})}.$$

Thus,

$$\begin{aligned}
\mathrm{td}(X)(t) &= \left(1 + \frac{1}{2}\gamma_1 t + \frac{1}{12}\gamma_1^2 t^2 + O(t^3)\right)\left(1 + \frac{1}{2}\gamma_2 t + \frac{1}{12}\gamma_2^2 t^2 + O(t^3)\right) \\
&= 1 + \frac{1}{2}(\gamma_1 + \gamma_2) t + \left(\frac{1}{12}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}\gamma_1\gamma_2 \right) t^2 + O(t^3) \\
&= 1 + \frac{1}{2} c_1 t + \left(\frac{1}{12}((\gamma_1 + \gamma_2)^2 - 2\gamma_1\gamma_2) + \frac{1}{4}\gamma_1\gamma_2 \right) t^2 + O(t^3) \\
&= 1 + \frac{1}{2} c_1 t + \frac{1}{12}(c_1^2 + c_2) t^2 + O(t^3)
\end{aligned}$$

Therefore, for a surface,

$$\mathrm{td}(X)(t) = 1 + \frac{1}{2} c_1 t + \frac{1}{12}(K^2 + c_2) t^2.$$

We are now ready to the Hirzebruch–Riemann–Roch theorem.

9.2 Hirzebruch–Riemann–Roch Theorem

Let $\mathcal{E}$ be a vector bundle of rank r on X, and let $\gamma_1, \ldots, \gamma_r$ be the Chern roots of $\mathcal{E}$. Write

$$\mathrm{ch}(\mathcal{E})(t) = \sum_{j=1}^{r} e^{\gamma_i t},$$

the *Chern character* of $\mathcal{E}$. We have

$$\mathrm{ch}(\mathcal{E})(t) = r + \sum_{k=1}^{\infty} \frac{1}{k!}\, s_k(\gamma_1, \ldots, \gamma_r) t^k,$$

where

$$s_k(\gamma_1, \ldots, \gamma_r) = \gamma_1^k + \cdots + \gamma_r^k.$$

Thus, we have

$$\mathrm{ch}(\mathcal{E})(t) = \mathrm{rk}(\mathcal{E}) + c_1 t + \frac{1}{2}(c_1^2 - 2c_2)t^2 + \cdots .$$

The computation from 1(a) show that

$$\mathrm{ch}(\mathcal{E} \otimes \mathcal{F})(t) = \mathrm{ch}(\mathcal{E})(t)\, \mathrm{ch}(\mathcal{F})(t).$$

We can also check that

$$\mathrm{ch}(\mathcal{E} \oplus \mathcal{F})(t) = \mathrm{ch}(\mathcal{E})(t) + \mathrm{ch}(\mathcal{F})(t).$$

Hence, the Chern character is a ring homomorphism

$$\mathrm{ch}\colon K_{\mathrm{vect}}(X) \to H^\bullet(X, \mathbb{Z}),$$

where $K_{\mathrm{vect}}(X)$ is the Grothendieck group associated with vector bundles. Let $\dim(X) = d$. Given a vector bundle $\mathcal{E}$ of rank r on X, let

$$\chi(X, \mathcal{E}) = \sum_{i=0}^{d} (-1)^i \dim H^i(X, \mathcal{E}),$$

the *Euler characteristic* of the vector bundle $\mathcal{E}$. Also, let $T(X, \mathcal{E})$ be the degree d part of

$$\mathrm{ch}(\mathcal{E})(t)\, \mathrm{td}(X)(t)$$

evaluated on $[X]$.

Theorem 9.4 *(Hirzebruch–Riemann–Roch) Let X be a complex, compact, nonsingular, projective variety and $\mathcal{E}$ a vector bundle of rank r on X. Then,*

$$\chi(X, \mathcal{E}) = T(X, \mathcal{E}).$$

In general, we denote the Chern classes of $\mathcal{E}$ by $e_1, \ldots, e_r$, and the Chern classes of X (really, of T_X) by $c_1, \ldots, c_n$. Let us unravel what Theorem 9.4 says in the case of a curve and of surface, when $\mathcal{E}$ is a line bundle.

(1) X is a curve, i.e., $\dim(X) = d = 1$, and $\mathcal{E} = \mathcal{O}_X(D)$, where D is a divisor. Then, we have

$$\begin{aligned} \mathrm{ch}(\mathcal{E})(t) &= 1 + Dt \\ \mathrm{td}(X)(t) &= 1 + \frac{1}{2}c_1(X)t \\ \mathrm{ch}(\mathcal{E})(t)\,\mathrm{td}(X)(t) &= 1 + \left(D + \frac{1}{2}c_1(X)\right)t. \end{aligned}$$

The degree 1 part evaluated at $[X]$ is

$$\left(D + \frac{1}{2}c_1(X)\right)[X] = \deg(D) + 1 - g.$$

Therefore, we get the Riemann-Roch theorem for curves:

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g.$$

Using Serre duality, we get the usual version

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(K - D)) = \deg(D) + 1 - g.$$

(2) Now, consider surfaces, i.e., $\dim(X) = 2$, and still a line bundle $\mathcal{E} = \mathcal{O}_X(D)$. Since $c_1 = -K$, We have

$$\begin{aligned} \mathrm{ch}(\mathcal{E})(t) &= 1 + e_1 t + \frac{1}{2}e_1^2 t^2 \\ \mathrm{td}(X)(t) &= 1 - \frac{1}{2}Kt + \frac{1}{12}(K^2 + c_2)t^2. \end{aligned}$$

The degree 2 part is

$$-\frac{1}{2}(K \cdot e_1)t + \left(\frac{1}{2}e_1^2 + \frac{1}{12}(K^2 + c_2)\right)t^2,$$

and evaluated at $[X]$, we get

$$\frac{1}{2}(D^2 - K \cdot D) + \frac{1}{12}(K^2 + c_2),$$

and thus, the Riemann-Roch for surfaces is

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) + \dim H^2(X, \mathcal{O}_X(D)) = \frac{1}{2}(D - K) \cdot D + \frac{1}{12}(K^2 + c_2).$$

Let us now give a quick proof of the Riemann-Roch theorem for curves and line bundles $\mathcal{O}_X(D)$.

Theorem 9.5 *Let X be a complete nonsingular curve over an algebraically closed field k. Then, if X has genus g and canonical class K, for any divisor D on X, we have*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(K - D)) = \deg(D) + 1 - g.$$

Proof. For every $P \in |D|$, we have the two exact sequences

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \kappa(P) \longrightarrow 0 \qquad (a)$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(P) \longrightarrow \kappa(P) \longrightarrow 0. \qquad (b)$$

First, assume that D is effective. Tensor (a) with $\mathcal{O}_X(D)$. We get

$$0 \longrightarrow \mathcal{O}_X(D - P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \kappa(P) \longrightarrow 0.$$

Using cohomology, take $\chi(X, -)$. We get

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D - P)) + 1.$$

We proceed by induction. If $D = 0$, the formula says

$$\dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = 0 + 1 - g.$$

However, by Serre duality,

$$\dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_X) = g,$$

the genus of X, and

$$H^0(X, \mathcal{O}_X) \cong k,$$

because X is a projective variety, and thus, $\dim H^0(X, \mathcal{O}_X) = 1$. By induction, we get

$$\chi(X, \mathcal{O}_X(D - P)) = \deg(D - P) + 1 - g = \deg(D) - 1 + 1 - g = \deg(D) - g,$$

and thus,

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g.$$

Now, assume that D is arbitrary. We can write $D = D_1 - D_2$, where $D_1, D_2 \geq 0$. For any $P \in |D_2|$, using (b), we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow \kappa(P) \longrightarrow 0.$$

Again, by taking $\chi(X, -)$, we get

$$\chi(X, \mathcal{O}_X(D + P)) = \chi(X, \mathcal{O}_X(D)) + 1. \qquad (*)$$

By induction, we get

$$\chi(X, \mathcal{O}_X(D + P)) = \deg(D + P) + 1 - g = \deg(D) + 1 + 1 - g = \deg(D) + 2 - g,$$

and by $(*)$,

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g.$$

Then, we reduce the proof in the case of a general divisor to the case an effective divisor, which completes the proof. $\square$

Appendix A

Sheaves and Ringed Spaces

A.1 Presheaves

Let X be a topological space. If $\mathcal{T}$ denotes the topology on X, then $\mathcal{T}$ is completely specified by and completely specifies a certain category which we shall denote by $\mathrm{Cat}\,\mathcal{T}$. The objects of $\mathrm{Cat}\,\mathcal{T}$ are the open sets in X. If U and V are open sets of X, we set

$$\mathrm{Hom}\,(U,V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V \\ \{\mathrm{incl}\} & \text{if } U \subseteq V \end{cases}$$

where $\{\mathrm{incl}\}$ is the set with one element: The natural inclusion map $U \hookrightarrow V$. Let $\mathbf{C}$ be an arbitrary category (for example $\mathbf{C}$ might be the category of sets, the category of groups, the category of rings, etc—the reader is urged to think of $\mathbf{C}$ as the category of sets until he becomes more facile with the material to be presented.)

Definition A.1 A *presheaf* $\mathcal{F}$*, with values in* $\mathbf{C}$ on X is a (contravariant) functor from $(\mathrm{Cat}\,\mathcal{T})^o$ to $\mathbf{C}$.

Observe that to give a presheaf on X (with values in $\mathbf{C}$) we must give for each open set U of X, and object $\mathcal{F}(U)$, of $\mathbf{C}$ and these objects must "fit together" according to the rule: If $V \subseteq U$, there is a map in $\mathbf{C}$, denoted ρ^U_V taking $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$—frequently called the *restriction from* U *to* V—so that if

$$V' \subseteq V \subseteq U, \quad \text{then} \quad \rho^U_{V'} = \rho^V_{V'} \circ \rho^U_V.$$

Here are some examples of preshaves, they are taken from the literally infinite number available in mathematics.

(1) X is an arbitrary topological space, $\mathbf{C}$ is the category of rings, and $\mathcal{F}$ is the presheaf given by:

$$\mathcal{F}(U) = \text{all real-valued continuous functions defined on } U.$$

(Clearly $\mathcal{F}(U)$ is a ring under the usual operations of addition and multiplication of functions). We must still give the collection of maps $\rho_V^U \in \mathrm{Hom}\,(\mathcal{F}(U), \mathcal{F}(V))$, but in this case our choice is clear:

$$\rho_V^U(f) = f \restriction V.$$

That is, ρ_V^U is what we are used to calling the restriction from U to V; in fact, the terminology associated with ρ_V^U in the general case comes from just this example. The presheaf $\mathcal{F}$ is called *the presheaf of germs of real-valued continuous functions on* X.

(2) Again X is to be an arbitrary topological space. Let $\mathbf{C}$ be any category and let A be any object of $\mathbf{C}$. For each open U in X, let $\mathcal{F}(U) = A$ and let ρ_V^U be the identity map $A \longrightarrow A$ for each $V \subseteq U$. These data define a presheaf called the *constant presheaf* on X and usually denoted, A. Of particular importance is the presheaf $\mathbb{Z}$, where $\mathbb{Z}$ is the *group* of integers—so $\mathbf{C}$ is the category of abelian groups in this case.

(3) Let X be the field of complex numbers with its usual topology. For each open set U of X, let $\mathcal{F}(U)$ be the holomorphic complex-valued functions on U (resp. the meromorphic complex-valued functions on U), and let ρ_V^U be the usual restriction of functions. We obtain the presheaf called the *presheaf of germs of holomorphic* (resp. *meromorphic*) *functions on* X.

(4) (Partial generalization of (2)) Let X be an arbitrary topological space; let $\mathbf{C}$ be the category of abelian groups—$\mathcal{AB}$. Given an open set U of X, let $\mathbb{Z}_U$ be defined as follows:

$$\mathbb{Z}_U(V) = \coprod_{\mathrm{Hom}\,(V,U)} \mathbb{Z} \;= \begin{cases} (0) & \text{if } V \not\subseteq U \\ \mathbb{Z} & \text{if } V \subseteq U \end{cases}$$

If $V' \subseteq V$ then $\mathbb{Z}_U(V) \longrightarrow \mathbb{Z}_U(V')$ is clear; it is the identity map if $V \subseteq U$ and the zero map otherwise. This prescription yields a presheaf $\mathbb{Z}_U$ for each open set U of X; hence, yields an infinite family of presheaves. When $U = X$, we obtain the presheaf $\mathbb{Z}$ of example (2).

Problem A.1 (Generalization of (4)). Let X be as in (4), $\mathbf{C}$ as in (4). Let $\mathcal{F}$ be a presheaf of sets on X and let A be an abelian group. For each U in X set

$$A_{\mathcal{F}}(U) = \coprod_{\mathcal{F}(U)} A = \{\text{functions: } \mathcal{F}(U) \longrightarrow A \text{ with finite support}\}.$$

Make $A_{\mathcal{F}}$ into a presheaf on X. How does one choose $\mathcal{F}, A$ in order that $A_{\mathcal{F}} = \mathbb{Z}_U$?

Now the presheaves on X with values in $\mathbf{C}$ form a category themselves which we will denote $\mathcal{P}(X, \mathbf{C})$. To see this, one need only define the notion of morphism between presheaves and check the required axioms. This is done as follows: Given $\mathcal{F}, \mathcal{G}$ objects of $\mathcal{P}(X, \mathbf{C})$, a morphism σ from $\mathcal{F}$ to $\mathcal{G}$ is a *consistent* collection of morphisms $\sigma(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each U in Cat $\mathcal{T}$. Consistency is understood in the sense that whenever $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\sigma(U)"] \arrow[d, "\rho_V^U"'] & \mathcal{G}(U) \arrow[d, "\rho_V^U"] \\ \mathcal{F}(V) \arrow[r, "\sigma(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes. So a morphism of presheaves is nothing but a "natural transformation" of functors. That this definition of morphism satisfies the category axioms is obvious.

If $\mathbf{C} = \mathcal{AB}$, we write $\mathcal{P}(X)$ instead of $\mathcal{P}(X, \mathcal{AB})$. Let $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups on X; let $\sigma\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Consider the following two functors $\mathcal{F}', \mathcal{G}''$ on X:

$$\begin{aligned} \mathcal{F}'(U) &= \operatorname{Ker}(\mathcal{F}(U) \xrightarrow{\sigma(U)} \mathcal{G}(U)) = \operatorname{Ker}\ \sigma(U) \\ \mathcal{G}''(U) &= \operatorname{Coker}(\mathcal{F}(U) \xrightarrow{\sigma(U)} \mathcal{G}(U)) = \operatorname{Coker}\ \sigma(U). \end{aligned}$$

One checks easily that $\mathcal{F}', \mathcal{G}''$ are presheaves of abelian groups on X and that we have canonical morphisms

$$\mathcal{F}' \longrightarrow \mathcal{F}, \quad \mathcal{G} \longrightarrow \mathcal{G}''.$$

The presehaf $\mathcal{F}'$ is called the *kernel* of σ and the presheaf $\mathcal{G}''$ is called the *cokernel* of σ. Given a sequence of presheaves

$$\mathcal{F}' \xrightarrow{\sigma} \mathcal{F} \xrightarrow{\tau} \mathcal{F}''$$

we shall say that this sequence is *exact* if and only if for *every* U of Cat $\mathcal{T}$, the corresponding sequence

$$\mathcal{F}'(U) \xrightarrow{\sigma(U)} \mathcal{F}(U) \xrightarrow{\tau(U)} \mathcal{F}''(U)$$

is exact (as a sequence of abelian groups). A moment's thought shows that $\mathcal{F}' \xrightarrow{\sigma} \mathcal{F}$ is *injective* (i.e. $\mathcal{F}'(U) \longrightarrow \mathcal{F}(U)$ is injective for all U) if and only if $\operatorname{Ker} \sigma = 0$, and a similar statement holds for *surjective*. In this way, the category $\mathcal{P}(X)$ behaves just like the category $\mathcal{AB}$; hence it is an abelian category.[1]

Problem A.2 Show that the presheaves $\mathbb{Z}_U$ form a system of generators for $\mathcal{P}(X)$ in the sense of Grothendieck [21]. Deduce that $\mathcal{P}(X)$ contains "sufficiently many injectives." How many of the axioms AB 1, 1^*, etc., can you prove for $\mathcal{P}(X)$?

[1]The reader who wants the precise definition of abelian category can consult Grothendieck [21] or Freyd [16]; he can also trivially check that $\mathcal{P}(X)$ is an abelian category according to these definitions.

A.2 Sheaves

A sheaf is a special type of presheaf. Briefly, the problem which sheaves solve is the representation of a consistent collection of local data on a topological space as a mathematical entity. That is, one frequently is given data valid in a neighborhood of each point $x \in X$, consistent in the sense that these data agree on overlaps, and one wishes to incorporate all the data into one mathematical object. On the surface, it appears that a presheaf is just the correct object; however, implicit in the requirement that our object represent the data is the requirement that it be reconstructable from (or be determined by) the local data. It is just this requirement which presheaves fail to satisfy. For example, if X is a space consisting of two disjoint, connected components, and if $\mathcal{F}, \mathcal{G}$ are the presheaves given by

$$\begin{aligned} \mathcal{F} &= \text{constant presheaf } \mathbb{Z} \text{ on } X \\ \mathcal{G}(U) &= \begin{cases} \mathbb{Z} & \text{if } U \text{ is in one component} \\ \mathbb{Z} \coprod \mathbb{Z} & \text{if } U \text{ intersects both components non-trivially} \end{cases} \end{aligned}$$

then $\mathcal{F}$ and $\mathcal{G}$ are "locally isomorphic" by a globally defined map; that is, they look the same in a suitable neighborhood of each point, but they are NOT the same presheaf.

The first sentence of this section betrays our point of view; we consider sheaves as "special" presheaves. There is another fruitful way to look at sheaves which is slightly more to an analyst's or topologist's taste. Both view-points give the same results and it is wise to know both as there are situations where one is technically simpler to handle than the other.

To define the notion of sheaf we need the concept of an (open) covering. Let $\{U_i \longrightarrow U\}$ be a family of morphisms in $\mathrm{Cat}\,\mathcal{T}$ (i.e., the U_i are open subsets of U). We say that the given family lies in $\mathrm{Cov}\,\mathcal{T}$ or is a covering of U if and only if

$$\bigcup_i U_i = U.$$

Suppose $\{U_i \longrightarrow U\} \in \mathrm{Cov}\,\mathcal{T}$; then for any presheaf $\mathcal{F}$ on X and any index i, we obtain a morphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(U_i)$. Therefore, by varying i, we obtain the morphism

$$\alpha\colon \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i).$$

(This assumes, of course, that the category $\mathbf{C}$ posseses products—a situation which holds for all the categories we will consider below, sets, groups, modules, etc.) For fixed i and varying j, we have the canonical morphisms $U_i \cap U_j \longrightarrow U_i$; hence, we deduce a morphism $\mathcal{F}(U_i) \longrightarrow \prod_i \mathcal{F}(U_i \cap U_j)$. Upon taking the product of these morphisms over all i, we get the morphism

$$\beta_1\colon \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

In the same manner, varying i not j and taking the product over all j, we obtain

$$\beta_2\colon \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

Given this preparation we can now state our definition.

Definition A.2 A *sheaf,* $\mathcal{F}$ on X is a presheaf which satisfies the axioms:

(S) For every family $\{U_i \longrightarrow U\} \in \operatorname{Cov}\mathcal{T}$ the sequence

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

(Recall that for categories $\mathbf{C}$ based on sets, a sequence

$$F' \xrightarrow{\alpha'} F \underset{\beta_2'}{\overset{\beta_1'}{\rightrightarrows}} F''$$

is *exact* if and only if α' maps F' bijectively onto the set of all elements of F whose image in F'' under β_1' and β_2' agree. Exactness makes sense in an arbitrary category as well—we omit the definition and refer the reader to [??].)

A few examples will clarify the intuitive content of the definition of a sheaf. Note that the question of exactness in (S) can be broken down into two questions:

(a) (Uniqueness): Are two elements $\xi, \eta \in \mathcal{F}(U)$ equal if when restricted to each set of an open covering they become equal?

(a) (Existence): Given a collection of elements $\xi_i \in \mathcal{F}(U_i)$ whose restrictions to the overlaps $U_i \cap U_j$ agree for every i and j, does there exist an element in $\mathcal{F}(U)$, say ξ, whose restriction to U_i is ξ_i for every i? That is, can we "patch together" the elements ξ_i to form a "globally defined" element ξ?

In the light of this remark, it is trivial to see that the presheaves of Examples 1 and 3 (of Section A.1) are sheaves. They will be called sheaves of germs of continuous functions (resp. holomorphic) functions hereafter. What about example 2? There is a serious reason why the constant presheaf A is not a sheaf (except in the trivial case).

Proposition A.1 *Let X be a locally connected space, and let $\mathcal{F}$ be a sheaf of sets on X. If U is any open subset of X and $\{U_i\}$ is the family of connected components of U, then $\{U_i \longrightarrow U\} \in \operatorname{Cov}\mathcal{T}$ and*

$$\mathcal{F}(U) \cong \prod_i \mathcal{F}(U_i).$$

Proof. The U_i are open in U by local connectedness, and clearly form a covering of U. If we apply axiom (S) to this covering, we deduce the exact sequence

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

Now, for $i \neq j$, $U_i \cap U_j = \emptyset$; hence $\mathcal{F}(U_i \cap U_j) = \{\emptyset\}$. It follows immediately that $\beta_1 = \beta_2$ for every element of $\prod_i \mathcal{F}(U_i)$, and the exactness of our sequence completes the proof. $\square$

The proposition demonstrates that the closest a sheaf can come to being constant is to be locally constant, that is, constant on connected open sets.

Our main object now will be to construct from each given presheaf of sets, a corresponding sheaf—called the *associated sheaf* to the presheaf.

Let U be an open set in X and let $\{U_i \longrightarrow U\}_{i\in I}$ and $\{U'_\lambda \longrightarrow U\}_{\lambda\in\Lambda}$ be two coverings of U. By a *map* ϵ from $\{U_i \longrightarrow U\}_{i\in I}$ to $\{U'_\lambda \longrightarrow U\}_{\lambda\in\Lambda}$ we mean a map of the index set I, to the index set Λ, say ϵ again, such that for every $i \in I$, we have $U_i \subseteq U'_{\epsilon(i)}$. Frequently, $\{U_i \longrightarrow U\}$ is called a *refinement* of $\{U'_\lambda \longrightarrow U\}$ if there is a map ϵ as above. Let $\mathcal{F}$ be a presheaf on X, and define for any open covering $\{U_i \longrightarrow U\}$ the set $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$ by

$$H^0(\{U_i \longrightarrow U\}, \mathcal{F}) = \operatorname{Ker} \left\{ \prod_i \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right\}.$$

Here, $\operatorname{Ker} \{\mathcal{F} \underset{\beta'}{\overset{\beta}{\rightrightarrows}} \mathcal{G}\}$ means the set of all $\xi \in \mathcal{F}$ such that $\beta(\xi) = \beta'(\xi)$ in $\mathcal{G}$.

Lemma A.2 *Let $\mathcal{F}$ be an arbitrary presheaf of sets on X, let U be any open subset of X, let $\{U_i \longrightarrow U\}$, $\{U'_\lambda \longrightarrow U\}$ be two coverings of U; finally let ϵ be a map from $\{U_i \longrightarrow U\}$ to $\{U'_\lambda \longrightarrow U\}$. Then ϵ induces a map ϵ^* from $H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{F})$ to $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$ and any two maps from $\{U_i \longrightarrow U\}$ to $\{U'_\lambda \longrightarrow U\}$ induce the same map on the sets H^0.*

Proof. The inclusions $U_i \longrightarrow U'_{\epsilon(i)}$ induce maps $\mathcal{F}(U'_{\epsilon(i)}) \longrightarrow \mathcal{F}(U_i)$ for each $i \in I$; hence we have a map $\prod_\lambda \mathcal{F}(U'_\lambda) \xrightarrow{\epsilon_1^*} \prod_i \mathcal{F}(U_i)$. In a similar manner, we obtain the map $\prod_{\lambda,\mu} \mathcal{F}(U'_\lambda \cap U'_\mu) \xrightarrow{\epsilon_2^*} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$, and the diagram

$$\begin{tikzcd}
\prod \mathcal{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathcal{F}(U_i \cap U_j) \\
\prod \mathcal{F}(U'_\lambda) \arrow[u, "\epsilon_1^*"] \arrow[r, shift left] \arrow[r, shift right] & \prod \mathcal{F}(U'_\lambda \cap U'_\mu) \arrow[u, "\epsilon_2^*"']
\end{tikzcd}$$

is commutative. It follows that ϵ^* exists as claimed.

We have now to prove the important part of the lemma which says that for any two maps ϵ and ϵ' from the covering $\{U_i \longrightarrow U\}$ to the covering $\{U'_\lambda \longrightarrow U\}$, we have $\epsilon^* = \epsilon'^*$.

If α, β are indices in either I or Λ, we shall let ρ^α_β denote the restriction map from $\mathcal{F}(U'_\alpha)$ to $\mathcal{F}(U'_\beta)$ when it exists, and in a similar way $\rho^\alpha_{\beta\gamma}$ will denote the restriction map from $\mathcal{F}(U'_\alpha)$ to $\mathcal{F}(U'_\beta \cap U'_\gamma)$ (and $\rho^{\alpha\beta}_\gamma$ denotes the restriction map from $\mathcal{F}(U'_\alpha \cap U'_\beta)$ to $\mathcal{F}(U_\gamma)$ when $U_\gamma \subseteq U'_\alpha \cap U'_\beta$). With this notation, the maps ϵ^*_1 and ϵ^*_2 can be rendered very explicitly. An element of $\prod \mathcal{F}(U'_\lambda)$ is a function whose value at $\lambda \in \Lambda$ is in the set $\mathcal{F}(U'_\lambda)$; hence, if $\xi \in \prod \mathcal{F}(U'_\lambda)$,

$$\begin{aligned}(\epsilon^*_1\xi)_i &= \rho_i^{\epsilon(i)}\xi_{\epsilon(i)}\\ (\epsilon'^*_1\xi)_i &= \rho_i^{\epsilon'(i)}\xi_{\epsilon'(i)}.\end{aligned}$$

Now, if $\xi \in H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{F})$, then

$$\rho^\lambda_{\lambda\mu}\xi_\lambda = (\beta_1\xi)_{\lambda\mu} = (\beta_2\xi)_{\lambda\mu} = \rho^\mu_{\lambda\mu}\xi_\mu. \tag{$*$}$$

Since $U_i \subseteq U'_{\epsilon(i)} \cap U'_{\epsilon'(i)}$, we deduce

$$\rho_i^{\epsilon(i)} = \rho_i^{\epsilon(i)\,\epsilon'(i)}\rho^{\epsilon(i)}_{\epsilon(i)\,\epsilon'(i)}; \quad \rho_i^{\epsilon'(i)} = \rho_i^{\epsilon(i)\,\epsilon'(i)}\rho^{\epsilon'(i)}_{\epsilon(i)\,\epsilon'(i)}.$$

Thus,

$$(\epsilon^*_1\xi)_i = \rho_i^{\epsilon(i)\,\epsilon'(i)}\rho^{\epsilon(i)}_{\epsilon(i)\,\epsilon'(i)}\xi_{\epsilon(i)} = \rho_i^{\epsilon(i)\,\epsilon'(i)}\rho^{\epsilon'(i)}_{\epsilon(i)\,\epsilon'(i)}\xi_{\epsilon'(i)} = (\epsilon'^*_1\xi)_i,$$

as required. Observe that equation $(*)$ was used in the middle equality. $\square$

Lemma A.2 is the most important special case of a general result on homotopies between maps in the theory of Čech Cohomology (Chapter B, Section B.3).

Our next objective is to define Čech "cohomology groups" using the notion of direct mapping family. Recall that if I is an index set which is a directed preorder and if we have a direct mapping family $\{F_i\}_{i\in I}$, which means that for all $i, j \in I$ with $i \leq j$, there is a map $\rho^i_j\colon F_i \to F_j$ so that

$$\begin{aligned}\rho^i_i &= \mathrm{id}\\ \rho^i_k &= \rho^j_k \circ \rho^i_j\end{aligned}$$

for all $i, j, k \in I$ with $i \leq j \leq k$, then the *direct limit* (or *inductive limit*), $\varinjlim F_i$, is defined as follows: First, form the disjoint union $\coprod_{i\in I} F_i$. Next, let $\sim$ be the equivalence relation on $\coprod_{i\in I} F_i$ defined by:

$$f_i \sim f_j \quad \text{iff} \quad \rho^i_k(f_i) = \rho^j_k(f_j) \quad \text{for some } k \in I \text{ with } k \geq i, j,$$

for any $f_i \in F_i$ and any $f_j \in F_j$. Finally, the direct limit $\varinjlim F_i$ is given by

$$\varinjlim_{i\in I} F_i = \Big(\coprod_{i\in I} F_i\Big)/\sim.$$

For every index $i \in I$, we have the canonical injection $\epsilon_i\colon F_i \to \coprod_{i\in I} F_i$, and thus, a canonical map $\pi_i\colon F_i \longrightarrow \varinjlim F_i$, namely

$$\pi_i\colon f \mapsto [\epsilon_i(f)]_\sim.$$

(Here, $[x]_\sim$ means equivalence class of x modulo $\sim$.) It is obvious that $\pi_i = \pi_j \circ \rho^i_j$ for all $i, j \in I$ with $i \leq j$. Note that if each F_i is a group or a ring, then $\varinjlim F_i$ is also a group or a ring. For example, in the case where each F_i is a group, we define addition by

$$[f_i] + [f_j] = [\rho^i_k(f_i) + \rho^j_k(f_j)], \quad \text{for any } k \in I \text{ with } k \geq i, j.$$

The direct limit $\varinjlim F_i$ is characterized by the important *universal mapping property*: For every G and every family of maps $\theta_i \colon F_i \to G$ so that $\theta_i = \theta_j \circ \rho^i_j$, for all $i, j \in I$ with $i \leq j$, there is a unique map, $\varphi \colon \varinjlim F_i \to G$, so that

$$\theta_i = \varphi \circ \pi_i, \quad \text{for all } i \in I.$$

We apply the above construction to the preorder of domination among open coverings.

Given two coverings $\{U_i \longrightarrow U\}$ and $\{U'_\lambda \longrightarrow U\}$, we shall write $\{U_i \longrightarrow U\} \geq \{U'_\lambda \longrightarrow U\}$ and say that $\{U_i \longrightarrow U\}$ *dominates* $\{U'_\lambda \longrightarrow U\}$ (or *refines* $\{U'_\lambda \longrightarrow U\}$) if there is a map $\{U_i \longrightarrow U\} \longrightarrow \{U'_\lambda \longrightarrow U\}$. Clearly, the relation of domination partially orders the coverings of U and turns the set of such coverings into a directed set. According to Lemma A.2, the sets $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$ form a direct mapping family on this directed set of coverings of U, for the map between sets $H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{F})$, $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$ *depends only on the domination relation.* Consequently, it is legitimate to pass to the direct limit over all coverings of U in the mapping family $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$. The direct limit is denoted $\check{H}^0(U, \mathcal{F})$ and is usually called the *zeroth Čech cohomology group of U with coefficients in $\mathcal{F}$*. For the present purposes, we adopt the slightly simpler notation $\mathcal{F}^{(+)}(U)$ for $\check{H}^0(U, \mathcal{F})$. Thus,

$$\mathcal{F}^{(+)}(U) = \varinjlim_{\{U_i \to U\}} H^0(\{U_i \to U\}, \mathcal{F})$$

(the direct limit over all coverings $\{U_i \to U\}$ of U.)

Remark: As pointed out by Serre (see FAC [47], Chapter 1, §3, Subsection 22), there is a set-theoretic difficulty when defining the direct limit $\mathcal{F}^{(+)}(U)$ with respect to the collection of *all* coverings of U, since the collection of all coverings of an open set is not a set (the index set is arbitrary). This difficulty can be circumvented by observing that any covering $\{U_i \longrightarrow U\}_{i \in I}$ is equivalent to a covering $\{U'_\lambda \longrightarrow U\}_{\lambda \in \Lambda}$ whose index set Λ is a subset of 2^X. Indeed, we can take for $\{U'_\lambda \longrightarrow U\}_{\lambda \in \Lambda}$ the *set* of all open subsets of X that belong to the family $\{U_i \longrightarrow U\}_{i \in I}$. As we noted earlier, if $\{U_i \longrightarrow U\}_{i \in I}$ and $\{U'_\lambda \longrightarrow U\}_{\lambda \in \Lambda}$ are equivalent, then there is a bijection between $H^0(\{U_i \longrightarrow U\}, \mathcal{F})$ and $H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{F})$, so that we can define

$$\mathcal{F}^{(+)}(U) = \varinjlim_{\{U_i \to U\}} H^0(\{U_i \to U\}, \mathcal{F})$$

with respect to coverings $\{U_i \to U\}$ whose index set I is a subset of 2^X. Another way to circumvent the problem is to use a device due to Godement ([18], Chapter 5, Section 5.8).

Now observe that $U \rightsquigarrow \mathcal{F}^{(+)}(U)$ is itself a presheaf. For if $V \subseteq U$ and if $\{U_i \longrightarrow U\}$ is a covering, then $\{U_i \cap V \longrightarrow V\}$ is a covering, and clearly there is a natural map

$H^0(\{U_i \longrightarrow U\}, \mathcal{F}) \longrightarrow H^0(\{U_i \cap V \longrightarrow V\}, \mathcal{F})$. From the universal mapping property of direct limits it follows that we obtain a map $\mathcal{F}^{(+)}(U) \longrightarrow \mathcal{F}^{(+)}(V)$. One checks very easily that these data do indeed describe a presheaf $\mathcal{F}^{(+)}$.

The point of all the above is that $\mathcal{F}^{(+)}$ while, in general, not a sheaf, is much closer to being a sheaf than $\mathcal{F}$ is. In fact, we shall show that $\mathcal{F}^{(+)}$ satisfies the following axiom weaker than axiom (S):

Axiom (+) If $\{U_i \longrightarrow U\} \in \operatorname{Cov} \mathcal{T}$, then $\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i)$ is injective.

Theorem A.3 *Let $\mathcal{F}$ be a presheaf of sets on the space X, then the presheaf $\mathcal{F}^{(+)}$ satisfies axiom (+). For any presheaf $\mathcal{G}$, if $\mathcal{G}$ satisfies axiom (+), then $G^{(+)}$ is a sheaf. Consequently, for any presheaf $\mathcal{F}$, $\mathcal{F}^{(+)(+)}$ is a sheaf. If $\mathcal{F}$ is an arbitrary presheaf, there is a natural map $\mathcal{F} \longrightarrow \mathcal{F}^{(+)}$ and the presheaf $\mathcal{F}^{(+)(+)}$ satisfies the following universal property: Every map of the presheaf $\mathcal{F}$ into a sheaf $\mathcal{G}$ factors uniquely through the canonical map of $\mathcal{F}$ into $\mathcal{F}^{(+)(+)}$.*

Proof. Let $\overline{\xi_1}, \overline{\xi_2}$ belong to $\mathcal{F}^{(+)}(U)$ and assume $\overline{\xi_1}$ and $\overline{\xi_2}$ have the same image in $\mathcal{F}^{(+)}(U_i)$ for each i. Since

$$\mathcal{F}^{(+)}(U) = \varinjlim H^0(\{V_\alpha \to U\}, \mathcal{F}),$$

the elements $\overline{\xi_1}$ and $\overline{\xi_2}$ can be represented by elements ξ_1, ξ_2 in $H^0(\{V_\alpha \to U\}, \mathcal{F})$ for some covering $\{V_\alpha \to U\}$ of U. When this is done, the image of $\overline{\xi_1}$ (resp. $\overline{\xi_2}$) in $\mathcal{F}^{(+)}(U_i)$ is represented by the image of ξ_1 (resp. ξ_2) in $H^0(\{V_\alpha \cap U_i \to U_i\}, \mathcal{F})$. However, $\overline{\xi_1}$ and $\overline{\xi_2}$ have the same image in $\mathcal{F}^{(+)}(U_i)$ for every i; hence, there is a covering $\{W_{\alpha i} \longrightarrow U_i\}$ dominating $\{V_\alpha \cap U_i \to U_i\}$ such that the images of ξ_1 and ξ_2 agree in $\prod_\alpha \mathcal{F}(W_{\alpha i})$. When both α and i vary, $\{W_{\alpha i} \longrightarrow U\}$ is a covering which dominates $\{V_\alpha \longrightarrow U\}$ and for which ξ_1, ξ_2 have equal image in $\prod_{\alpha, i} \mathcal{F}(W_{\alpha i})$. It follows immediately from the definition of $\mathcal{F}^{(+)}$ that $\overline{\xi_1} = \overline{\xi_2}$.

Now assume that the presheaf $\mathcal{G}$ satisfies axiom (+). We contend that the map $H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{G}) \longrightarrow H^0(\{U_i \longrightarrow U\}, \mathcal{G})$ induced by a refinement of coverings $\{U_i \longrightarrow U\} \geq \{U'_\lambda \longrightarrow U\}$ is *always injective*. To see this, let ϵ be the map of coverings $\{U_i \longrightarrow U\} \longrightarrow \{U'_\lambda \longrightarrow U\}$, and consider the diagram of coverings

$$\begin{tikzcd}
& \{U_i \cap U'_\lambda\} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\
\{U_i\} \arrow[rr, "\epsilon"] \arrow[dr] & & \{U'_\lambda\} \arrow[dl] \\
& U &
\end{tikzcd}$$

Since $\{U_i \cap U'_\lambda \longrightarrow U'_\lambda\}$ for fixed λ is a covering, and since $\mathcal{G}$ satisfies (+), we deduce that

$$\theta\colon \prod_\lambda \mathcal{G}(U'_\lambda) \rightrightarrows \prod_{i,\lambda} \mathcal{G}(U_i \cap U'_\lambda)$$

is injective. However, θ when restricted to $H^0(\{U'_\lambda \longrightarrow U\}, \mathcal{G})$ is precisely the map π'^*; hence, π'^* is an injection. But Lemma A.2 shows that $\pi^* \circ \epsilon^* = \pi'^*$; hence, ϵ^* is indeed an injection.

Let $\{U_i \longrightarrow U\}$ be a covering, and let $\overline{\xi} \in H^0(\{U_i \longrightarrow U\}, \mathcal{G}^{(+)})$ be given. We must show that $\overline{\xi}$ is the image of some element of $\mathcal{G}^{(+)}(U)$. Let $\overline{\xi}_i$ be the i^{th} component of $\overline{\xi}$, (so $\overline{\xi}_i \in \mathcal{G}^{(+)}(U_i)$), and choose for each i a covering $\{V_{\alpha i} \longrightarrow U_i\}$ and an element $\xi_i \in H^0(\{V_{\alpha i} \longrightarrow U_i\}, \mathcal{G})$ representing $\overline{\xi}_i$. Then we have the following diagram of coverings

$$\begin{tikzcd}
\{V_{\alpha i}\} \arrow[d] & \{V_{\alpha i} \cap U_j\} \arrow[l] \arrow[d] & \{V_{\alpha i} \cap V_{\beta j}\} \arrow[l] \arrow[d] \\
\{U_i\} \arrow[d] & \{U_i \cap U_j\} \arrow[l] \arrow[d] & \{U_i \cap U_{\beta j}\} \arrow[l] \arrow[d] \\
U & \{U_j\} \arrow[l] & \{V_{\beta j}\}. \arrow[l]
\end{tikzcd}$$

The element ξ_i induces an element $\xi_{i,j}^{(1)} \in H^0(\{V_{\alpha i} \cap U_j \longrightarrow U_i \cap U_j\}, \mathcal{G})$, and similarly, ξ_j induces an element $\xi_{i,j}^{(2)} \in H^0(\{U_i \cap V_{\beta j} \longrightarrow U_i \cap U_j\}, \mathcal{G})$. Since $\overline{\xi} \in H^0(\{U_i \longrightarrow U\}, \mathcal{G}^{(+)})$, the elements $\xi_{i,j}^{(1)}$ and $\xi_{i,j}^{(2)}$ represent the same element of $\mathcal{G}^{(+)}(U_i \cap U_j)$. Therefore, $\xi_{i,j}^{(1)}$ and $\xi_{i,j}^{(2)}$ "become equal" in some covering of $U_i \cap U_j$ which is a common refinement of the coverings

$$\{V_{\alpha i} \cap U_j \longrightarrow U_i \cap U_j\}, \quad \{U_i \cap V_{\beta j} \longrightarrow U_i \cap U_j\}.$$

However, we have just proven that the induced maps on H^0 are always injections (since $\mathcal{G}$ satisfies $(+)$), and it follows from this that $\xi_{i,j}^{(1)}$ and $\xi_{i,j}^{(2)}$ become equal in *any* common refinement of the above coverings. In particular, $\xi_{i,j}^{(1)}$ and $\xi_{i,j}^{(2)}$ become equal in $\prod_{\alpha,\beta} \mathcal{G}(V_{\alpha i} \cap V_{\beta j})$, which proves that $\xi \in H^0(\{V_{\alpha i} \longrightarrow U\}, \mathcal{G})$. Hence, $\xi \in \mathcal{G}^{(+)}(U)$ as required, and $\mathcal{G}^{(+)}$ therefore is a sheaf.

The identity covering $\{U \longrightarrow U\}$ is dominated by every covering; hence we obtain the map

$$\mathcal{F}(U) = H^0(\{U \longrightarrow U\}, \mathcal{F}) \longrightarrow \lim\ H^0(\{U_i \longrightarrow U\}, \mathcal{F}) = \mathcal{F}^{(+)}(U).$$

Moreover, if $\mathcal{G}$ is a sheaf, and $\{U_i \longrightarrow U\}$ is a covering of U, then consideration of the commutative diagram

$$\begin{tikzcd}
\prod \mathcal{F}(U_i \cap U_j) \arrow[r] & \prod \mathcal{G}(U_i \cap U_j) \\
\prod \mathcal{F}(U_i) \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & \prod \mathcal{G}(U_i) \arrow[u, shift left] \arrow[u, shift right] \\
\mathcal{F}(U) \arrow[r] \arrow[u] & \mathcal{G}(U) \arrow[u]
\end{tikzcd}$$

shows that any map $\mathcal{F} \longrightarrow \mathcal{G}$ factors through $\mathcal{F}^{(+)}$ in a unique way. This proves the universal mapping property of the sheaf associated to a presheaf, and completes the proof of Theorem A.3. $\square$

Notation: The sheaf associated to the presheaf $\mathcal{F}$ will be denoted $\mathcal{F}^{\#}$. The sheaf $\mathcal{F}^{\#}$ is sometimes called the *sheafification of* $\mathcal{F}$.

There is a second way of constructing the sheaf associated to a given presheaf, this is the method of *étalé* spaces—it is more classical than the double limit method adopted above. If X is a topological space, then the pair (E, π) consisting of a topological space E and a surjective map $\pi\colon E \to X$ is an *étalé* space over X if and only if π is a local homeomorphism. If (E, π) is a such a space and U is an open subset of X, then a continuous map $\sigma\colon U \to E$ is called a *section* of E if and only if $(\pi \circ \sigma)(x) = x$ for every $x \in U$. (The word "section" is short for *cross-section* and the origin of this word is obvious from the diagram showed in Figure A.1).

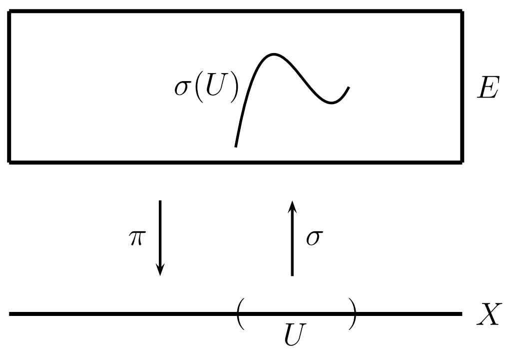

Figure A.1: A section of some *étalé* space

Given (E, π) over X, let $E_0(U)$ $(= \Gamma(U, E))$ denote the set of sections of E over U. The family of sets $E_0(U)$ is a presheaf in the obvious way, and it is trivial to verify that E_0 is actually a sheaf. So with every *étalé* space (E, π) over X, one has a canonical sheaf associated, E_0, usually called the *sheaf of germs of continuous sections of* E. The nub of the second method consists in associating to the presheaf $\mathcal{F}$ an *étalé* space $(\widetilde{\mathcal{F}}, \pi)$ and then in passing to the sheaf $\widetilde{\mathcal{F}}_0$.

Let $x \in X$ be a chosen point and let J_x denote the family of open sets of X containing x. The set J_x is directed by defining that $U \geq V$ whenever $U \subseteq V$. If $\mathcal{F}$ is a presheaf on X, we may then form the direct limit

$$\varinjlim_{U \in J_x} \mathcal{F}(U) = \mathcal{F}_x,$$

which is called the *stalk of* $\mathcal{F}$ *at* x. Observe that if $\mathcal{F}$ is a sheaf of groups or rings, $\mathcal{F}_x$ is a group (resp. ring) for each $x \in X$.

The notion of stalk allows us to give a very precise formulation of the principle that sheaves are determined locally on X. This is

Proposition A.4 *Let $\mathcal{F}, \mathcal{G}$ be sheaves of sets over X, let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism. For each $x \in X$, the morphism φ induces a map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$; φ is an isomorphism if and only if φ_x is an isomorphism for all $x \in X$.*

Proof. Choose $x \in X$, and let $U \in J_x$. Then φ gives rise to a map $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$, which when coupled with the canonical $\mathcal{G}(U) \longrightarrow \mathcal{G}_x$, gives us the mapping $\mathcal{F}(U) \longrightarrow \mathcal{G}_x$. it is easy to check that the latter map commutes with restriction to smaller open sets; so it factors through the direct limit, $\mathcal{F}_x$ of the sets $\mathcal{F}(U)$. Clearly, if φ is an isomorphism so is φ_x for each x.

Now suppose φ_x is an injection for each x. Given an open set U, let ξ, η be chosen in $\mathcal{F}(U)$ and assume that $\varphi(\xi) = \varphi(\eta)$. For each $x \in U$, $\varphi(\xi)_x = \varphi(\eta)_x$ (here $\varphi(\xi)_x$ is the image of $\varphi(\xi)$ in $\mathcal{G}_x$, etc.). As $\varphi_x(\xi_x) = \varphi(\xi)_x$, we deduce that $\xi_x = \eta_x$ for each $x \in U$. This means that there exist open sets U_x for each $x \in U$, with $x \in U_x$ and $\rho^U_{U_x}(\xi) = \rho^U_{U_x}(\eta)$. But the family $\{U_x \longrightarrow U\}$ is an open covering, and ξ, η go to the same element of $\prod_x \mathcal{F}(U_x)$ under the mapping $\mathcal{F}(U) \longrightarrow \prod_x \mathcal{F}(U_x)$. Since $\mathcal{F}$ is a sheaf, this implies that $\xi = \eta$.

Finally, let $\alpha \in \mathcal{G}(U)$ be chosen, then as φ_x is surjective for each $x \in U$, there exist elements ξ_x in $\mathcal{F}_x$ such that $\varphi_x(\xi_x) = \alpha_x$. The elements ξ_x arise from elements $\xi'_x \in \mathcal{F}(U_x)$– where the sets U_x are open neighborhoods of x. Since $\varphi(\xi'_x)_x = \alpha_x$, there is a smaller neighborhood V_x of x such that $\rho^{U_x}_{V_x}(\xi'_x) = \rho^U_{V_x}(\alpha)$. We may therefore assume that $U_x = V_x$. If y is another point in U, let ξ'_{xy} be the restriction of ξ'_x to $U_x \cap U_y$ and let ξ'_{yx} be the similar restriction of ξ'_y. Then

$$\varphi(\xi'_{xy}) = \rho^U_{U_x \cap U_y}(\alpha) = \varphi(\xi'_{yx}).$$

Since φ is injective (by the above paragraph), this shows that $\xi'_{xy} = \xi'_{yx}$. It follows from the second part of the sheaf axiom for $\mathcal{F}$ that there exists an element $\xi \in \mathcal{F}(U)$ such that $\rho^U_{U_x}(\xi) = \xi'_x$ for every $x \in U$. Clearly, $\varphi(\xi) = \alpha$, which proves that φ is surjective. $\square$

Theorem A.5 *Let $\mathcal{F}$ be a presheaf of sets on X. There is associated to $\mathcal{F}$ an étalé space, $(\widetilde{\mathcal{F}}, \pi)$ in a canonical way such that for every $x \in X$,*

$$\mathcal{F}_x = \pi^{-1}(x).$$

If $\widetilde{\mathcal{F}}_0$ is the sheaf of germs of sections of $\widetilde{\mathcal{F}}$, then as a sheaf, $\widetilde{\mathcal{F}}_0$ is canonically isomorphic to $\mathcal{F}^{\#}$. Every sheaf is the sheaf of germs of sections of its associated étalé space.

Proof. For each $x \in X$ form the stalk, $\mathcal{F}_x$, at x. Let $\widetilde{\mathcal{F}}$ be the disjoint union of the sets $\mathcal{F}_x$ as x varies over X, and define $\pi\colon \widetilde{\mathcal{F}} \to X$ by the rule: π *takes all of* $\mathcal{F}_x$ *onto* x. Then the equation $\mathcal{F}_x = \pi^{-1}(x)$ is automatic, and all that is lacking is the definition of a suitable topology on $\widetilde{\mathcal{F}}$. If the open set U is given and $x \in U$ is chosen, then there is a map $\rho_{U,x}\colon \mathcal{F}(U) \longrightarrow \mathcal{F}_x$. Let $s \in \mathcal{F}(U)$, and let $\rho_{U,x}(s)$ be denoted $\widetilde{s}(x)$. For fixed U, and varying x, we obtain $\widetilde{s}(x) \in \mathcal{F}_x$ (each x); hence, $\widetilde{s}$ is a function from U to $\widetilde{\mathcal{F}}$. Moreover, by the definition of π, we have $(\pi \circ \widetilde{s})(x) = x$ for each $x \in U$. It is easily seen that if $t = \rho^U_V(s)$, for $V \subseteq U$, then $\widetilde{t}$ is the restriction of $\widetilde{s}$ to V in the *sense of functions*. We give $\widetilde{\mathcal{F}}$ the *finest* *topology that renders the maps $\widetilde{s}$ continuous* for every $s \in \mathcal{F}(U)$ and every $U \in \operatorname{Cat}\mathcal{T}$. Thus a set $E \subseteq \widetilde{\mathcal{F}}$ is open if and only if

$$(\forall U)(\forall s \in \mathcal{F}(U))(\{x \in U \mid \widetilde{s}(x) \in E\} \text{ is open in } X).$$

If U is open in X, and V is another open set in X, then for every $s \in \mathcal{F}(V)$, we have $\widetilde{s}^{-1}(\pi^{-1}(U)) = U \cap V$; hence, π is continuous. Moreover, $\widetilde{s}(U)$ is open in $\widetilde{\mathcal{F}}$ by definition, and π maps $\widetilde{s}(U)$ homeomorphically onto U. Thus, $(\widetilde{\mathcal{F}}, \pi)$ is an *étalé* space, as required.

The mapping $s \mapsto \widetilde{s}$ takes $\mathcal{F}(U)$ into $\widetilde{\mathcal{F}}_0(U)$ for each U, and is a map of the presheaf $\mathcal{F}$ into the sheaf $\widetilde{\mathcal{F}}_0$. As such it factors through the associated sheaf $\mathcal{F}^{\#}$ to $\mathcal{F}$, and we obtain the map of sheaves $\mathcal{F}^{\#} \longrightarrow \widetilde{\mathcal{F}}_0$. If we show that $\mathcal{F}^{\#}_x = \mathcal{F}_x$ and $(\widetilde{\mathcal{F}}_0)_x = \mathcal{F}_x$, then Proposition A.4 will imply that $\mathcal{F}^{\#}$ is isomorphic to $\widetilde{\mathcal{F}}_0$. Since $\mathcal{F}^{(+)} = \mathcal{F}$ whenever $\mathcal{F}$ is a sheaf, the remaining statement of Theorem A.5 will follow from the equality $\mathcal{F}^{\#} = \widetilde{\mathcal{F}}_0$.

The equality $(\widetilde{\mathcal{F}}_0)_x = \mathcal{F}_x$ follows immediately from the definitions. We now show that $\mathcal{F}^{(+)}_x = \mathcal{F}_x$, which will complete the proof. If s and t are two elements of $\mathcal{F}_x$ whose images s' and t' are equal in $\mathcal{F}^{(+)}_x$, then they are representing elements $\sigma, \tau \in \mathcal{F}(U)$—for some U containing x—whose images $\sigma', \tau' \in \mathcal{F}^{(+)}(U)$ are equal. Hence, there is a covering $\{U_\alpha \longrightarrow U\}$ such that $\sigma'_\alpha = \tau'_\alpha$ for every α (where σ'_α (resp. τ'_α) is the α^{th} component of σ' (resp. τ')). But the point x belongs to one of the U_α, and for this α, the elements σ'_α and τ'_α are representatives of s and t; hence, $s = t$. Given any element s of $\mathcal{F}^{(+)}_x$, it is represented by some element σ of $\mathcal{F}^{(+)}(U)$—U being an open neighborhood of x. The element σ, in turn, is represented by a family (σ_α) of elements of $\mathcal{F}(U)$ corresponding to a covering $\{U_\alpha \longrightarrow U\}$. One of the U_α contains x, and for this α, the element σ_α represents a element r of $\mathcal{F}_x$. Clearly, the image of r in $\mathcal{F}^{(+)}$ is s. $\square$

Remark: Theorem A.5 show that the functor $(E, \pi) \rightsquigarrow E_0$ is an equivalence of categories between the category of *étalé* spaces over X and the category of sheaves of sets over X.

Given a sheaf, $\mathcal{F}$, over a topological space X, for every open subset U of X, the set (resp. group, ring, etc.), $\mathcal{F}(U)$, is called the *set (resp. group, ring, etc.) of sections of $\mathcal{F}$ over U* and is also denoted $\Gamma(U, \mathcal{F})$, the notation being justified by the fact every sheaf is the sheaf of germs of sections of its *étalé* space. A section $\sigma \in \mathcal{F}(X) = \Gamma(X, \mathcal{F})$ over X is called a *global section of $\mathcal{F}$ over X*. For every section $\sigma \in \mathcal{F}(U) = \Gamma(U, \mathcal{F})$, where U is any open subset of X, for every $x \in U$, we let σ_x (or occasionally $\sigma(x)$) denote the image of σ under the canonical map $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$ (the *germ of σ over U at x*). A sheaf $\mathcal{G}$ over X is a *subsheaf of the sheaf $\mathcal{F}$* if and only if $\mathcal{G}(U)$ is a subset of $\mathcal{F}(U)$ (resp., subgroup, if $\mathcal{F}$ is a sheaf of abelian groups) for every open subset U of X, and the restriction maps $\rho_{\mathcal{G}}{}^U_V$ (resp. homomorphisms) are induced (by restriction and corestriction) by the corresponding maps $\rho_{\mathcal{F}}{}^U_V$ (for all open U, V with $V \subseteq U$). The reader should check that the *étalé* space $\widetilde{\mathcal{G}}$ associated with $\mathcal{G}$ is an open subset of the *étalé* space $\widetilde{\mathcal{F}}$ associated with $\mathcal{F}$, and that conversely, the sheaf of sections associated with an open subset of an *étalé* space E is a subsheaf of the sheaf of section of E. Assume further that $\mathcal{F}$ is a sheaf of abelian groups. The *support of the sheaf $\mathcal{F}$* is the set, $\operatorname{Supp}\mathcal{F}$, of all $x \in X$ such that $\mathcal{F}_x \neq (0)$.

Problem A.3 Let $\mathcal{F}$ be a sheaf and (E, π) its associated *étalé* space. Let U be an open subset of X, and let s and t be sections of E over U. Show that the set of all $x \in U$ for which $s(x) = t(x)$ is an open subset of U. If the topology on E is Hausdorff, prove that the set where $s(x) = t(x)$ is *closed* in U. Hence, show that if E is Hausdorff, two sections which agree at a point agree in the whole connected component of that point (analytic continuation of sections). If X is a complex analytic manifold and $\mathcal{F}$ is its sheaf of germs of holomorphic functions, is E Hausdorff? Answer the same question if X is a topological space and $\mathcal{F}$ is the sheaf of germs of continuous functions on X.

Problem A.4 Let $\mathcal{F}$ be a sheaf over X and suppose furthermore that $\mathcal{F}$ satisfies axiom (S) for arbitrary sets and arbitrary coverings from X. (That is, remove all mention of openness from axiom (S).) Show that for each set T in X, $\mathcal{F}(T) =\ (\Gamma(T, \widetilde{F}))$ is the set $\prod_{x\in T} \mathcal{F}_x$.

A.3 The Category $\mathcal{S}(X)$, Construction of Certain Sheaves

By $\mathcal{S}(X)$ we shall mean the full subcategory (i.e., all morphisms) of $\mathcal{P}(X)$ formed by the sheaves (of abelian groups) of $\mathcal{P}(X)$. (Similarly for $\mathcal{S}(X, \mathcal{E})$, where $\mathcal{E}$ stands for the category of sets.) We have two functors relating $\mathcal{S}$ and $\mathcal{P}$:

(a) $i\colon \mathcal{S} \to \mathcal{P}$, the functor which regards a sheaf as a presheaf, and

(b) $\#\colon \mathcal{P} \to \mathcal{S}$, the functor which assigns to each presheaf its associated sheaf.

Theorem A.6 *The category $\mathcal{S}(X)$ is an abelian category. The functor i is left-exact and the functor $\#$ is exact. Moreover, the functors i and $\#$ are adjoint (in the sense of Kan [35]), that is*
$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{G}, i\mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{S}}(\mathcal{G}^{\#}, \mathcal{F})$$
for every presfeaf $\mathcal{G}$ and every sheaf $\mathcal{F}$ over X.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X, and let $\theta\colon \mathcal{F} \to \mathcal{G}$ be a morphism. We can form the two presheaves
$$\begin{aligned} U &\rightsquigarrow \mathrm{Ker}\,(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)) \\ U &\rightsquigarrow \mathrm{Coker}\,(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)). \end{aligned}$$
Of these, a simple argument shows that the first is a sheaf, while the second need not be a sheaf. Define Ker θ to be the former sheaf and Coker θ to be the sheaf associated with the latter presheaf. The reader may check that with these definitions of Ker θ and Coker θ, $\mathcal{S}(X)$ forms an abelian category.

That i is left-exact follows immediately from the fact that the presheaf kernel and sheaf kernel coincide. The adjointness property of the functors i and $\#$ is merely another way of stating the universal mapping property of the sheaf associated to a presheaf. It remains only to prove that $\#$ is an exact functor. Now, the functor $(+)\colon \mathcal{P}(X) \to \mathcal{P}(X)$ given by $\mathcal{F} \rightsquigarrow \mathcal{F}^{(+)}$ is left-exact as one easily checks. It follows from this that $i \circ \# = (+) \circ (+)$ is also left-exact. However, as i is left-exact and fully faithful (i.e., $i(\mathcal{F}) = (0)$ if and only if $\mathcal{F} = (0)$), one finds that $\#$ is left-exact. So all that is necessary is to prove that $\#$ is right-exact. Let $\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of presheaves, and observe that by the adjointness of $\#$ and i we have an isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{F}, i\mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{S}}(\mathcal{F}^{\#}, \mathcal{G})$$

for every sheaf $\mathcal{G}$ over X. Hence, in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{P}}(\mathcal{F}'', i\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{P}}(\mathcal{F}, i\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{P}}(\mathcal{F}', i\mathcal{G}) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{S}}(\mathcal{F}''^{\#}, \mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathcal{S}}(\mathcal{F}^{\#}, \mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathcal{S}}(\mathcal{F}'^{\#}, \mathcal{G}),
\end{tikzcd}$$

exactness in the top row implies exactness in the bottom row. But we have assumed $\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is exact, so that the top row of our diagram is exact. Thus, for every sheaf $\mathcal{G}$, the bottom row of the diagram is exact; this implies that the sequence $\mathcal{F}'^{\#} \longrightarrow \mathcal{F}^{\#} \longrightarrow \mathcal{F}''^{\#} \longrightarrow 0$ is exact. $\square$

A less category–theoretic proof of the exactness of $\#$ may be given as a consequence of the following proposition.

Proposition A.7 *Let $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X)$ and let $\theta\colon \mathcal{F} \to \mathcal{G}$ be a morphism. Then θ is injective (resp. surjective, bijective) if and only if for every $x \in X$, the induced map $\theta_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, bijective). A sheaf $\mathcal{F}$ is zero if and only if all its stalks are zero.*

Proof. Since passage to stalks is an exact functor (trivial because a presheaf and its associated sheaf have the same stalks), and application of the last statement to the sheaves Ker θ, Coker θ will yield the first statements. Consequently, only the last statement requires proof.

Now, obviously, if $\mathcal{F}$ is zero so are all its stalks. Assume all the stalks of $\mathcal{F}$ are zero. Given any open set U, and any $\xi \in \mathcal{F}(U)$, since ξ_x is zero in $\mathcal{F}_x$ for each $x \in U$, there is an open neighborhood U_x of x in U such that $\xi'_x = \rho^U_{U_x}(\xi)$ vanishes. But $\{U_i \longrightarrow U\}$ is a covering and ξ goes to zero under the map $\mathcal{F}(U) \longrightarrow \prod_x \mathcal{F}(U_x)$. As $\mathcal{F}$ is a sheaf, ξ is zero—completing the proof. $\square$

The proof that $\#$ is exact is now obvious. Namely, if $\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is exact in $\mathcal{P}$, then certainly

$$\mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

is exact as a sequence of abelian groups. As $\mathcal{F}^{\#}_x = \mathcal{F}_x$, etc., we deduce that

$$\mathcal{F}'^{\#}_x \longrightarrow \mathcal{F}^{\#}_x \longrightarrow \mathcal{F}''^{\#}_x \longrightarrow 0$$

is exact for every x. Proposition A.7 shows that $\mathcal{F}'^{\#} \longrightarrow \mathcal{F}^{\#} \longrightarrow \mathcal{F}''^{\#} \longrightarrow 0$ is exact in $\mathcal{S}$.

The proof of Theorem A.6 illustrates a very basic principle: *If one wishes to make a cetain (functorial) construction on sheaves, one first does the construction for presheaves, and then passes to the associated sheaf*. The following illustrations of this process are worth more than any further explanation.

(a) *Cokernel of a map of sheaves* (as in Theorem A.6). If $\theta\colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, form the presheaf cokernel

$$U \rightsquigarrow \operatorname{Coker}(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)),$$

and pass to the associated sheaf. The result is Coker θ.

(b) *Direct sum of sheaves*. Let $\{\mathcal{F}_\alpha\}$ be a family of sheaves and form the "presheaf direct sum of the $\mathcal{F}_\alpha$," i.e., the presheaf

$$U \rightsquigarrow \coprod_\alpha \mathcal{F}_\alpha(U).$$

The associated sheaf to this presheaf is the direct sum of the $\mathcal{F}_\alpha$ in $\mathcal{S}(X)$. Hence, $\mathcal{S}(X)$ has direct sums.

(c) *Direct products of sheaves*. Let $\{\mathcal{F}_\alpha\}$ be a family of sheaves and form the "presheaf direct product of the $\mathcal{F}_\alpha$," i.e., the presheaf

$$U \rightsquigarrow \prod_\alpha \mathcal{F}_\alpha(U).$$

In this case, one obtains a sheaf, so $U \rightsquigarrow \prod_\alpha \mathcal{F}_\alpha(U)$ is the direct product of the $\mathcal{F}_\alpha$ in $\mathcal{S}(X)$. Hence, $\mathcal{S}(X)$ has products.

(d) *Direct limits of sheaves*. Let $\{\mathcal{F}_\lambda\}$ be a direct mapping family of sheaves over the directed index set Λ. The "presheaf direct limit"' of the $\mathcal{F}_\lambda$ is the presheaf given by

$$U \rightsquigarrow \varinjlim_\lambda \mathcal{F}_\lambda(U).$$

Its associated sheaf is the direct limit of the $\mathcal{F}_\lambda$ in $\mathcal{S}(X)$. Observe (exercise) that for every $x \in X$,

$$\left(\varinjlim_\lambda \mathcal{F}_\lambda\right)_x = \varinjlim_\lambda (\mathcal{F}_\lambda)_x.$$

(e) *Projective (Inverse) limits of sheaves*. Let $\{\mathcal{F}_\lambda\}$ be an inverse mapping family of sheaves over the directed index set Λ. The "presheaf inverse limit"' of the $\mathcal{F}_\lambda$,

$$U \rightsquigarrow \varprojlim_\lambda \mathcal{F}_\lambda(U)$$

is actually a sheaf; hence it is the inverse limit of the $\mathcal{F}_\lambda$ in $\mathcal{S}(X)$.

(f) *Torsion subsheaf of a sheaf*. Let $\mathcal{F}$ a sheaf, then $\mathcal{F}(U)$ has a torsion subgroup $\mathrm{t}\mathcal{F}(U)$ for each U. The presheaf

$$U \rightsquigarrow \mathrm{t}\mathcal{F}(U)$$

is actually a sheaf, called the *torsion subsheaf*, $\mathrm{t}\mathcal{F}$, of $\mathcal{F}$. The quotient $\mathcal{F}/\mathrm{t}\mathcal{F}$ of $\mathcal{F}$ by its torsion subsheaf (i.e., the cokernel of $0 \longrightarrow \mathrm{t}\mathcal{F} \longrightarrow \mathcal{F}$) is called the *torsion-free quotient of* $\mathcal{F}$. It is an easy exercise to verify that $(\mathcal{F}/\mathrm{t}\mathcal{F})(U)$ is a torsion free group for every U. One obtains the decomposition

$$0 \longrightarrow \mathrm{t}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathrm{t}\mathcal{F} \longrightarrow 0$$

of every sheaf into its torsion and torsion-free pieces.

(g) *Tensor-product of two sheaves*. Let $\mathcal{F}, \mathcal{G}$ be two sheaves, and form the presheaf tensor product of $\mathcal{F}$ and $\mathcal{G}$ (over $\mathbb{Z}$),

$$U \rightsquigarrow \mathcal{F}(U) \otimes \mathcal{G}(U).$$

The sheaf associated to this presheaf is the tensor product of $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{S}(X)$. Observe (exercise) that for every $x \in X$,

$$(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes \mathcal{G}_x.$$

(h) *Sheaf of germs of homomorphisms from $\mathcal{F}$ to $\mathcal{G}$*. Let $\mathcal{F}, \mathcal{G}$ be two sheaves over X. If U is an open subset subset of X, we may define two "new sheaves" $\mathcal{F} \upharpoonright U$ and $G \upharpoonright U$, called the restrictions of $\mathcal{F}$ (resp. $\mathcal{G}$) to U, as follows:

$$(\mathcal{F} \upharpoonright U)(V) = \mathcal{F}(V), \quad (\mathcal{G} \upharpoonright U)(V) = \mathcal{G}(V) \quad \text{for } V \subseteq U.$$

These are *sheaves over* U. Consider the presheaf

$$U \rightsquigarrow \mathrm{Hom}_{\mathcal{S}(U)}(\mathcal{F} \upharpoonright U, \mathcal{G} \upharpoonright U).$$

It is actually a sheaf, called the sheaf of germs of homomorphisms from $\mathcal{F}$ to $\mathcal{G}$, and denoted $\mathcal{H}om\,(\mathcal{F}, \mathcal{G})$. We have a canonical homomorphism

$$\mathcal{H}om\,(\mathcal{F}, \mathcal{G})_x \longrightarrow \mathrm{Hom}\,(\mathcal{F}_x, \mathcal{G}_x),$$

but this is neither injective nor surjective in general.

Problem A.5 Show that an injective sheaf is injective as a presheaf, i.e., the functor i preserves injectives.

Problem A.6 For any open set U of X, define the *section functor*, Γ_U, by

$$\Gamma_U(\mathcal{F}) = \mathcal{F}(U).$$

(Sometimes, the set $\mathcal{F}(U)$ is denoted $\Gamma(U, \mathcal{F})$.) Show that Γ_U is a left-exact functor.

Problem A.7 If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}$ is an exact sequence of sheaves, and if $\mathcal{C}$ denotes the *presheaf* cokernel of $\mathcal{F}' \longrightarrow \mathcal{F}$, prove that $\mathcal{C}$ satisfies axiom (+). Deduce that $\mathcal{C}^{(+)}$ is the sheaf cokernel of $\mathcal{F}' \longrightarrow \mathcal{F}$.

A.4 Direct and Inverse Images of Sheaves

Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous map from X to Y.

(a) *Direct Image*. Let $\mathcal{F}$ be a sheaf on X. For each open set V in Y, the set $f^{-1}(V)$ is open in X. Define a *presheaf on* Y by

$$V \rightsquigarrow \mathcal{F}(f^{-1}(V)).$$

This is *actually a sheaf*, called the *direct image of* $\mathcal{F}$ *by* f, denoted $f_*\mathcal{F}$. Thus

$$\Gamma(V, f_*\mathcal{F}) = (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) = \Gamma(f^{-1}(V), \mathcal{F}).$$

(b) *Inverse Image*. Let $\mathcal{G}$ be a sheaf on Y. If $x \in X$, consider the set $\mathcal{G}_{f(x)}$. There is a sheaf, $\mathcal{F}$, whose stalk at $x \in X$ is the set $\mathcal{G}_{f(x)}$; it is called the *inverse image of* $\mathcal{G}$ *by* f, denoted $f^*\mathcal{G}$. Thus, $f^*\mathcal{G}$ is a sheaf over X, and $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$. (This is an example of a construction where knowledge of the stalk-theoretic approach to sheaf theory is very helpful.)

To define the sections of the inverse image of $\mathcal{G}$ over an open set, one proceeds as follows: Given an open set U of X, let I_U^f be the family of all open V in Y with the property that $f(U) \subseteq V$. Now, the sets $\mathcal{G}(V)$ form a direct mapping family as V ranges over I_U^f; hence, we obtain the presheaf, $f_P\mathcal{G}$, on X by setting

$$f_P\mathcal{G}(U) = \varinjlim_{V \in I_U^f} \mathcal{G}(V).$$

The sheaf associated to this presheaf is $f^*\mathcal{G}$. To see this, note that the stalk of $f^*\mathcal{G}$ at $x \in X$ is just the limit $\varinjlim_{U \in J_x} (f_P\mathcal{G})(U)$ (recall, J_x denotes the family of open sets of X containing x.) Thus,

$$(f_P\mathcal{G})_x = \varinjlim_{U \in J_x} \varinjlim_{V \in I_U^f} \mathcal{G}(V) = \mathcal{G}_{f(x)},$$

the latter equality because f is continuous from X to Y.

Remark: Suppose that f is an open map (as well as being continuous). Then the directed set I_U^f has a final element, $f(U)$; hence the presheaf $f_P\mathcal{G}$ is given by $U \rightsquigarrow \mathcal{G}(f(U))$. It is easy to see that, under these circumstances, $f_P\mathcal{G}$ satisfies axiom (+). However, even if f is open, $f_P\mathcal{G}$ need not be a sheaf. If f is a monomorphisms (into), then $f_P\mathcal{G}$ is a sheaf; hence coincides with $f^*\mathcal{G}$.

If $f\colon X \to Y$ is the inclusion map, i.e., X is a subspace of Y, then $f^*\mathcal{G}$ is called the *restriction of* $\mathcal{G}$ *to* X, and is denoted $\mathcal{G} \upharpoonright X$. If $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is the sheaf on Y given by

$$\Gamma(U, f_*\mathcal{F}) = \Gamma(U \cap X, \mathcal{F}).$$

Theorem A.8 *Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous map. Then the functor $f_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ is left-exact, the functor $f^*\colon \mathcal{S}(Y) \to \mathcal{S}(X)$ is exact, and these two functors are adjoint, i.e., there is a natural isomorphism*

$$\theta(\mathcal{F}, \mathcal{G})\colon \mathrm{Hom}_{\mathcal{S}(X)}(f^*\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{S}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

for every $\mathcal{F} \in \mathcal{S}(X)$ and $\mathcal{G} \in \mathcal{S}(Y)$.

Proof. That f_* is left-exact is trivial. To prove that f^* is exact, one examines the stalk-theoretic definition and uses Proposition A.7. The details are easy and will be omitted. Let us prove that f^* is the left-adjoint of f_*. Let ξ take $f^*\mathcal{G}$ to $\mathcal{F}$ in $\mathcal{S}(X)$. There is a map of presheaves $f_P\mathcal{G} \longrightarrow f^*\mathcal{G}$, and this, when coupled with ξ yields a map of presheaves $\xi'\colon f_P\mathcal{G} \to \mathcal{F}$. Hence, for every open set U in X, we obtain a map

$$\xi'(U)\colon f_P\mathcal{G}(U) \to \mathcal{F}(U).$$

Now for every open V of Y, such that $f(U) \subseteq V$, we have a mapping $\mathcal{G}(V) \longrightarrow f_P\mathcal{G}(U)$. Hence, for all U and V with $f(U) \subseteq V$, we deduce a map

$$\xi'(U, V)\colon \mathcal{G}(V) \to \mathcal{F}(U).$$

Given V, let $U = f^{-1}(V)$. Certainly $F(U) \subseteq V$, so we obtain from the above a map

$$\widetilde{\xi}(V)\colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$$

which is easily seen to be a map of sheaves. Set $\theta(\mathcal{F}, \mathcal{G})(\xi)$ equal to $\widetilde{\xi}$. The procedure above, run in reverse with obvious modifications, yields a map inverse to $\theta(\mathcal{F}, \mathcal{G})$, and completes the proof. $\square$

Corollary A.9 *Let the hypotheses be as in Theorem A.8. Let $\mathcal{F}$ be a sheaf on X, $\mathcal{G}$ a sheaf on Y. Then there are canonical maps*

$$\begin{aligned} \mathrm{can}_X f&\colon f^*f_*\mathcal{F} \longrightarrow \mathcal{F} \\ \mathrm{can}_Y f&\colon \mathcal{G} \longrightarrow f_*f^*\mathcal{G}. \end{aligned}$$

Proof. In the theorem, set $\mathcal{G} = f_*\mathcal{F}$. Then

$$\theta(\mathcal{F}, \mathcal{G})\colon \mathrm{Hom}_{\mathcal{S}(X)}(f^*f_*\mathcal{F}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}(Y)}(f_*\mathcal{F}, f_*\mathcal{F})$$

is an isomorphism. The inverse image of the identity map $f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$ under θ is $\mathrm{can}_X f$. In a similar manner one constructs $\mathrm{can}_Y f$. $\square$

Example. let X be an arbitrary topological space, let Y be the one point space, and let $f\colon X \to Y$ be the map collapsing all of X to the one point of Y. If $\mathcal{F}$ is a sheaf on X, then

$f_*\mathcal{F}$ is the sheaf on Y whose stalk at the unique point of Y is the set $\mathcal{F}(X)$. If $\mathcal{G}$ is a sheaf on Y (i.e., if the set $\mathcal{G}(Y)$ is given), and if U is a nonempty open subset of X, then

$$f_P\mathcal{G}(U) = \mathcal{G}(f(U)) = \mathcal{G}(Y).$$

Hence, $f_P\mathcal{G}$ is the constant presheaf (with value $\mathcal{G}(Y)$) on X, and $f^*\mathcal{G}$ is the sheaf generated by the constant presheaf $U \rightsquigarrow \mathcal{G}(Y)$.

Let us apply this to the case in which $\mathcal{G} = f_*\mathcal{F}$ for a sheaf $\mathcal{F}$ on X. Then $f^*f_*\mathcal{F}$ is the sheaf generated by the constant presheaf $U \rightsquigarrow \mathcal{F}(X)$ (U open in X), and our canonical map $f^*f_*\mathcal{F} \longrightarrow \mathcal{F}$ is exactly the one induced by the map of presheaves $\mathcal{F}(X) = f_Pf_*\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ given by the restriction from X to U. It follows that $\mathrm{can}_X f$ is *neither injective nor surjective in general*.

Problem A.8 Give an example to show that $\mathrm{can}_Y f$ is neither injective nor surjective in general.

Problem A.9 Let X, Y, Z be three topological spaces, let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous maps, and let $h = g \circ f$. Show that as functors, h_* and $g_* \circ f_*$ are equal, and that $h^* = f^* \circ g^*$.

Problem A.10 A space X is *irreducible* if it is not the union of two closed proper subspaces. (We assume $X \neq \emptyset$.) A sheaf $\mathcal{F}$ on any space X is *locally associated to a constant presheaf* if every point of X has an open neighborhood U such that $\mathcal{F} \upharpoonright U$ is the sheaf associated to some constant presheaf on U. Suppose that X is irreducible. Prove that the following three conditions are equivalent:

(a) $\mathcal{F}$ is a constant presheaf on X.

(b) $\mathcal{F}$ is the sheaf associated to a constant presheaf on X.

(c) $\mathcal{F}$ is locally associated to a constant presheaf.

A.5 Locally Closed Subspaces

A subspace Y of a topological space X is *locally closed* in X if and only if for each $y \in Y$, there is an open neighborhood of y *in* X, say $U(y)$, such that $Y \cap U(y)$ is closed in $U(y)$.

Example.

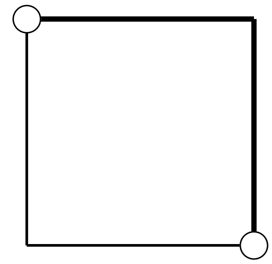

Figure A.2: Example of a locally closed set

A square whose interior and bold face edge excluding circled points make up Y.

Remark: The following are equivalent:

(a) X is locally closed in X.

(b) Y is the intersection of an open set and a closed set in X.

(c) Y is open in its closure, $\overline{Y}$.

Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. If M is an arbitrary subspace of X, the *support of a section* $\sigma \in \Gamma(M, \mathcal{F} \upharpoonright M)$ is the set, $\mathrm{Supp}\,\sigma$, of all $x \in M$ such that $\sigma_x \neq 0$.

Lemma A.10 *Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let M be an arbitrary subspace of X and let $\sigma \in \Gamma(M, \mathcal{F} \upharpoonright M)$. Then the support of σ is closed in M.*

Proof. Look at the complement, C, of the support of σ in M. If $x \in C$, $\sigma_x = 0$. Now, $\mathcal{F}_x$ is the inductive limit over all open neighborhoods U (of x *in* M) of the groups $(\mathcal{F} \upharpoonright M)(U)$. The element $\sigma \in \Gamma(M, \mathcal{F})$ induces on each such U its restriction, $\sigma \upharpoonright U$, in $\Gamma(U, \mathcal{F} \upharpoonright M)$, and σ_x is the image of these restrictions in the direct limit $\mathcal{F}_x$. Since $\sigma_x = 0$, and since $0 \in \mathcal{F}_x$ is the image of the element 0 in $(\mathcal{F} \upharpoonright M)(U)$ for each U, it follows by the definition of direct limit that there exists an open U in M such that $\sigma \upharpoonright U = 0$. Hence, this open neighborhood U (of x in M) lines in C, and this proves that C is open in M. $\square$

Locally closed subspaces are extremely important in algebraic geometry and sheaf theory because of the following extension theorem.

Theorem A.11 *Let Y be a locally closed subspace of X, and let $\mathcal{F}$ be a sheaf of abelian groups on Y. Then there exists a unique sheaf $\overline{\mathcal{F}}$ on X such that $\overline{\mathcal{F}} \upharpoonright Y = \mathcal{F}$ and $\overline{\mathcal{F}} \upharpoonright (X - Y) = (0)$.*

Remark: The sheaf $\overline{\mathcal{F}}$ is called $\mathcal{F}$ *extended by zero outside* Y.

Proof. Let us first prove that $\overline{\mathcal{F}}$ is unique. This will be done by characterizing the group $\overline{\mathcal{F}}(U)$ in terms of the sheaf $\mathcal{F}$. Assume then that we have a sheaf $\overline{\mathcal{F}}$ on X which when restricted to Y gives $\mathcal{F}$ and when restricted to $X - Y$ is zero. Consider the map

$$\rho^U_{U\cap Y} \colon \overline{\mathcal{F}}(U) \longrightarrow \overline{\mathcal{F}}(U \cap Y) = \mathcal{F}(U \cap Y).$$

(Here, $\overline{\mathcal{F}}(U \cap Y)$ is the group of sections over $U \cap Y$ of the *étalé* space associated with $\overline{\mathcal{F}}$.) Suppose that $\sigma \in \overline{\mathcal{F}}(U)$ and $\rho^U_{U\cap Y}(\sigma) = 0$. For any $x \in U$, when $x \notin Y$, $\sigma_x \in (\overline{\mathcal{F}} \restriction (X-Y))_x = (0)$ and when $x \in Y$, $\sigma_x = \rho^U_{U\cap Y}(\sigma)_x = 0$. Hence, $\sigma_x = 0$ for all $x \in U$, that is, $\rho^U_{U\cap Y}$ is injective. Which elements of $\mathcal{F}(U\cap Y)$ come from $\overline{\mathcal{F}}(U)$? Obviously, exactly those elements σ of $\mathcal{F}(U \cap Y)$ which can be extended *continuously* to functions $\overline{\sigma} \colon U \to \widetilde{\overline{\mathcal{F}}}$ which vanish outside Y. By Lemma A.10, the support of such a $\overline{\sigma}$ is closed in U. But the support of $\overline{\sigma}$ is exactly the support of σ; so we deduce that the elements σ of $\mathcal{F}(U \cap Y)$ which come from $\overline{\mathcal{F}}(U)$ are exactly those whose support is closed in U (not only in $U \cap Y$). Hence,

$$\overline{\mathcal{F}}(U) = \{\sigma \in \mathcal{F}(U \cap Y) \mid \operatorname{Supp}\sigma \text{ is closed in } U\}, \tag{$*$}$$

and this proves the uniqueness of $\mathcal{F}$.

Actually it does more, for the presheaf $U \rightsquigarrow \overline{\mathcal{F}}(U)$, where $\overline{\mathcal{F}}(U)$ is *defined* by $(*)$ is easily seen to be a sheaf. We claim that this sheaf is the required extension by zero of $\mathcal{F}$. If $x \in X - \overline{Y}$, then clearly $\overline{\mathcal{F}}_x = (0)$; so $\overline{\mathcal{F}}$ vanishes outside $\overline{Y}$. Suppose $x \in \overline{Y} - Y$, and let U be an open neighborhood of x in X. If $\sigma \in \overline{\mathcal{F}}(U)$, then $\operatorname{Supp}\sigma$ is closed in U. As x is not in Y, x in not in $\operatorname{Supp}\sigma$. Consequently, there exists an open subset, V, of U, such that $x \in V$ and $V \cap \operatorname{Supp}\sigma = \emptyset$. Since U is open in X, so is V; hence $\overline{\mathcal{F}}_x$, which is the limit of $\overline{\mathcal{F}}(V)$ over all such V, is zero.

Finally, we must prove that $\overline{\mathcal{F}} \restriction Y$ is $\mathcal{F}$. It is here that we must use the local closedness of Y—the rest of the proof being valid with no hypotheses on Y. If π is the inclusion mapping $Y \hookrightarrow X$, then $\overline{\mathcal{F}} \restriction Y$ is precisely $\pi^*\overline{\mathcal{F}}$. From Theorem A.8, we deduce the isomorphism

$$\operatorname{Hom}_{\mathcal{S}(Y)}(\pi^*\overline{\mathcal{F}}, \mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{S}(X)}(\overline{\mathcal{F}}, \pi_*\mathcal{F}).$$

However, we know from Section A.4 that $\pi_*\mathcal{F}$ is the sheaf $U \rightsquigarrow \mathcal{F}(U\cap Y)$; hence, there exists a morphism from $\overline{\mathcal{F}}$ to $\pi_*\mathcal{F}$. This proves that we have a morphism from $\overline{\mathcal{F}} \restriction Y = \pi^*\mathcal{F}$ to $\mathcal{F}$ as sheaves on Y; and to prove these sheaves isomorphic, we need to prove only that our morphism gives an isomorphism on the stalks. Let $y \in Y$, then *as* Y *is locally closed*, there is an open neighborhood, U, of y in X such that $U \cap Y$ is *closed in* U. The same is true for any open subset V of U. But then equation $(*)$ shows that for such V, $\overline{\mathcal{F}}(V) = \mathcal{F}(V \cap Y)$; hence for $y \in Y$, we have $\overline{\mathcal{F}}_y = \mathcal{F}_y$. $\square$

Let Y be locally closed in X, and let $\mathcal{F}$ be a sheaf of abelian groups on X. By the *truncation of* $\mathcal{F}$ *outside* Y, we mean the sheaf $\mathcal{F}_Y$ defined by

$$\mathcal{F}_Y = \overline{\mathcal{F} \restriction Y}.$$

Observe that $\mathcal{F}_Y \restriction Y = \mathcal{F} \restriction Y$ and $\mathcal{F} \restriction (X - Y) = (0)$.

If Y is open in X, then from Section A.4 one has

$$(\mathcal{F} \restriction Y)(U \cap Y) = \mathcal{F}(U \cap Y)$$

for any open subset U of X. Consequently,

$$\mathcal{F}_Y(U) = \{\sigma \in \mathcal{F}(U \cap Y) \mid \operatorname{Supp} \sigma \text{ is closed in } U\}.$$

If $\sigma \in \mathcal{F}(U \cap Y)$, and $\operatorname{Supp} \sigma$ is closed in U, then $\overline{\sigma}$ defined by $\overline{\sigma} \restriction (U \cap Y) = \sigma$ and $\overline{\sigma} = 0$ outside Y, is an element of $\mathcal{F}(U)$, and this gives us an *injection* $\mathcal{F}_Y \longrightarrow \mathcal{F}$ whenever Y is open in X. If Y is closed in X, then it follows from the definition of $\mathcal{F}_Y$ that
$\mathcal{F}_Y(U) = (\mathcal{F} \restriction Y)(U \cap Y)$.

Theorem A.12 *Let Y be a closed subspace of X, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Then the sequence*

$$0 \longrightarrow \mathcal{F}_{X-Y} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0 \tag{$**$}$$

is exact, where the map $\mathcal{F} \longrightarrow \mathcal{F}_Y$ is the restriction of sections.

Proof. Let U be open in X, then, as above, $\mathcal{F}(U) = (\mathcal{F} \restriction Y)(U \cap Y)$. Hence, the map $\sigma \mapsto \sigma \restriction (U \cap Y)$ of $\mathcal{F}(U)$ to $\mathcal{F}_Y(U)$ is defined; it is clearly surjective (examine the stalks). Now an element σ of $\mathcal{F}(U)$ goes to zero in $\mathcal{F}_Y(U)$ if and only if its support is contained in $U - (U \cap Y)$. Since $U - (U \cap Y) = U \cap (X - Y)$, this shows that σ goes to zero in $\mathcal{F}_Y(U)$ if and only if it comes from $\mathcal{F}_{X-Y}(U)$. $\square$

Remark: The exact sequence $(**)$ will be called *the exact sequence associated to the closed subspace Y*.

Our results on extension of sheaves by zero give us another characterization of locally closed subspaces. This characterization clearly shows that Theorem A.11 is valid only for locally closed subspaces.

Proposition A.13 *Let Y be a subspace of the topological space X. Then the following are equivalent:*

(a) Y is locally closed in X.

(b) Given any sheaf, $\mathcal{F}$, of abelian groups on X there exists a sheaf $\mathcal{F}_Y$ on X such that $\mathcal{F}_Y \restriction Y = \mathcal{F} \restriction Y$ and $\mathcal{F}_Y \restriction (X - Y) = (0)$.

Proof. $(a) \Rightarrow (b)$ is the content of Theorem A.11 and succeeding remarks. Let us prove $(b) \Rightarrow (a)$. Let $\mathcal{F}$ be the "constant" sheaf $\mathbb{Z}$ on X (i.e., the sheaf associated to the constant presheaf $\mathbb{Z}$) and assume (b) holds for $\mathcal{F}$. If $y \in Y$, then $(\mathcal{F}_Y)_y$ contains the element 1; hence, there exists an open neighborhood $U(y)$ of y *in* X and a section $s \in \Gamma(U(y), \mathcal{F}_Y)$ such that $s(y) = 1$. By choosing $U(y)$ small enough, we may assume that $s \restriction (U(y) \cap Y) \equiv 1$. On $U(y) - (U(y) \cap Y)$, the section s must vanish by hypotesis (b); *hence $U(y) \cap Y$ is the support of s over $U(y)$*. Lemma A.10 shows that $U(y) \cap Y$ is closed in $U(y)$, as required. $\square$

Proposition A.14 *Let X be a topological space, and let $\mathcal{F}$ be the sheaf associated to the constant presheaf A—where A is a ring. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$ such that $\mathcal{G}_x$ is an ideal in $\mathcal{F}_x = A$ for each $x \in X$. Then for each $a \in A$, the set of all $x \in X$ such that $a \in \mathcal{G}_x$ is open in X. If A is noetherian and $x \in X$, then for all y sufficiently close to x, we have $\mathcal{G}_x \subseteq \mathcal{G}_y$.*

Proof. Since $\mathcal{F}$ is associated to the constant presheaf A the sections of $\mathcal{F}$ are precisely the locally constant functions with values in A. To say that $a \in \mathcal{G}_x$ is to say that there exists a section of $\mathcal{F}$ having the value a at x. It follows that this section has the value a near x, that is,

$$U_a = \{x \mid a \in \mathcal{G}_x\}$$

is open. If A is noetherian, and x is given, then $\mathcal{G}_x$ has a finite basis $a_1, \ldots, a_r$. The intersection $U_{a_1} \cap \cdots \cap U_{a_r} = U$ is the set of all y such that $\mathcal{G}_x \subseteq \mathcal{G}_y$. Since U is open, we are done. $\square$

Corollary A.15 *Under the same hypotheses as Proposition A.14, save that $A = \mathbb{Z}$, each ideal $\mathcal{G}_x$ corresponds to a number $n(x) \geq 0$, and the function $x \mapsto n(x)$ has the following properties:*

(a) For all y close to x, $n(y)$ divides $n(x)$.

(b) For all $n \geq 0$, the set of all x such that $n(x)$ divides n is open in X.

If X is compact, the function $n(x)$ takes only finitely many values.

Proposition A.14 and its corollary will be used to prove Theorem A.16 below, which is of interest in the cohomology of algebraic varieties. We denote the sheaf associated to the constant presheaf $\mathbb{Z}$ by $\mathcal{Z}$.

Theorem A.16 *Let X be a compact space and let $\mathcal{G}$ be a subsheaf of the sheaf $\mathcal{Z}$ on X. Then $\mathcal{G}$ possesses a finite composition series whose quotients have the form $\mathbb{Z}_Y$ for locally closed subspaces Y of X.*

Proof. Each $\mathcal{G}_x$ has the form $n(x)\mathcal{Z}$ where the function $n(x)$ satisfies properties (a) and (b) of the corollary. Let U be the set of all $x \in X$ such that $n(x) \neq 0$. Since $\mathcal{Z}$ is a Hausdorff *étalé* space, the set U is open in X. For each integer $r \geq 1$, the condition $n(x) \leq r$ is an *open condition* (i.e., is satisfied on an open set) because every divisor of an integer is less than or equal to that integer. Hence,

$$U_r = \{x \in U \mid n(x) \leq r\}$$

is an open subset of U, and we have the ascending chain

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U.$$

Since X is compact, there exists an integer $n \geq 1$ such that $U_n = U$. Let $\mathcal{G}_r$ be the sheaf $\mathcal{G}_{U_r}$, then we have the composition series

$$\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{G}.$$

Clearly, $\mathcal{G}_1$ is isomorphic to $\mathcal{Z}_{U_1}$. Given $r > 1$, let Y_r be the locally closed subspace $U_r - U_{r-1}$. The sheaf $\mathcal{G}_r/\mathcal{G}_{r-1}$ is concentrated on Y_r, and as $n(x)$ is constantly equal to r on Y_r, it is obvious that $\mathcal{G}_r/\mathcal{G}_{r-1}$ is isomorphic to $\mathcal{Z}_{Y_r}$ (as abelian groups!). $\square$

Problem A.11 Let Y be an arbitrary subspace of X. Show that every sheaf of abelian groups $\mathcal{F}$ on Y is the restriction of a sheaf $\widetilde{\mathcal{F}}$ on X. (Hint: If $\pi\colon Y \to X$ is the inclusion, let $\widetilde{\mathcal{F}} = \pi_*\mathcal{F}$.) Show that $\pi_*\mathcal{F}$ is not concentrated (in general) on Y unless Y is closed.

Problem A.12 Let Y and Z be locally closed in X and let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that $(\mathcal{F}_Y)_Z = \mathcal{F}_{Y\cap Z}$. Deduce that $\mathcal{F}_Y$ is both a quotient of a subsheaf of $\mathcal{F}$ and a subsheaf of a quotient of $\mathcal{F}$. Prove also that $\mathcal{F}_Y = \mathcal{Z}_Y \otimes_{\mathbb{Z}} \mathcal{F}$. Deduce that the functor $\mathcal{F} \rightsquigarrow \mathcal{F}_Y$ is exact.

Problem A.13 Show that the sheaves $\mathcal{Z}_U$, for any open subset U of X, are those associated to the presheaves $\mathbb{Z}_U$ of Example 4, Section A.1. Deduce that the sheaves $\mathcal{Z}_U$ are a set of generators for the category $\mathcal{S}(X)$ (in the sense of [Grothendieck [21]). Prove moreover that every sheaf $\mathcal{F} \in \mathcal{S}(X)$ is a quotient of a direct sum of the $\mathcal{Z}_U$.

Problem A.14 A category (abelian, with generators) is called *locally notherian* when its generators are noetherian objects (i.e., every ascending chain of subobjects is eventually stationary). Is $\mathcal{S}(X)$ a locally noetherian category?

Problem A.15 Prove the converse of Proposition A.14. That is, show that any subsheaf $\mathcal{G}$ of the sheaf $\mathcal{F}$ for which

$$U_a = \{x \mid a \in \mathcal{G}_x\}$$

is always open (all $a \in A$) has the property that $\mathcal{G}_x$ is an ideal for every $x \in X$. Deduce that there is a one to one correspondence between sheaves of ideals of $\mathbb{Z}$ and functions $x \mapsto n(x)$ from X to the non-negative integers having the property that for y close to x, $n(y)$ divides $n(x)$.

A.6 Ringed Spaces, Sheaves of Modules

Definition A.3 A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. The space X is called the *underlying space* of the ringed space; the sheaf $\mathcal{O}_X$ is called the *structure sheaf* of the ringed space.

In what follows, we shall always assume that $\mathcal{O}_X$ is a sheaf of commutative rings with unity. This assumption is not necessary, but it is the situation most often encountered in geometry. By abuse of notation, a ringed space $(X, \mathcal{O}_X)$ will often be denoted X—except when this will cause cause confusion because of several possibilities for the structure sheaf.

Ringed spaces form a category if one defines morphisms as follows: Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be ringed spaces. By a *morphism* φ *from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ we mean a pair $(|\varphi|, \widetilde{\varphi})$, where $|\varphi|$ is a continuous map $X \longrightarrow Y$ and $\widetilde{\varphi}$ is a homomorphism of sheaves of rings $\mathcal{O}_Y \longrightarrow |\varphi|_* \mathcal{O}_X$ (or, what is the same, a homomorphism $\widetilde{\widetilde{\varphi}}$ from $|\varphi|^* \mathcal{O}_Y$ to $\mathcal{O}_X$). Observe that the map on the sheaf level is contravariant to the map of ringed spaces, while the map of underlying spaces is covariant to the map of ringed spaces.

Remark: We will also use the notation φ^{alg} or even φ^{a} for $\widetilde{\varphi}$.

If M is a subspace of X, then $(M, \mathcal{O}_X \restriction M)$ is a ringed space, and it is trivial to verify that there exists a canonical map of ringed spaces $(M, \mathcal{O}_X \restriction M) \longrightarrow (X, \mathcal{O}_X)$. This map is called the *injection* of M into X. If $(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$ is a map of ringed spaces, the composition $(M, \mathcal{O}_X \restriction M) \longrightarrow (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is called the *restriction* of φ to $(M, \mathcal{O}_X \restriction M)$. A trivial verification shows that if $|\varphi|$ is injective and $\widetilde{\widetilde{\varphi}}$ is surjective, then φ is a monomorphism in the category of ringed spaces [??].

Let X be a ringed space. An $\mathcal{O}_X$*-module* or *sheaf of modules over* X is a sheaf of abelian groups, $\mathcal{F}$, such that for every open U in X, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module in a functorial way (i.e., in a way compatible with restriction to smaller open sets). More explicitly, this means that the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d, "(\rho_{\mathcal{O}}{}^U_V, \rho_{\mathcal{F}}{}^U_V)"'] & \mathcal{F}(U) \arrow[d, "\rho_{\mathcal{F}}{}^U_V"] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes for any two open subsets U, V with $V \subseteq U$.). It is obvious that $\mathcal{O}_X$*-modules form an abelian category*. The sheaf $\mathcal{O}_X$ is an $\mathcal{O}_X$-module, and any subsheaf of $\mathcal{O}_X$ which is an $\mathcal{O}_X$-module under the obvious action is called a *sheaf of ideals* on X or an $\mathcal{O}_X$*-ideal*.

Ringed spaces and sheaves of modules are fundamental to the study of modern geometry–whether it be algebraic geometry, differential geometry, several complex variables, etc. Here are examples of ringed spaces.

(1) Let X be a topological space and let $\mathcal{O}_X$ be the sheaf of germs of continuous (resp. differentiable, C^∞) real (resp. complex) valued functions on X. Then $(X, \mathcal{O}_X)$ is a ringed space. In particular, if X is a C^p-manifold, we may take $\mathcal{O}_X$ to be the sheaf of germs of C^p-functions on X. Hence, for each $x \in X$, $\mathcal{O}_{X,x}$ is the ring of equivalence classes of C^p-functions locally defined at x under the relation: *f is equivalent to g if and only if there exists a suitable neighborhood of x, say U, such that $f \restriction U = g \restriction U$.*

(2) If X is a complex analytic manifold (think of an open subset of $\mathbb{C}$), let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on X. Then $(X, \mathcal{O}_X)$ is a ringed space. The rings $\mathcal{O}_{X,x}$ are integral domains for each $x \in X$. Let $\mathcal{M}_x$ be the fraction field of $\mathcal{O}_{X,x}$, and let $\mathcal{M}$ be the union of the $\mathcal{M}_x$. Then $\mathcal{M}$ is a sheaf if the topology is chosen so that

$$\Gamma(U, \mathcal{M}) = \left\{ \begin{array}{l} \sigma \mid \exists f, g \in \mathcal{O}_X(U))(g(x) \neq 0 \quad \text{for all } x \in U) \\ \text{and} \\ (\forall x \in U)\left(\sigma(x) = \dfrac{f(x)}{g(x)}\right). \end{array} \right\}$$

The sheaf $\mathcal{M}$ is called the *sheaf of germs of meromorphic functions on* X; its sections over U are called *meromorphic functions on* U. The sheaf $\mathcal{M}$ is an $\mathcal{O}_X$-module, indeed, $\mathcal{O}_X$ is a subsheaf of $\mathcal{M}$ and $\mathcal{M}$ is really an $\mathcal{O}_X$-algebra.

For each U open in X, let $\mathcal{O}_X^*(U)$ (resp. $\mathcal{M}^*(U)$) denote the invertible sections of $\mathcal{O}_X$ (resp. $\mathcal{M}$) over U. Then, $\mathcal{O}_X^*$, $\mathcal{M}^*$ are sheaves of abelian groups on X, and $\mathcal{O}_X^*$ is a subsheaf of $\mathcal{M}^*$. The quotient sheaf $\mathcal{M}^*/\mathcal{O}_X^*$ is called the *sheaf of germs of divisors on* X; its sections over X are the *divisors on* X. Since the sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}_X^* \longrightarrow 0$$

is exact, to give a global section of $\mathcal{M}^*/\mathcal{O}_X^*$ (i.e., a divisor on X) is to give a covering $\{U_i \longrightarrow X\}$ of X and a collection of invertible meromorphic functions $f_i \in \mathcal{M}^*(U_i)$ such that on $U_i \cap U_j$, the function f_i/f_j is an *invertible holomorphic* function. Each meromorphic function f induces a divisor on X in an obvious way; this divisor is called a *principal divisor of* X, or the *divisor of the function* f. Two functions yield the same divisor if and only if their ratio is a nonzero global holomorphic function. When X is compact, it is known that such functions must be constant ("Liouville's theorem"); hence, two meromorphic functions on a compact, complex analytic manifold have the same divisor if and only if their ratio is constant.

(3) Let A be a commutative ring with unity. If X is the set of all prime ideals of A (*the prime spectrum of* A), then X is a topological space in a natural way. Namely, we give X the *Zariski Topology* defined by: The set, V, of prime ideals $x \in X$ is closed in X if and only if there exists an ideal $\mathfrak{A}$ of A, such that V is precisely the set of prime ideals of A which contain $\mathfrak{A}$. Since V depends on $\mathfrak{A}$, we write $V(\mathfrak{A})$ for V. Observe that the sets $U_f = \{x \in X \mid f \notin x\}$ are open for every $f \in A$, and that these sets are a basis for the Zariski topology on X. Every element f of A may be considered a function on X by decreeing that the value of f at x is the residue of f modulo the prime ideal x. So

$$f(x) = f(\mathrm{mod}\, x) \quad \text{in} \quad A/x = \kappa(x).$$

From this it follows that f vanishes at x if and only if $f \in x$; hence, U_f is precisely the set of point of X where f does *not* vanish.

For each $x \in X$, let A_x be the localization of A at the prime ideal x. Given an open subset U of X, we set $\widetilde{A}(U) = \Gamma(U, \widetilde{A})$ equal to the ring of all functions $\widetilde{f}$ from U to $\bigcup\{A_x \mid x \in U\}$ which satisfy

(1) $\widetilde{f}(x) \in A_x$, for all $x \in U$, and

(2) $(\forall x \in U)(\exists \text{ open } V \text{ containing } x, \text{ and } f, g \in A)$ such that

(a) $U_f \subseteq V$ and

(b) For all $x \in V$, $\widetilde{f}(x) = \dfrac{g(x)}{f(x)}$ *in* A_x.

One checks without difficulty that $U \rightsquigarrow \widetilde{A}(U)$ is a sheaf whose stalk at x is A_x. (See Example 2 and the sheaf $\mathcal{M}$.) The sheaf $\widetilde{A}$, now called $\mathcal{O}_X$, is called the *sheaf of germs of holomorphic functions on* X. The pair $(X, \mathcal{O}_X)$ is a ringed space, denoted $\operatorname{Spec} A$, and is called an *affine scheme*.

When X is an affine variety in the sense of Chapter 1, then the affine ring, $A(X)$, of X determines an affine scheme $(Y, \mathcal{O}_Y)$ as above. In this case, X corresponds to the subspace of Y which consists of all the maximal ideals of $A(X)$, and $(X, \mathcal{O}_Y \restriction X)$ determines and is determined by X. Hence every affine variety is a ringed space. Actually X determines and is determined by $(Y, \mathcal{O}_Y)$, as we shall show in Chapter ??.

If X is irreducible, each $\mathcal{O}_{X,x}$ is an integral domain, and we can repeat the definitions and arguments of Example 2 to obtain the meromorphic functions and divisors on X. In this case, the sheaf $\mathcal{M}$ is merely the sheaf associated to the constant presheaf $k(X)$–where, as in Chapter 1, $k(X)$ is the field of rational functions on X.

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Using Section A.3 as a model, one easily constructs the sheaves $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ as well as the group $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. For example, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf associated to the presheaf

$$U \rightsquigarrow \mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

The stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ at $x \in X$ is $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$, while we have a canonical map

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is neither injective nor surjective in general. The functor $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is right-exact (in both arguments), commutes with direct limits (hence, with arbitrary direct sums), and $\mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{G}$ (resp. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X$) is canonically isomorphic to $\mathcal{G}$ (resp. $\mathcal{F}$). The functors $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are left-exact (in both arguments), and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G})$ is canonically isomorphic to $\mathcal{G}$. For each $\mathcal{O}_X$-module, $\mathcal{F}$, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual of* $\mathcal{F}$ and is usually denoted $\check{\mathcal{F}}$, or $\mathcal{F}^D$. If I is any set of indices, the direct sum of copies of $\mathcal{O}_X$ indexed by the set I is denoted $\mathcal{O}_X^{(I)}$; an $\mathcal{O}_X$-module is *free* if it is of the form $\mathcal{O}_X^{(I)}$ for some index set I.

Let $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ be ringed spaces, and let $\varphi\colon (X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $|\varphi|_*\mathcal{O}_X$ is a sheaf of rings on Y and $|\varphi|_*\mathcal{F}$ is a $|\varphi|_*\mathcal{O}_X$-module. However, we have a map $\widetilde{\varphi}\colon \mathcal{O}_Y\to|\varphi|_*\mathcal{O}_X$ of sheaves of rings on Y, and this permits us to make $|\varphi|_*\mathcal{F}$ into an $\mathcal{O}_Y$-module. This $\mathcal{O}_Y$-module will be called the *direct image* of $\mathcal{F}$ by φ, and will be denoted $\varphi_*\mathcal{F}$.

In the same way, $|\varphi|^*\mathcal{O}_Y$ is a sheaf of rings on X and $|\varphi|^*\mathcal{G}$ is a $|\varphi|^*\mathcal{O}_Y$-module. Now there exists a map $\widetilde{\widetilde{\varphi}}\colon |\varphi|^*\mathcal{O}_Y\to\mathcal{O}_X$ of sheaves of rings, so that $\mathcal{O}_X$ may be considered a $|\varphi|^*\mathcal{O}_Y$-module. This being said, the tensor product of $|\varphi|^*\mathcal{O}_Y$-modules

$$|\varphi|^*\mathcal{G}\otimes_{|\varphi|^*\mathcal{O}_Y}\mathcal{O}_X$$

is defined and *is an $\mathcal{O}_X$-module* in a natural way. This $\mathcal{O}_X$-module will be called the *inverse image* of $\mathcal{G}$ by φ, and will be denoted $\varphi^*\mathcal{G}$.

The following properties of the operations φ_* and φ^* are easily checked and will be left to the reader as exercises.

(a) φ_* and φ^* are functors, φ_* is left-exact and φ^* is right-exact.

(b) If $\mathcal{F},\mathcal{F}'$ are $\mathcal{O}_X$-modules, then there is a canonical homomorphism (or functors)
$$\varphi_*(\mathcal{F})\otimes_{\mathcal{O}_Y}\varphi_*(\mathcal{F}')\longrightarrow\varphi_*(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{F}').$$

(c) If $\mathcal{G},\mathcal{G}'$ are $\mathcal{O}_Y$-modules, then there is a canonical *isomorphism* (or functors)
$$\varphi^*(\mathcal{G})\otimes_{\mathcal{O}_X}\varphi^*(\mathcal{G}')\xrightarrow{\sim}\varphi^*(\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{G}').$$

(d) Hypotheses as in (b), then there is a canonical homomorphism (of functors) of $\mathcal{O}_X$-modules
$$\varphi_*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}'))\longrightarrow\mathcal{H}om_{\mathcal{O}_Y}(\varphi_*(\mathcal{F}),\varphi_*(\mathcal{F}')).$$

(e) Hypotheses as in (c), then there is a canonical homomorphism (of functors) of $\mathcal{O}_X$-modules
$$\varphi^*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G},\mathcal{G}'))\longrightarrow\mathcal{H}om_{\mathcal{O}_X}(\varphi^*(\mathcal{G}),\varphi^*(\mathcal{G}')).$$
(See Theorem A.17 below for the essential step in the proof.)

(f) The functor φ^* commutes with direct limits and arbitrary direct sums.

(g) If $\psi\colon (Y,\mathcal{O}_Y)\to(Z,\mathcal{O}_Z)$ is another map of ringed spaces, then $(\psi\circ\varphi)_*=\psi_*\circ\varphi_*$ and $(\psi\circ\varphi)^*=\varphi^*\circ\psi^*$.

Theorem A.17 *Let $(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then there is a canonical isomorphism (of functors)*

$$\mathrm{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \varphi_*\mathcal{F}).$$

Hence, the functors φ^* and φ_* are adjoint.

Proof. Given a homomorphism $\xi\colon \varphi^*\mathcal{G} \to \mathcal{F}$ as $\mathcal{O}_X$-modules, the canonical homomorphism $|\varphi|^*\mathcal{G} \longrightarrow \varphi^*\mathcal{G}$ yields by composition a homomorphism of $|\varphi|^*\mathcal{O}_Y$-modules $\xi'\colon |\varphi|^*\mathcal{G} \to \mathcal{F}$. Theorem A.8 show that we have a map $\mathcal{G} \longrightarrow |\varphi|_*\mathcal{F}$ of abelian sheaves, say ξ'', and one checks that ξ'' is really a map of $\mathcal{O}_Y$-modules $\mathcal{G} \longrightarrow \varphi_*\mathcal{F}$.

Conversely, from a map $\eta\colon \mathcal{G} \to \varphi_*\mathcal{F}$ of $\mathcal{O}_Y$-modules, we deduce a map $\widetilde{\eta}\colon |\varphi|^*\mathcal{G} \to \mathcal{F}$ of $|\varphi|^*\mathcal{O}_Y$-modules. Upon tensoring with $\mathcal{O}_X$, we obtain a map $\widetilde{\eta}^*\colon \varphi^*\mathcal{G} \to \mathcal{F}$ of $\mathcal{O}_X$-modules. It is easy to see that $\xi \mapsto \xi''$ and $\eta \mapsto \widetilde{\eta}^*$ are inverse isomorphisms, and the proof is complete. $\square$

Remark: Theorem A.17 has the same corollary as Theorem A.8.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. *To give a homomorphism $\mathcal{O}_X \longrightarrow \mathcal{F}$ is the same as giving a global section of* $\mathcal{F}$, i.e., an element of $\Gamma(X, \mathcal{F})$. So

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \xrightarrow{\sim} \Gamma(X, \mathcal{F}).$$

This follows from the fact that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}$, but it is worthwhile to recall the proof. If ξ is an $\mathcal{O}_X$-homomorphism of $\mathcal{O}_X$ to $\mathcal{F}$, then $\xi(X)$ maps $\Gamma(X, \mathcal{O}_X)$ to $\Gamma(X, \mathcal{F})$. The image of the unit section, 1, in $\Gamma(X, \mathcal{O}_X)$ is a section $s = \xi(X)(1)$ in $\Gamma(X, \mathcal{F})$. If U is open in X and t is a section of $\mathcal{O}_X$ over U, then $t = t \cdot 1$, and, as ξ is an $\mathcal{O}_X$-homomorphism, we obtain

$$\xi(U)(t) = t \cdot \xi(U)(1) = t \cdot \rho_U^X(s).$$

This shows that $\xi \mapsto \xi(X)(1)$ is an injection, and a trivial argument shows it is bijective as well.

If we apply the italicized statement to the sheaf $\mathcal{O}_X^{(I)}$, we obtain

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F}) \xrightarrow{\sim} \prod_I \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \xrightarrow{\sim} \Gamma(X, \mathcal{F})^I.$$

Hence, *there is a one to one correspondence between $\mathcal{O}_X$-homomorphisms $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$ and families of global sections $\{s_i\}_{i\in I}$ of $\mathcal{F}$.*

We say that $\mathcal{F}$ *is generated by the sections* $\{s_i\}_{i\in I}$ if and only if the corresponding map $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$ is surjective. This amounts to saying that for each $x \in X$, the stalk $\mathcal{F}_x$ is generated (as $\mathcal{O}_{X,x}$-module) by the elements $\{(s_i)_x \mid i \in I\}$. We say that $\mathcal{F}$ *is generated by its (global) sections* if and only if there is some subset I of $\Gamma(X, \mathcal{F})$ such that $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$ is surjective.

Remark: There exist $\mathcal{O}_X$-modules which are not generated by their sections, *even locally*. For example, take for X the space of real numbers and for $\mathcal{O}_X$ the constant sheaf $\mathcal{Z}$. Let U be the open subset of X consisting of the complement of the origin $x = 0$, and let $\mathcal{F} = \mathcal{Z}_U$ (notation as in Section A.5). Let V be any connected neighborhood of the origin, then $\mathcal{F} \restriction V$ has only the zero section over V, hence is not generated by its sections over V—no matter how small V.

If $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module generated by its section, then the map $\mathrm{can}_X\, \varphi\colon \varphi^*\varphi_*\mathcal{F} \to \mathcal{F}$ is *surjective*. For if $\{s_i\}_{i\in I}$ is a generating family of sections for $\mathcal{F}$, then the elements $s_i \otimes 1$ are sections of $\varphi^*\varphi_*\mathcal{F}$ whose images under $\mathrm{can}_X\varphi$ are exactly the sections s_i, $i \in I$. The converse is not generally true, for if it were, every sheaf would be generated by its sections (take $\varphi = \mathrm{id}$), and this is false as we know.

In algebraic geometry and related topics such as complex analytic manifolds, the ringed spaces which arise are most often *local ringed spaces*. By this we mean that for each $x \in X$, the stalk, $\mathcal{O}_{X,x}$, of $\mathcal{O}_X$ at x is a local ring. In this case, we let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$, and let $\kappa(x)$ be the residue field at x, so $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be local ringed spaces. If $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then $\widetilde{\widetilde{\varphi}}$ maps $|\varphi|^*\mathcal{O}_Y$ to $\mathcal{O}_X$. In particular, for $x \in X$, $\widetilde{\widetilde{\varphi}}_x$ maps $\mathcal{O}_{Y,|\varphi|(x)}$ to $\mathcal{O}_{X,x}$. Now it may happen that $\widetilde{\widetilde{\varphi}}_x$ does not map $\mathfrak{m}_{|\varphi|(x)}$ into $\mathfrak{m}_x$; hence, does *not* induce a map $\kappa(|\varphi|(x))$ to $\kappa(x)$. This situtation vitiates most of the advantages inherent to the study of local ringed spaces (as opposed to ringed spaces), so it is to be avoided. The best way to do this, is to consider *only* those morphisms φ for which $\widetilde{\widetilde{\varphi}}_x$ maps $\mathfrak{m}_{|\varphi|(x)}$ into $\mathfrak{m}_x$ (hence, induces an *injection* $\kappa(|\varphi|(x)) \longrightarrow \kappa(x)$.) Such morphisms are called *local morphisms*. The collection of all local ringed spaces and local morphisms between them forms a category (a subcategory of the category of ringed spaces) denoted $\mathcal{LRS}$. *In the sequel, whenever we have a morphism of local ringed spaces we shall always assume that it is a local morphism.*

One final topic in the pot-pourri of results of this section concerns the *gluing of ringed spaces*. Suppose we are given a collection $(X_i, \mathcal{O}_{X_i})$ of ringed spaces. Assume that for each pair (i, j), we have open sets $U_{ij} \subseteq X_i$, $U_{ji} \subseteq X_j$, and an isomorphism $\varphi_{ji}\colon (U_{ij}, \mathcal{O}_{X_i} \restriction U_{ij}) \to (U_{ji}, \mathcal{O}_{X_j} \restriction U_{ji})$ of ringed spaces. We subject this data to the three conditions

(a) For all i, $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}$,

(b) For all triples (i, j, k), the map

$$\varphi'_{ji} = \varphi_{ji} \restriction U_{ij} \cap U_{ik}\colon (U_{ij} \cap U_{ik}, \mathcal{O}_{X_i} \restriction U_{ij} \cap U_{ik}) \longrightarrow (U_{jk} \cap U_{ji}, \mathcal{O}_{X_j} \restriction U_{jk} \cap U_{ji})$$

is an isomorphism, and

(c) For all triples (i, j, k), $\varphi'_{ik} = \varphi'_{ij} \circ \varphi'_{jk}$.

(Condition (c) is calleld the *gluing condition* for the morphisms φ_{ij}).

Then we claim that there exists a ringed space $(X, \mathcal{O}_X)$ and an open subspace X'_j of X such that, each ringed space $(X'_j, \mathcal{O}_X \restriction X'_j)$ is naturally isomorphic to $(X_j, \mathcal{O}_X)$. The ringed space $(X, \mathcal{O}_X)$ is said to be obtained from the collection $(X_i, \mathcal{O}_{X_i})$ by *gluing along the* U_{ij} *via the* φ_{ji}.

To obtain $(X, \mathcal{O}_X)$ we first construct X. Let Z be the disjoint union of the spaces X_i with the obvious topology. On Z we introduce a relation $\sim$ by: If $x \in X_i$, $y \in X_j$, then $x \sim y$ if and only if $x \in U_{ij}$, $y \in U_{ji}$, and $|\varphi|_{ji}(x) = y$. Our axioms imply that $\sim$ is an equivalence relation, and $Z/\sim$ with the quotient topology is the space X. The space X possesses open subspaces, X'_i, homeomorphic to X_i for all i.

For the structure sheaf $\mathcal{O}_X$, first note that our axioms show that the three sets

$$U_{ij} \cap U_{ik}, \quad U_{jk} \cap U_{ji}, \quad U_{ki} \cap U_{kj}$$

are all homeomorphic to $X'_i \cap X'_j \cap X'_k$ (under the homeomorphisms $X_i \longleftrightarrow X'_i$, etc.). Hence, we may transfer the structure of ringed spaces to each X'_i; call the transfered sheaf $\mathcal{O}_{X'_i}$. If $x \in X$, then for at least one i, $x \in X'_i$; and the stalk $\mathcal{O}_{X'_i,x}$ is independent of which i we choose. Since every small open neighborhood of x is in some X'_j, there is a unique way to make a sheaf on X whose stalks at x is $\mathcal{O}_{X'_i,x}$. This is the sheaf $\mathcal{O}_X$, and we have $\mathcal{O}_X \restriction X'_i = \mathcal{O}_{X'_i}$. It follows that $(X, \mathcal{O}_X)$ fulfills our claim.

A.7 Quasi-Coherent and Coherent Sheaves. Sheaves with Various Finiteness Properties

Let $(X, \mathcal{O}_X)$ be a ringed space. Not all $\mathcal{O}_X$-modules are equally important. Experience in analytic and algebraic geometry has shown that prime importance be accorded to the coherent and quasi-coherent $\mathcal{O}_X$-modules. Along with these two types there are important related classes, the sheaves of finite type and finite presentation.

Definition A.4 Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The sheaf $\mathcal{F}$ is of *finite type* over X if and only if for every $x \in X$, there is an open neighborhood U of x and an integer n (perhaps depending upon U), such that $\mathcal{F} \restriction U$ is a homomorphic image of $\mathcal{O}_X^n \restriction U$. The sheaf $\mathcal{F}$ is of *finite presentation* over X if and only if for every $x \in X$, there is an open neighborhood U of x and integers m, n (perhaps depending upon U), such that $\mathcal{F} \restriction U$ is the cokernel of a map $\mathcal{O}_X^m \restriction U \longrightarrow \mathcal{O}_X^n \restriction U$.

Observe that these properties are *local on* X. That is, finite type means that *locally* $\mathcal{F}$ is a homomorphic image of finitely many copies of $\mathcal{O}_X$, and similarly for finite presentation. To verify these conditions at x, one may work in any small open neighborhood of the point x.

Observe as well that: Any homomorphic image of an $\mathcal{O}_X$-module of finite type is of finite type, finite direct sums of $\mathcal{O}_X$-modules of finite type are of finite type, and if $\mathcal{F}'$, $\mathcal{F}''$ are $\mathcal{O}_X$-modules of finite type and

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type.

Problem A.16 If $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module for each $x \in X$. The converse is false.
Hint. Look at the example of the second remark in Section A.6.

If $\mathcal{F}$ is any $\mathcal{O}_X$-module, the *support* of $\mathcal{F}$ is the set

$$\mathrm{Supp}\,\mathcal{F} = \{x \in X \mid \mathcal{F}_x \neq (0)\}.$$

Proposition A.18 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type, and let $s_1, \ldots, s_n$ be sections of $\mathcal{F}$ over an open neighborhood, U, of a point $x \in X$ whose images in $\mathcal{F}_x$ generate $\mathcal{F}_x$. Then there exists an open neighborhood $V \subseteq U$ of x, such that $\mathcal{F} \restriction V$ is generated by $s_1 \restriction V_1, \ldots, s_n \restriction V$. In particular, the support of an $\mathcal{O}_X$-module of finite type is closed.*

Proof. Since $\mathcal{F}$ is of finite type, there exists a neighborhood of x, which we may suppose to be U, and a finite family of sections $\sigma_1, \ldots, \sigma_r$ which generate $\mathcal{F}$ over U. As $x \in U$, and as $(s_1)_x, \ldots, (s_n)_x$ generate $\mathcal{F}_x$, there exist sections ξ_{ij} of $\mathcal{O}_X$ over some open set $W \subseteq U$ such that at x,

$$(\sigma_j)_x = \sum_i (\xi_{ij})_x (s_i)_x.$$

This implies that on some neighborhood V of x, $V \subseteq W \subseteq U$, we have $\sigma_j = \sum_i \xi_{ij} s_i$. Thus, for any $y \in V$, $(\sigma_j)_y$ is a linear combination of the $(s_i)_y$; as the $(\sigma_j)_y$ generate $\mathcal{F}_y$, so do the $(s_i)_y$. If $x \notin \mathrm{Supp}\,\mathcal{F}$, then the zero section generates $\mathcal{F}_x$, so the zero section generates $\mathcal{F}$ in a neighborhood of x, i.e., for all y near x, $\mathcal{F}_y = 0$. $\square$

Suppose $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and that $\mathcal{G}$ is an $\mathcal{O}_Y$-module of finite type. Then $\varphi^*\mathcal{G}$ is an $\mathcal{O}_X$-module of finite type. To see this, let x be a chosen point of X and let V be an open neighborhood of $|\varphi|(x)$ in Y such that the sequence

$$\mathcal{O}_Y^r \restriction V \longrightarrow \mathcal{G} \restriction V \longrightarrow 0$$

is exact. Since $|\varphi|^*$ commutes with restriction and is exact, we obtain that

$$(|\varphi|^*\mathcal{O}_Y)^r \restriction |\varphi|^{-1}(V) \longrightarrow |\varphi|^*\mathcal{G} \restriction |\varphi|^{-1}(V) \longrightarrow 0$$

is exact. Now tensor the above with $\mathcal{O}_X$; we obtain that

$$\mathcal{O}_X^r \restriction |\varphi|^{-1}(V) \longrightarrow \varphi^*\mathcal{G} \restriction |\varphi|^{-1}(V) \longrightarrow 0$$

is exact, as required.

The direct image of an $\mathcal{O}_X$-module of finite type need *not* be of finite type as $\mathcal{O}_Y$-module.

Sheaves of finite presentation behave somewhat better than sheaves of finite type. Of course finite direct sums of $\mathcal{O}_X$-modules of finite presentation are of finite presentation. The same is true of the inverse image of an $\mathcal{O}_X$-module of finite presentation by an adaptation of the argument used above for sheaves of finite type.

Proposition A.19 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the canonical homomorphism*

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$$

is an isomorphism. If we assume only that $\mathcal{F}$ be of finite type, then the canonical homomorphism is injective.

Proof. If $\mathcal{F} = \mathcal{O}_X$ the result is trivial. Hence, it is also true for $\mathcal{F} = \mathcal{O}_X^n$, where n is a positive integer. Now assume only that $\mathcal{F}$ is of finite presentation. On an open neighborhood U of x, we have the exact sequence

$$\mathcal{O}_X^m \restriction U \longrightarrow \mathcal{O}_X^n \restriction U \longrightarrow \mathcal{F} \restriction U \longrightarrow 0.$$

Since we pass to the limit over smaller and smaller open V contained in U, we might as well assume $X = U$ and supress U in the argument. (The above argument will be condensed in the future to the catch-phrase: "The probelm is local on X, so we may assume that ...")

Now the stalk functor is exact, and the functors

$$\mathcal{H}om_{\mathcal{O}_X}(-,\mathcal{G}), \quad \mathrm{Hom}_{\mathcal{O}_{X,x}}(-,\mathcal{G}_x)$$

are left-exact. If we apply these functors in the indicated order to the exact sequence, we deduce the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))_x \arrow[r] \arrow[d, "\theta_1"] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n,\mathcal{G}))_x \arrow[r] \arrow[d, "\theta_2"] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^m,\mathcal{G}))_x \arrow[d, "\theta_3"] \\
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) \arrow[r] & \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^n,\mathcal{G}_x) \arrow[r] & \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^m,\mathcal{G}_x).
\end{tikzcd}$$

Since θ_2, θ_3 are isomorphisms (as we have observed), the five lemma shows that θ_1 is an isomorphism. Were $\mathcal{F}$ merely of finite type, our diagram would be missing the righthand column, and we could only conclude that θ_1 is injective. $\square$

Corollary A.20 *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules of finite presentation. Suppose there is a point x such that $\mathcal{F}_x$ is isomorphic to $\mathcal{G}_x$. Then there is a neighborhood, U, of x such that $\mathcal{F} \restriction U$ is isomorphic to $\mathcal{G} \restriction U$.*

Proof. Let $\xi\colon \mathcal{F}_x \to \mathcal{G}_x$ and $\eta\colon \mathcal{G}_x \to \mathcal{F}_x$ be inverse isomorphisms. Proposition A.19 shows that there is an open neighborhood, V, of x and homomorphisms φ, ψ of $\mathcal{F} \restriction V \longrightarrow \mathcal{G} \restriction V$, resp. $\mathcal{G} \restriction V \longrightarrow \mathcal{F} \restriction V$, which induce ξ, resp. η at x. The compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are endomorphisms of $\mathcal{G}$, resp. $\mathcal{F}$ which become the identity automorphism at x. It follows from this that there is a neighborhood $U \subseteq V$ of x on which $\varphi \circ \psi$ and $\psi \circ \varphi$ become the identity. $\square$

Definition A.5 An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* (QC) on X if and only if for every $x \in X$ there is an open neighborhood, U, of x such that $\mathcal{F} \restriction U$ is the cokernel of a map $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)}$, where the sets I and J may depend upon U.

The notion of quasi-coherence is a generalization of that of finite presentation to which it reduces when the sets I and J are guaranteed finite for a covering family of open sets of X. *Of course, arbitrary direct sums of QC modules are QC, and inverse images of QC modules are QC*.

In algebraic geometry, we shall see that the QC sheaves play a role which is the direct generalization of that played by modules over a ring. In this context, it is natural to ask for an analog in sheaf theory of the family *noetherian* modules over a ring. This leads to the most important class of $\mathcal{O}_X$-modules, the *coherent* $\mathcal{O}_X$-modules.

Definition A.6 An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if and only if it is of finite type and satisfies the following condition: For every open subset $U \subseteq X$, every integer $n > 0$, and every homomorphism $\xi\colon \mathcal{O}_X^n \restriction U \to \mathcal{F} \restriction U$, the sheaf $\mathrm{Ker}\,\xi$ is of finite type.

Observe that the following statements hold:

(1) If $\mathcal{F}$ is coherent, then $\mathcal{F}$ is of finite presentation and the support of $\mathcal{F}$ is closed.

(2) A finitely presented sheaf need not be coherent because $\mathcal{O}_X$ need not be coherent.

(3) If $\mathcal{F}$ is a coherent sheaf and $\mathcal{F}'$ is an $\mathcal{O}_X$-submodule then $\mathcal{F}'$ is coherent **provided** $\mathcal{F}'$ is of finite type.

(4) The inverse image of a coherent $\mathcal{O}_Y$-module need *not* be a coherent $\mathcal{O}_X$-module.

Theorem A.21 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_X$-modules. If any two of the three modules are coherent, so is the third.

Proof. (a) Suppose that $\mathcal{F}$, $\mathcal{F}''$ are coherent. To show that $\mathcal{F}'$ is coherent, we need show only that $\mathcal{F}'$ is of finite type. Given $x \in X$, since $\mathcal{F}$ is of finite type, there is a neighborhood U of x, an integer $n > 0$, and a surjection

$$\xi \colon \mathcal{O}_X^n \restriction U \longrightarrow \mathcal{F} \restriction U \longrightarrow 0.$$

Since the problem is local on X, we may assume $U = X$; hence, we may suppress U. Consider the following diagram:

$$\begin{tikzcd}
 & & \operatorname{Ker} \xi \arrow[r, "\eta"] \arrow[d] & \operatorname{Ker}(\theta \circ \xi) \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{O}_X^n \arrow[r, "\mathrm{id}"] \arrow[d, "\xi"] & \mathcal{O}_X^n \arrow[r] \arrow[d, "\theta\circ\xi"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r, "\theta"'] & \mathcal{F}'' \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{F}''$ is coherent, $\operatorname{Ker}(\theta \circ \xi)$ is of finite type. The snake lemma yields the exact sequence.

$$\operatorname{Ker}(\theta \circ \xi) \longrightarrow \mathcal{F}' \longrightarrow 0,$$

so $\mathcal{F}'$ is of finite type, as required.

(b) Suppose that $\mathcal{F}'$, $\mathcal{F}$ are coherent. Clearly $\mathcal{F}''$ is of finite type. Let U be open in X, $\xi \colon \mathcal{O}_X^n \restriction U \to \mathcal{F}'' \restriction U$, and let K be the kernel of ξ over U. Our problem is local on X, so given $x \in X$ there is a neighborhood V of x and a *surjection* $\eta \colon \mathcal{O}_X^m \restriction V \to \mathcal{F}' \restriction V$. By restricting attention to V, we may assume that $V = X$. Because $\mathcal{O}_X^n$ is free, we may lift ξ to a map of $\mathcal{O}_X^n$ to $\mathcal{F}$; and hence, we obtain the commutative diagram

$$\begin{tikzcd}
 & & \operatorname{Ker} \theta \arrow[r] \arrow[d] & K \arrow[d] & \\
 & \mathcal{O}_X^m \arrow[r] \arrow[d, "\eta"] & \mathcal{O}_X^{n+m} \arrow[r] \arrow[d, "\theta"] & \mathcal{O}_X^n \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0 \\
 & 0. & & &
\end{tikzcd}$$

The snake lemma yields the exact sequence

$$\operatorname{Ker} \theta \longrightarrow K \longrightarrow 0,$$

and, as $\operatorname{Ker} \theta$ is of finite type (because $\mathcal{F}$ is coherent), so is K.

(c) Finally, suppose $\mathcal{F}'$, $\mathcal{F}''$ are coherent. Certainly $\mathcal{F}$ is of finite type. Let U be open in X, $\xi\colon \mathcal{O}_X^n \restriction U \to \mathcal{F} \restriction U$, and let K be the kernel of ξ. As usual, we may assume $U = X$. Let τ be the map $\mathcal{F} \longrightarrow \mathcal{F}''$, and let J be the kernel of the map $\tau \circ \xi\colon \mathcal{O}_X^n \longrightarrow \mathcal{F}''$. Since $\mathcal{F}''$ is coherent, J is of finite type; moreover, by construction the map $J \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{F}$ factors through $\mathcal{F}'$. Given $x \in X$, there is a small neighborhood V of x and a surjection $\mathcal{O}_X^p \restriction V \longrightarrow J \restriction V \longrightarrow 0$. Once again we may assume $V = X$. Let u be the composition $\mathcal{O}_X^p \longrightarrow J \longrightarrow \mathcal{O}_X^n$, then the map ξ induces a map $\overline{\xi}$ of Coker u into $\mathcal{F}''$. Moreover, by construction the mapping $\overline{\xi}$ is an *injection*. We obtain the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ker}\ \eta \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_X^p \arrow[r, "u"'] \arrow[d, "\eta"] & \mathcal{O}_X^n \arrow[r] \arrow[d, "\xi"] & \mathrm{Coker}\ u \arrow[r] \arrow[d, "\overline{\xi}"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0,
\end{tikzcd}$$

and the snake lemma yields the exact sequence

$$\mathrm{Ker}\ \eta \longrightarrow K \longrightarrow 0.$$

Since $\mathcal{F}'$ is coherent, Ker η is of finite type, and we are done. $\square$

Corollary A.22 *Any finite direct sum of coherent $\mathcal{O}_X$-modules is a coherent $\mathcal{O}_X$-module.*

Corollary A.23 *If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules and $\xi\colon \mathcal{F} \to \mathcal{G}$ is a homomorphism, then* Ker ξ, Coker ξ, *and* Im ξ *are coherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-submodules of a coherent $\mathcal{O}_X$-module $\mathcal{H}$, then $\mathcal{F} + \mathcal{G}$ and $\mathcal{F} \cap \mathcal{G}$ are coherent.*

Proof. The image of ξ is a submodule of finite type of the coherent $\mathcal{O}_X$-module $\mathcal{G}$, so it is coherent. Theorem A.21 applied to the exact sequences

$$0 \longrightarrow \mathrm{Ker}\ \xi \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im}\ \xi \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Im}\ \xi \longrightarrow \mathcal{G} \longrightarrow \mathrm{Coker}\ \xi \longrightarrow 0$$

shows that Ker ξ and Coker ξ are coherent. The sheaf $\mathcal{F} + \mathcal{G}$ is of finite type, and is a submodule of $\mathcal{H}$; hence, it is coherent. Since the sequence

$$0 \longrightarrow \mathcal{F} \cap \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}/\mathcal{G}$$

is exact, Theorem A.21 shows that $\mathcal{F} \cap \mathcal{G}$ is coherent. (Use the exact sequence $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{G}$ to show that $\mathcal{H}/\mathcal{G}$ is coherent.) $\square$

Proposition A.24 *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. Since $\mathcal{F}$ is coherent it is of finite presentation, and since the question is local on X, we may assume that $\mathcal{F}$ has a global finite presentation. Thus,

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{F} \longrightarrow 0$$

is exact. Tensor this with $\mathcal{G}$ over $\mathcal{O}_X$; we obtain

$$\mathcal{G}^p \longrightarrow \mathcal{G}^q \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0.$$

Corollary A.23 shows that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.

The same argument repeated with $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ instead of $- \otimes_{\mathcal{O}_X} \mathcal{G}$ yields the exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}^q \longrightarrow \mathcal{G}^p,$$

and another application of Corollary A.23 completes the proof. $\square$

Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. We shall say that $\mathcal{A}$ *is coherent* if and only if $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X$ *is coherent (as sheaf of rings) if and only if for every open set* U *and every homomorphism* $\xi\colon \mathcal{O}_X^n \restriction U \to \mathcal{O}_X \restriction U$*, the kernel of* ξ *is of finite type as* $\mathcal{O}_X$*-module*. (Clearly, $\mathcal{O}_X$ is of finite type.) This statement says that the $\mathcal{O}_X$-module of relations among any *finite* family of sections of $\mathcal{O}_X$ (over U) is finitely generated.

Remark: The last statement contains the historical origins of the notion of coherence. M. Oka proved that the sheaf of germs of holomorphic functions on a complex analytic manifold is coherent in the above sense (of relations); this was the starting point of the investigations of Serre and Cartan on complex anayltic manifolds. For more details see the historical notes at the end of this chapter.

Proposition A.25 *Suppose that* $\mathcal{O}_X$ *is coherent. Then an* $\mathcal{O}_X$*-module* $\mathcal{F}$ *is coherent if and only if it is of finite presentation.*

Proof. If $\mathcal{F}$ is coherent, it is of finite presentation. Suppose $\mathcal{F}$ is of finite presentation, then locally on X the sequence

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{F} \longrightarrow 0$$

is exact. By Corollary A.23, $\mathcal{F}$ is coherent. $\square$

Given a ringed space $(X, \mathcal{O}_X)$, for any open subset U of X, the sheaf $\mathcal{O}_X \restriction U$ is also denoted $\mathcal{O}_U$.

Proposition A.26 *Let* $\mathcal{O}_X$ *be coherent, let* $\mathcal{F}$ *be a coherent* $\mathcal{O}_X$*-module, and let* M *be a submodule of finite type of* $\mathcal{F}_x$ *for some point* $x \in X$*. Then there exists an open neighborhood* U *of* x *and a coherent* $\mathcal{O}_U$*-module* $\mathcal{G}$ *of* $\mathcal{F} \restriction U$ *such that* $\mathcal{G}_x \xrightarrow{\sim} M$*.*

Proof. Since $\mathcal{F}$ is coherent it is of finite type; hence at x, there is an integer $q > 0$ such that

$$\mathcal{O}_{X,x}^q \longrightarrow \mathcal{F}_x \longrightarrow 0$$

is exact. Now we know that M is of finite type, so that there is an integer $v > 0$ and a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ \mathcal{O}_{X,x}^v \arrow[r] \arrow[d, dotted, "j"'] & M \arrow[r] \arrow[d, dotted, "i"] & 0 \\ \mathcal{O}_{X,x}^q \arrow[r] & \mathcal{F}_x \arrow[r] & 0 \end{tikzcd}$$

where the dotted map j exists because $\mathcal{O}_{X,x}^v$ is a free module. Since $\mathcal{O}_X$ is coherent, so is $\mathcal{O}_X^v$; hence $\mathcal{O}_X^v$ is of finite presentation. It follows from Proposition A.19, that j is induced by a homomorphism $\lambda\colon \mathcal{O}_X^v \restriction U \to \mathcal{O}_X^q \restriction U$ for some small neighborhood U of x. We may choose U small enough so that $\mathcal{O}_X^q \restriction U \longrightarrow \mathcal{F} \restriction U \longrightarrow 0$ is exact. Then we obtain the diagram

$$\mathcal{O}_X^v \restriction U \xrightarrow{\lambda} \mathcal{O}_X^q \restriction U \xrightarrow{\pi} \mathcal{F} \restriction U \longrightarrow 0.$$

Let K be the kernel of $\pi \circ \lambda$ and θ be the injection $K \longrightarrow \mathcal{O}_X^v \restriction U$. Set $\mathcal{G}$ equal to Coker θ. Then $\mathcal{G}_x \widetilde{\longrightarrow} M$ by construction, $\mathcal{G}$ is of finite type, and being an $\mathcal{O}_U$-submodule of $\mathcal{F} \restriction U$, $\mathcal{G}$ is coherent. □

Corollary A.27 *In order that $\mathcal{O}_X$ be coherent, it is necessary that the intersection of two finitely generated ideals of $\mathcal{O}_{X,x}$ be finitely generated for every $x \in X$.*

Proposition A.28 *Suppose $\mathcal{O}_X$ is coherent and M is an arbitrary $\mathcal{O}_{X,x}$-module of finite presentation. Then there exists an open neighborhood U of x and a coherent $\mathcal{O}_U$-module, $\mathcal{F}$, such that $\mathcal{F}_x \widetilde{\longrightarrow} M$.*

Proof. The sequence

$$\mathcal{O}_{X,x}^p \xrightarrow{\xi} \mathcal{O}_{X,x}^q \longrightarrow M \longrightarrow 0$$

is exact for some integers p and q. Since $\mathcal{O}_X$ is coherent, the module $\mathcal{O}_X^p$ is of finite presentation, so the map ξ is induced by a homomorphism $u\colon \mathcal{O}_X^p \restriction U \to \mathcal{O}_X^q \restriction U$ for some open neighborhood U of x (by Proposition A.19). Let $\mathcal{F} = \text{Coker } u$, then $\mathcal{F}$ is coherent and $\mathcal{F}_x \widetilde{\longrightarrow} M$. □

Theorem A.29 *Let $\mathcal{O}_X$ be coherent and let $\mathcal{J}$ be a coherent sheaf of ideals. Then an $\mathcal{O}_X/\mathcal{J}$-module $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is coherent as an $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X/\mathcal{J}$ is coherent as a sheaf of rings.*

Proof. Clearly $\mathcal{F}$ is of finite type as $\mathcal{O}_X$-module if and only if $\mathcal{F}$ is of finite type as $\mathcal{O}_X/\mathcal{J}$-module. Suppose $\mathcal{F}$ is coherent as $\mathcal{O}_X$-module, and let $\xi\colon (\mathcal{O}_X/\mathcal{J})^n \restriction U \to \mathcal{F} \restriction U$ be a given homomorphism. Let K'' be the kernel of ξ and let K be the kernel of the composed map

$$\mathcal{O}_X^n \restriction U \longrightarrow (\mathcal{O}_X/\mathcal{J})^n \restriction U \longrightarrow \mathcal{F} \restriction U.$$

Then the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}^n \arrow[r] \arrow[d, "\mathrm{id}"] & K \arrow[r] \arrow[d] & K'' \arrow[d] & \\
0 \arrow[r] & \mathcal{J}^n \arrow[r] \arrow[d] & \mathcal{O}_X^n \arrow[r] \arrow[d] & (\mathcal{O}_X/\mathcal{J})^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathcal{F} \arrow[r, "\mathrm{id}"] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

(where we have assumed $U = X$) and the snake lemma yields the exact sequence

$$0 \longrightarrow \mathcal{J}^n \longrightarrow K \longrightarrow K'' \longrightarrow 0.$$

This shows that K'' is of finite type, i.e., $\mathcal{F}$ is $\mathcal{O}_X/\mathcal{J}$-coherent. In particular, since $\mathcal{J}$ is coherent, $\mathcal{O}_X/\mathcal{J}$ is $\mathcal{O}_X$-coherent; hence $\mathcal{O}_X/\mathcal{J}$ is coherent as $\mathcal{O}_X/\mathcal{J}$-module, i.e., as a sheaf of rings.

Conversely, if $\mathcal{F}$ is $\mathcal{O}_X/\mathcal{J}$-coherent it is of finite presentation—and we may assume $\mathcal{F}$ possesses a global finite presentation. Then the exact sequence

$$(\mathcal{O}_X/\mathcal{J})^p \longrightarrow (\mathcal{O}_X/\mathcal{J})^q \longrightarrow \mathcal{F} \longrightarrow 0,$$

the fact (proven above) that $\mathcal{O}_X/\mathcal{J}$ is coherent (over $\mathcal{O}_X$) and Corollary A.23, show that $\mathcal{F}$ is $\mathcal{O}_X$-coherent. $\square$

The inverse image of a coherent sheaf is, in general, *not* coherent. However, there is one important case in which it is coherent.

Proposition A.30 *Let $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, and assume $\mathcal{O}_X$ is a coherent sheaf of rings. Then the inverse image of any coherent $\mathcal{O}_Y$-module is a coherent $\mathcal{O}_X$-module.*

Proof. Any coherent $\mathcal{O}_Y$-module, $\mathcal{F}$, is of finite presentation. Since the problem is local on X, we may assume $\mathcal{F}$ possesses a global finite presentation:

$$\mathcal{O}_Y^p \longrightarrow \mathcal{O}_Y^q \longrightarrow \mathcal{F} \longrightarrow 0.$$

Apply the right-exact functor φ^* to this sequence; we deduce

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \varphi^*\mathcal{F} \longrightarrow 0$$

is exact. As $\mathcal{O}_X$ is coherent, we are done. $\square$

Remarks:

(1) If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules which are locally isomorphic (that is, for every $x \in X$, there is an open neighborhood U of x and an isomorphism $\varphi\colon \mathcal{F} \restriction U \to \mathcal{G} \restriction U$), and if one is QC, coherent, of finite type, or of finite presentation, so is the other. As the reader has observed, this remark has been used extensively in the above proofs.

(2) The finiteness notions of this section do not behave well under direct image. In fact, we shall give sufficient conditions for the direct image of a QC sheaf to be QC in Chapter 3, only for the category of "preschemes." The corresponding problem for coherent sheaves is very much deeper, and we shall give the solution (due to Serre and Grothendieck [??]) in Chapter 7.

Problem A.17 Let $\mathcal{O}_X$ be coherent. Prove that an $\mathcal{O}_X$-submodule of $\mathcal{O}_X^n$ is coherent if and only if it is of finite type. Deduce that the sheaf of relations among a finite number of sections of a coherent sheaf is a coherent sheaf.

Problem A.18 Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{J}$ a sheaf of ideals of $\mathcal{O}_X$. Define a canonical map $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$. The image of this map will be called $\mathcal{J}\mathcal{F}$. Now assume $\mathcal{F}$ is coherent and $\mathcal{J}$ is coherent. Show that $\mathcal{J}\mathcal{F}$ is coherent.

Problem A.19 There is a canonical homomorphism

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

for any $\mathcal{O}_X$-module $\mathcal{F}$. It is defined by sending $s \in \mathcal{O}_X(U)$ to multiplication by s in $\mathrm{Hom}_{\mathcal{O}_X \restriction U}(\mathcal{F} \restriction U, \mathcal{F} \restriction U)$. By definition the *annihilator* of $\mathcal{F}$ is the kernel, $\mathcal{J}$, of the canonical homomorphism $\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. Suppose $\mathcal{O}_X$ is coherent and $\mathcal{F}$ is coherent as $\mathcal{O}_X$-module. Prove that the annihilator, $\mathcal{J}$, of $\mathcal{F}$, is a coherent $\mathcal{O}_X$-module. Prove also that $\mathcal{J}_x$ is the annihilator of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

Problem A.20 Let $(X, \mathcal{O}_X)$ be a ringed space, and assume X is compact. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules with $\mathcal{G}$ of finite type. If $\mathcal{F}$ is the direct limit of $\mathcal{O}_X$-modules $\mathcal{F}_\lambda$, and if there is a *surjection* $u\colon \mathcal{F} \to \mathcal{G}$, show that for some λ, the induced map $\mathcal{F}_\lambda \longrightarrow \mathcal{G}$ is surjective.

Problem A.21 There is one case for which direct images behave well with respect to the finiteness properties: finite type, QC, coherent. Let $(Y, \mathcal{O}_Y) \xrightarrow{\varphi} (X, \mathcal{O}_X)$ be a morphism of ringed spaces; assume that Y is a *closed* subspace of X and that $\mathcal{O}_X = \varphi_* \mathcal{O}_Y$. Show that a necessary and sufficient condition that an $\mathcal{O}_Y$-module, $\mathcal{F}$, be of finite type (resp. QC, coherent) is that the $\mathcal{O}_X$-module $\varphi_* \mathcal{F}$ be of finite type (resp., QC, coherent.).

A.8 Locally Free Sheaves

Definition A.7 Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if and only if for each $x \in X$, there is an open neighborhood U of x such that $\mathcal{F} \restriction U$ is isomorphic to $\mathcal{O}_X^{(I)} \restriction U$ for some index set I. (Of course, I may depend upon U). If for each U, I is a finite set, then $\mathcal{F}$ is *locally free of finite rank*. If for each U the finite sets I have the same number of elements, n, then $\mathcal{F}$ *is locally free of rank* n. A localy free sheaf of rank one is called an *invertible* sheaf.

The following properties of locally free sheaves are easily deduced from Definition A.7 and the material of Section A.7.

(1) If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of finite rank, then for all $x \in X$, $\mathcal{F}_x$ is a finitely generated free $\mathcal{O}_{X,x}$-module, say of rank $n(x)$. The function $x \mapsto n(x)$ is locally constant; hence *if X is connected, any locally free $\mathcal{O}_X$-module of finite rank is a localy free $\mathcal{O}_X$-module of rank n for a unique integer n.*

(2) Every locally free $\mathcal{O}_X$-module is QC. If $\mathcal{O}_X$ is coherent so is every locally free $\mathcal{O}_X$-module of finite rank.

(3) If $\mathcal{F}$ is locally free, the functor $\mathcal{G} \rightsquigarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ from $\mathcal{O}_X$-modules to $\mathcal{O}_X$-modules is exact.

Henceforth, *all locally free $\mathcal{O}_X$-modules will be of finite rank, except for explicit mention to the contrary.*

Proposition A.31 *Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then there is a canonical homomorphism (of functors)*

$$\mathcal{F}^D \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

(Recall, $\check{\mathcal{F}} = \mathcal{F}^D$, the dual of $\mathcal{F}$, is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$). When $\mathcal{F}$ is locally free, this homomorphism is an isomorphism.

Proof. Since $\mathcal{F}^D = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, we need to define a homomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Let U be open in X, let $u \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)) = \mathrm{Hom}_{\mathcal{O}_X \restriction U}(\mathcal{F} \restriction U, \mathcal{O}_X \restriction U)$, and let $g \in \mathcal{G}(U)$. Assign to the pair (u, g) that homomorphism of $\mathcal{F} \restriction U$ into $\mathcal{G} \restriction U$ which at x maps $f_x \in \mathcal{F}_x$ to $u_x(f_x)g_x$ (where g_x is the image of g in $\mathcal{G}_x$ and u_x is the image of u in $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x})$). To check that this mapping is an isomorphism when $\mathcal{F}$ is locally free, note that as the problem is local on X, we may assume $\mathcal{F} = \mathcal{O}_X^n$. Since the map is functorial, we may even assume that $n = 1$. In this case, the lefthand side is canonically isomorphic to $\mathcal{G}$, as is the righthand side; and out map is the identity. $\square$

Corollary A.32 *If $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module so is its dual $\mathcal{F}^D$. Moreover, there is a canonical isomorphism*

$$\mathcal{F}^D \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{O}_X;$$

and $\mathcal{F}^{DD}$ is isomorphic to $\mathcal{F}$.

Proof. The question of invertibility being local, we may suppose that $\mathcal{F} = \mathcal{O}_X$. Then $\mathcal{F}^D$ is also $\mathcal{O}_X$; hence $\mathcal{F}^D$ is invertible. By Proposition A.31, we have a canonical isomorphism

$$\mathcal{F}^D \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F});$$

so we must prove that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ is isomorphic to $\mathcal{O}_X$. (It is certainly *locally* isomorphic to $\mathcal{O}_X$). However, we have define (Problem 3, Section A.7) a canonical homomorphism $\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ for *any* $\mathcal{O}_X$-module $\mathcal{F}$. Once a homomorphism is globally defined, to check that it is an isomorphism is a local problem. Hence, we may assume that $\mathcal{F} = \mathcal{O}_X$; in this case, the result is trivial.

Now for any $\mathcal{O}_X$-module, $\mathcal{F}$, there is a canonical homomorphism $\mathcal{F} \rightsquigarrow \mathcal{F}^{DD}$. If $\mathcal{F}$ is invertible, so is $\mathcal{F}^{DD}$, therefore, upon assuming (as we may) that $\mathcal{F} = \mathcal{O}_X$, we deduce that $\mathcal{F}^D \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{O}_X$. $\square$

Proposition A.33 *Suppose that $(X, \mathcal{O}_X)$ is a local ringed space and that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type. If there exists an $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is isomorphic to $\mathcal{O}_X$, then for all $x \in X$, $\mathcal{F}_x$ is a module isomorphic to $\mathcal{O}_{X,x}$. If both $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, then $\mathcal{F}$ is invertible.*

Proof. For each $x \in X$, we have $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$; so (for given x) our hypotheses amount to: Let M and M' be A-modules, where A is a local ring and M is of finite type and assume $M \otimes_A M' \simeq A$. We must prove, $M \simeq A$. Now,

$$A/\mathfrak{m}_A \simeq A/\mathfrak{m}_A \otimes_A (M \otimes_A M') \simeq M/\mathfrak{m}_A M \otimes_{A/\mathfrak{m}_A} M'/\mathfrak{m}_A M',$$

where, as usual, $\mathfrak{m}_A$ is the maximal ideal of A. It follows from this that both $M/\mathfrak{m}_A M$ and $M'/\mathfrak{m}_A M'$ are one dimensional vector spaces over $A/\mathfrak{m}_A$. Thus, $M = A\xi + \mathfrak{m}M$, for some element ξ of M not in $\mathfrak{m}_A M$. Nakayama's lemma implies that $M = A\xi$. Moreover, the annihilator of ξ in A will also annihilate $M \otimes_A M' \simeq A$; hence, it is zero. This proves $a \mapsto \xi a$ is an isomorphism of A onto M, as required. When $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, they are of finite presentation, so Proposition A.19 shows that $\mathcal{F}$ is invertible. $\square$

Remark: If $\mathcal{F}$ and $\mathcal{F}'$ are invertible sheaves, so is their tensor product (the question is local, so we may assume $\mathcal{F} = \mathcal{O}_X$ and it becomes trivial). For $n \geq 1$, let $\mathcal{F}^{\otimes n}$ denote the tensor product of n copies of $\mathcal{F}$ and let $\mathcal{F}^{\otimes(-n)} = (\mathcal{F}^{-1})^{\otimes n}$.[2] If we set $\mathcal{F}^{\otimes 0} = \mathcal{O}_X$, then the foregoing results yield a canonical functorial isomorphism

$$\mathcal{F}^{\otimes m} \otimes \mathcal{F}^{\otimes n} \simeq \mathcal{F}^{\otimes(m+n)}$$

[2] $\mathcal{F}^{-1} = \mathcal{F}^D$ for invertible sheaves.

for *every* pair of integers (m, n). Were the class of invertible sheaves on X a set, it would form a group under the tensor product. (Actually, the equivalence classes of isomorphic invertible sheaves on X form a group, as shown in Section 4.4.) Continuing to talk as if invertible sheaves form a group, we see that $\mathcal{O}_X$ plays the role of the identity element, and that Proposition A.33 shows that the terminology "invertible" is well chosen. For that proposition shows that a "formally" invertible sheaf (in the sense of the "group under $\otimes$") over a local ringed space is really invertible, provided proper finiteness assumptions are made.

Proposition A.34 *Let $(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then the inverse image of a locally free (resp. invertible) $\mathcal{O}_Y$-module is a locally free (resp. invertible) $\mathcal{O}_X$-module. For locally free $\mathcal{O}_Y$-modules, inverse image commutes with taking the duals. Consequently, for every integer n, $\varphi^*(\mathcal{G}^{\otimes n})$ is canonically isomorphic to $(\varphi^*(\mathcal{G}))^{\otimes n}$ (for any sheaf $\mathcal{G}$ on Y).*

Proof. If two $\mathcal{O}_Y$-modules are locally isomorphic, so are their inverse images. Since inverse image commutes with direct sums, and since $\varphi^*(\mathcal{O}_Y) = \mathcal{O}_X$, the first statements are proved.

According to property (e) of Section A.6, there is a canonical homomorphism

$$\varphi^*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y)) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\varphi^*(\mathcal{G}), \mathcal{O}_X).$$

When $\mathcal{G}$ is locally free, this map is an isomorphism as one sees by checking locally—in which case one may assume $\mathcal{G} = \mathcal{O}_Y^n$. The last statement follows from the definition and from the fact that inverse image commutes with tensor product (property (c) of Section A.6). $\square$

If $(X, \mathcal{O}_X)$ is a local ringed space and U is an open set in X then for any $\mathcal{O}_X$-module $\mathcal{F}$ and any section s over U, we set $\overline{s}(x)$ equal to the residue class of $s(x)$ modulo $\mathfrak{m}_x\mathcal{F}_x$ (Here, $s(x)$ is the image of s in $\mathcal{F}_x$, for $x \in U$; earlier, we also used the notation s_x.) We shall say that $\overline{s}(x)$ is the *value* of s at x, and that s *vanishes at* x if $\overline{s}(x)$ is zero, i.e., $s(x) \in \mathfrak{m}_x\mathcal{F}_x$.

Let U_s be the set of all $x \in U$ such that $\overline{s}(x) \neq 0$.

Locally free sheaves have special properties on local ringed space. One such property is expressed in Proposition A.33; here is another.

Proposition A.35 *Let $(X, \mathcal{O}_X)$ be a local ringed space, let $\mathcal{F}$ be an invertible $\mathcal{O}_X$-module, and let f be a global section of $\mathcal{F}$ (i.e., a section over X). Then for each $x \in X$ the following three properties are equivalent:*

(a) f_x generates $\mathcal{F}_x$.

(b) $\overline{f}(x) \neq 0$ (i.e., $x \in X_f$).

(c) There is an open neighborhood U of x and a section g of $\mathcal{F}^{-1}$ over U, such that the canonical image of $f \otimes g$ in $\Gamma(U, \mathcal{O}_X)$ is the identity section.

Proof. The problem is local on X; so we may assume that $\mathcal{F} = \mathcal{O}_X$. In this case (a) and (b) are clearly equivalent, and (c) implies (b). If (b) holds, then f_x is invertible at x; that is, there is an element $\gamma \in \mathcal{O}_{X,x}$ with $f_x\gamma = 1$. But this implies that there is an open neighborhood U of x, and a section $g \in \Gamma(U, \mathcal{O}_X)$ such that $g_x = \gamma$, and $fg = 1$ on U. This is property (c) for the case $\mathcal{F} = \mathcal{O}_X$. $\square$

Corollary A.36 *Let $(X, \mathcal{O}_X)$ be a local ringed space, and let f be a section of $\mathcal{O}_X$ over an open set U. Then there exists a section g of $\mathcal{O}_X$ over U_f such that $gf = 1$ on U_f. That is, if a section f of $\mathcal{O}_X$ does not vanish on a set, it is invertible there.*

Proof. By property (c) of Proposition A.35, f is locally invertible on U_f. But all these inverses may be patched together to give a "global" inverse on U_f because our module is globally isomorphic to $\mathcal{O}_X$ (not just locally) and $\Gamma(U, \mathcal{O}_X)^*$ is a group for each open U. (Here, $\Gamma(U, \mathcal{O}_X)^*$ is the set of invertible sections on U.) $\square$

Corollary A.37 *Under the hypotheses of Proposition A.35, the set X_f is open in X.*

Problem A.22 Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules and suppose that $\mathcal{F}''$ is locally free. Then for each $x \in X$, there is an open set U containing x such that $\mathcal{F} \restriction U$ is isomorphic to the direct sum $\mathcal{F}' \restriction U \coprod \mathcal{F}'' \restriction U$.

Problem A.23 Let the hypotheses be as in Proposition A.35, and let $\mathcal{F}'$ be a second invertible sheaf on X with global section g. Prove that $X_f \cap X_g = X_{f \otimes g}$.

Problem A.24 Assume $(X, \mathcal{O}_X)$ is a local ringed space. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free $\mathcal{O}_X$-modules and let $\xi \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism. Then a necessary and sufficient condition that there exist an open neighborhood U of $x \in U$, such that $\xi \restriction U$ be injective and $\mathcal{G} \restriction U$ be the direct sum of $\xi(\mathcal{F}) \restriction U$ and a locally free submodule $\mathcal{H}$ of $\mathcal{G} \restriction U$ is that $\xi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ induce an *injection* of vector spaces $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \longrightarrow \mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x$.

Appendix B

Cohomology

B.1 Flasques and Injective Sheaves, Resolutions

We shall lay the foundations, here, for the cohomology theories of the next two sections. In what follows, $(X, \mathcal{O}_X)$ will be a ringed space and $\mathcal{S}(X, \mathcal{O}_X)$ will denote the abelian category of $\mathcal{O}_X$-modules. Of course, $\mathcal{S}(X)$ is an abbreviations for $\mathcal{S}(X, \mathcal{Z})$.

The notion of injective resolution is basic to the foundations of cohomology theory. Let $\mathbf{C}$ be an abelian category (the reader should think of $\mathbf{C}$ as $\mathcal{S}(X, \mathcal{O}_X)$). Recall that an object, I, in $\mathbf{C}$ is an *injective* if for every pair of objects, A and B, in $\mathbf{C}$, every morphism, $f\colon A \to I$, and every monomorphism, $h\colon A \to B$, there is some (not necessarily unique) morphism, $\widehat{f}\colon B \to I$, so that

$$f = \widehat{f} \circ h,$$

as in the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[dl, dotted, "\widehat{f}"] \\ & I & \end{tikzcd}$$

It is well-known that I is an injective iff $\mathrm{Hom}\,(-, I)$ is (right) exact.

Definition B.1 Let F be an object of an abelian category, $\mathbf{C}$. An *acyclic resolution of F* is an exact sequence

$$0 \longrightarrow F \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \cdots$$

in $\mathbf{C}$. An *injective resolution of F* is an acyclic resolution for which all the Q_i (with $i \geq 0$) are injective objects of $\mathbf{C}$.

We shall say that $\mathbf{C}$ *has enough injectives* whenever each object, F, of $\mathbf{C}$ is isomorphic to a subobject of an injective (object) of $\mathbf{C}$; that is, whenever we are given F, there should be an injective, Q, and an exact sequence

$$0 \longrightarrow F \longrightarrow Q, \qquad \text{in } \mathbf{C}.$$

The standard sufficient condition for the existence of injective resolutions in C is the following:

Proposition B.1 *Let* $\mathbf{C}$ *be an abelian category with enough injectives. Then every object,* F*, of* $\mathbf{C}$ *has an injective resolution.*

Proof. By hypothesis there is an exact sequence

$$0 \longrightarrow F \xrightarrow{i} Q_0$$

with Q_0 injective. If F_1 is the cokernel of i, then F_1 may be embedded in an injective object, Q_1,

$$0 \longrightarrow F_1 \xrightarrow{i_1} Q_1.$$

Let F_2 be the cokernel of i_1, and repeat the argument. An obvious induction yields the exact sequence

$$0 \longrightarrow F_j \xrightarrow{i_j} Q_j \xrightarrow{\xi_j} F_{j+1} \longrightarrow 0, \qquad j = 0, 1, \cdots$$

with Q_j injective for all $j \geq 0$ and with $F_0 = F$ and $i_0 = i$. If we set

$$u_j = i_{j+1} \circ \xi_j \colon Q_j \longrightarrow Q_{j+1},$$

then the sequence

$$0 \longrightarrow F \longrightarrow Q_0 \xrightarrow{u_0} Q_1 \xrightarrow{u_1} Q_2 \xrightarrow{u_2} Q_3 \longrightarrow \cdots$$

is the desired injective resolution of F. $\square$

Remark: Suppose X is a class of objects of $\mathbf{C}$. We may speak of X*-resolutions* (acyclic resolutions in which each Q_i is an object of X), and may also ask if $\mathbf{C}$ possesses enough X-objects. The same argument as given in Proposition B.1 show that if $\mathbf{C}$ possesses enough X-objects, then every object of $\mathbf{C}$ has an X-resolution.

Now, injective resolutions would be useless were it not for the fact that they possess a "quasi-uniqueness" property. To explain this, we need the notion of homotopy between maps of resolutions.

Definition B.2 Let F and F' be objects of C, and let

$$0 \longrightarrow F \longrightarrow Q_0 \xrightarrow{u_0} Q_1 \longrightarrow \cdots \tag{$*$}$$

and

$$0 \longrightarrow F' \longrightarrow Q_0' \xrightarrow{u_0'} Q_1' \longrightarrow \cdots \tag{$*$}'$$

be acyclic resolutions of F and F'. Suppose that $\xi\colon F \to F'$ is a morphism. By a *morphism* (ξ_i) *from* $(*)$ *to* $(*)'$ *over* ξ, we mean a collection of morphisms

$$\xi_i\colon Q_i \to Q_i', \qquad i = 0, 1, \cdots$$

such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d, "\xi"] & Q_0 \arrow[r] \arrow[d, "\xi_0"] & Q_1 \arrow[r] \arrow[d, "\xi_1"] & \cdots \\ 0 \arrow[r] & F' \arrow[r] & Q_0' \arrow[r] & Q_1' \arrow[r] & \cdots \end{tikzcd}$$

is commutative. A *homotopy*, s, between two morphisms (ξ_i) and (η_i) of $(*)$ to $(*)'$ (over ξ), denoted $(\xi_i) \sim (\eta_i)$, is a collection of morphisms

$$s_i\colon Q_i \to Q_{i-1}' \qquad i = 1, 2, \cdots$$

such that for every $i \geq 1$,

$$\eta_i - \xi_i = u_{i-1}' \circ s_i + s_{i+1} \circ u_i,$$

while $\eta_0 - \xi_0 = s_1 \circ u_0$, as illustrated in the diagram below:

$$\begin{tikzcd} \cdots \arrow[r] & Q_{i-1} \arrow[r, "u_{i-1}"] \arrow[d, "\rho_{i-1}"'] & Q_i \arrow[r, "u_i"] \arrow[dl, "s_i"'] \arrow[d, "\rho_i"'] & Q_{i+1} \arrow[r] \arrow[dl, "s_{i+1}"'] \arrow[d, "\rho_{i+1}"] & \cdots \\ \cdots \arrow[r] & Q_{i-1}' \arrow[r, "u_{i-1}'"'] & Q_i' \arrow[r, "u_i'"'] & Q_{i+1}' \arrow[r] & \cdots, \end{tikzcd}$$

where $\rho_i = \eta_i - \xi_i$. If a homotopy between (ξ_i) and (η_i) exists, we say they are *homotopic*.

Here is the"quasi-uniqueness" of injective resolutions.

Theorem B.2 *Let F and F' be objects of C, let $(*)$ be an acyclic resolution of F and assume $(*)'$ is an injective resolution of F'. If $\xi\colon F \to F'$ is a morphism then there exists a morphism (ξ_i) from $(*)$ to $(*)'$ over ξ. Any two morphisms over ξ are homotopic. Consequently, any two injective resolutions of F have the same homotopy type; that is, there are morphisms (ξ_i) and (η_i) from one to the other and back over the identity whose compositions are homotopic to the identity.*

Proof. We shall construct the morphism (ξ_i) by induction on i. To begin with, the morphisms $\xi\colon F \to F'$ and $F' \to Q_0'$ yield a morphism $F \to Q_0'$. Since Q_0' is injective, and $(*)$ is acyclic, the diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & Q \arrow[dl, dotted, "\xi_0"] \\ & Q_0' & \end{tikzcd}$$

may be completed by the addition of the morphism ξ_0, as shown. Now, assume by induction that the morphisms ξ_i have been constructed for $i < n$ in such a way that the required diagram is commutative, and consider the following diagram:

$$\begin{tikzcd} Q_{n-2} \arrow[r] & Q_{n-1} \arrow[r] \arrow[d, "u'_{n-1}\circ\xi_{n-1}"'] & Q_n \arrow[dl, dotted, "\xi_n"] \\ & Q'_n & \end{tikzcd}$$

Since Q'_n is injective, since $(*)$ is exact, and since the composition map

$$Q_{n-2} \longrightarrow Q_{n-1} \longrightarrow Q'_n$$

is zero, there is an extension of $u'_{n-1} \circ \xi_{n-1}$ to a morphism $\xi_n\colon Q_n \to Q'_n$. Existence of a morphism over ξ is thereby assured.

Suppose that (ξ_i) and (η_i) are morphisms over ξ. To construct a homotopy, we shall once again proceed by induction on i. The diagram

$$\begin{tikzcd} F \arrow[r] & Q_0 \arrow[r, "u_0"] \arrow[d, "\eta_0-\xi_0"'] & Q_1 \arrow[dl, dotted, "s_1"] \\ & Q'_0 & \end{tikzcd}$$

may be completed by the addition of the morphism s_1 as shown, because the upper sequence is exact, Q'_0 is injective, and $F \longrightarrow Q_0 \longrightarrow Q'_0$ is the zero map. Assume by induction that the morphisms s_i have been constructed for $i < n$ and that they satisfy the required conditions. In diagram

$$\begin{tikzcd} Q_{n-2} \arrow[r] & Q_{n-1} \arrow[r, "u_0"] \arrow[d, "\zeta"'] & Q_n \arrow[dl, dotted, "s_n"] \\ & Q'_{n-1} & \end{tikzcd}$$

where ζ is the morphism $\eta_{n-2} - \xi_{n-2} - u'_{n-2} \circ s_{n-1}$, the upper row is exact, the morphism $Q_{n-1} \longrightarrow Q_{n-1} \xrightarrow{\zeta} Q'_{n-1}$ is zero (as one checks), and Q'_{n-1} is injective. It follows that there exists a morphism $s_n\colon Q_n \to Q'_{n-1}$ rendering the above diagram commutative. The sequence (s_n) constructed in this manner is the desired homotopy.

If we are given two injective resolutions of F

$$0 \longrightarrow F \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots \tag{$**$}$$

and

$$0 \longrightarrow F' \longrightarrow Q'_0 \longrightarrow Q'_1 \longrightarrow \cdots \tag{$**$}'$$

then the identity map may be raised in both directions to morphisms of resolutions, say $(\xi_i)\colon (**) \to (**)'$ and $(\eta_i)\colon (**)' \to (**)$. The compositions $(\xi_i \circ \eta_i)$, $(\eta_i \circ \xi_i)$ must then be homotopic to the identity, as the identity morphism is a lifting of $F \xrightarrow{\mathrm{id}} F$ in each resolution $(**)$, $(**)'$. $\square$

Remarks:

(1) For the first two statements of Theorem B.2, $(*)'$ need not be an injective resolution of F', but merely an *injective complex* over F'. That is, each Q'_i should be injective but in place of exactness, we require only that $u'_n \circ u'_{n-1} = 0$ for every n.

(2) The reader should not despair over the lack of intuitive meaning in the notions of resolution and homotopy. We will see the intuitive content of these notions, as well as an honest uniqueness theorem (to be deduced from Theorem B.2) in the next section.

Proposition B.3 *Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence in the category* $\mathbf{C}$*, and let*

$$0 \longrightarrow F' \longrightarrow Q'_0 \longrightarrow Q'_1 \longrightarrow \cdots$$

and

$$0 \longrightarrow F'' \longrightarrow Q''_0 \longrightarrow Q''_1 \longrightarrow \cdots$$

be injective resolutions of F' and F''. Then there exists an injective resolution

$$0 \longrightarrow F \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots$$

of F, such that for each i, the sequence $0 \longrightarrow Q'_i \longrightarrow Q_i \longrightarrow Q''_i \longrightarrow 0$ is exact and the obvious diagrams commute.

Proof. The proof is entirely straightforward, and is an excellent exercise in the techniques of this section. It will be left to the reader. $\square$

We now come back to earth and consider the category $\mathcal{S}(X, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules.

Proposition B.4 *The category $\mathcal{S}(X, \mathcal{O}_X)$ possesses enough injectives. Consequently, every $\mathcal{O}_X$-module has an injective resolution of $\mathcal{O}_X$-modules.*

First Proof. The category $\mathcal{S}(X, \mathcal{O}_X)$ is abelian with good direct sums and generators. (The sheaves $\mathcal{O}_X \restriction U$ for all open U in X are generators as one checks.) It follows from [21], Theorem 1.10.1, that $\mathcal{S}(X, \mathcal{O}_X)$ possesses enough injectives. $\square$

Second Proof. Given any $\mathcal{O}_X$-module, $\mathcal{F}$, for each $x \in X$, choose an injective $\mathcal{O}_{X,x}$-module $\mathcal{Q}_x$ and a monomorphism $\mathcal{F}_x \longrightarrow \mathcal{Q}_x$. Consider the sheaf, $\mathcal{Q}$, whose sections over the open set U are given by $\prod_{x\in U} \mathcal{Q}_x$. Clearly, we have a monomorphism $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q}$, and it remains only to show that $\mathcal{Q}$ is an injective sheaf. This is the content of

Lemma B.5 *If for each $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{Q}_x$ is injective, then the sheaf*

$$U \rightsquigarrow \prod_{x \in U} \mathcal{Q}_x$$

is injective.

Proof. Let $\mathcal{Q}^x$ be the sheaf on X whose sections over U are given by

$$\Gamma(U, \mathcal{Q}^x) = \begin{cases} \mathcal{Q}_x & \text{if } x \in U \\ (0) & \text{if } x \notin U. \end{cases}$$

Then, $\mathcal{Q}$ is the product of the sheaves $\mathcal{Q}_x$ for all $x \in X$. However, to give a homomorphism of a sheaf $\mathcal{F}$ into one of the sheaves $\mathcal{Q}^x$ is the same as giving a homomorphism of the stalks: $\mathcal{F}_x \longrightarrow \mathcal{Q}_x$. It follows immediately from this that each sheaf $\mathcal{Q}^x$ is injective. The product of injectives being an injective, we conclude that $\mathcal{Q}$ is an injective sheaf. $\square$

Proposition B.6 *Let $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{Q}$ be an injective $\mathcal{O}_X$-module. If φ^* is an exact functor (for example, if $\mathcal{O}_X = \mathcal{O}_Y = \mathcal{Z}$) then $\varphi_*\mathcal{Q}$ is an injective $\mathcal{O}_Y$-module.*

Proof. If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}$ is an exact sequence of $\mathcal{O}_Y$-modules, we must prove that the sequence

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \varphi_*\mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}', \varphi_*\mathcal{Q}) \longrightarrow 0$$

is exact. By Theorem A.17,

$$\mathrm{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{G}, \mathcal{Q}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \varphi_*\mathcal{Q}).$$

Since φ^* is exact, and since $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module, the sequence

$$\mathrm{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{F}, \mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{F}', \mathcal{Q}) \longrightarrow 0$$

is exact. This completes the proof. $\square$

Remark: Necessary and sufficient conditions for the exactness of φ^* will be investigated in Chapter ??, when we discuss flatness and faithful flatness.

The analog of Proposition B.6 for inverse image is false in general. However, there is one case in which it is true.

Proposition B.7 *Let U be an open subset of X and let $\varphi\colon (U, \mathcal{O}_X \restriction U) \to (X, \mathcal{O}_X)$ be the canonical inclusion map. If $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module then $\varphi^*\mathcal{Q}$ is an injective $\mathcal{O}_U$-module.*

Proof. Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}$ be an exact sequence of $\mathcal{O}_U$-modules. Because U is open in X, it follows from the discussion in Section A.5 that there is an *isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \varphi^*\mathcal{Q}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\overline{\mathcal{G}}, \mathcal{Q})$$

for any $\mathcal{O}_U$-module $\mathcal{G}$. Since $0 \longrightarrow \overline{\mathcal{F}'} \longrightarrow \overline{\mathcal{F}}$ is exact, and since $\mathcal{Q}$ is $\mathcal{O}_X$-injective, the sequence

$$\mathrm{Hom}_{\mathcal{O}_X}(\overline{\mathcal{F}}, \mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\overline{\mathcal{F}'}, \mathcal{Q}) \longrightarrow 0$$

is exact. Thus, the sequence

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}, \varphi^*\mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}', \varphi^*\mathcal{Q}) \longrightarrow 0$$

is exact, as required. $\square$

Injective sheaves, while perfect for foundations, are too large to be handled in a computable way. We shall see that in order to compute cohomology groups of a sheaf, one is required in principle to find an injective resolution of this sheaf. This is practically impossible to do in an explicit manner. Consequently, we wish to find a more manageable class of sheaves in which to take resolutions, being assured that at the same time the cohomology groups which will arise from these new resolutions will agree with those coming from injective resolutions. A class of sheaves having just these properties has been introduced by R. Godement [18]. This is the class of *flasque*[1] sheaves.

Definition B.3 An $\mathcal{O}_X$-module, $\mathcal{F}$, is *flasque* if and only if, for every open subset U of X, the map

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$$

is surjective.

The requirement in Definition B.3 is that every section of $\mathcal{F}$ over an open subset of X be extendable to a global section of $\mathcal{F}$. This requirement does appear to be of local nature. Appearances are deceiving, however, as the following proposition shows.

Proposition B.8 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flasque, so is $\mathcal{F} \upharpoonright U$ for every open subset U of X. Conversely, if for every $x \in X$, there is a neighborhood, U, such that $\mathcal{F} \upharpoonright U$ is flasque, then $\mathcal{F}$ is flasque.*

Proof. The first statement is trivial, let us prove the converse. Given any open set V of X, let s be a section of $\mathcal{F}$ over V. Let T be the set of all pairs (U, σ), where U is an open in X containing V, and σ is an extension of s to U. Partially order T in the obvious way, and observe that T is inductive. Zorn's lemma provides us with a maximal extension of s to a section σ over an open set U_0. Were U_0 not X, there would exist an open set W in X not contained in U_0 such that $\mathcal{F} \upharpoonright W$ is *flasque*. Thus we could extend the section $\rho^{U_0}_{U_0 \cap W}(\sigma)$ to a section σ' of $\mathcal{F}$. Since σ and σ' agree on $U_0 \cap W$ by construction, their common extension to $U_0 \cup W$ extends s, a contradiction. $\square$

[1] A very loose translation of the French word *flasque* is "flabby" or "limp."

Proposition B.9 *Every $\mathcal{O}_X$-module may be embedded in a canonical functorial way into a flasque $\mathcal{O}_X$-module. Consequently, every $\mathcal{O}_X$-module has a canonical flasque resolution (i.e., a resolution by flasque $\mathcal{O}_X$-modules.)*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and define a presheaf $C^0(X, \mathcal{F})$ by

$$U \rightsquigarrow \prod_{x \in U} \mathcal{F}_x.$$

It is immediate that $C^0(X, \mathcal{F})$ is actually a sheaf and that we have an *injection* of $\mathcal{O}_X$-modules $\mathcal{F} \longrightarrow C^0(X, \mathcal{F})$. A section of $C^0(X, \mathcal{F})$ over any open set U is a collection (s_x) of elements indexed by U, each s_x lying over the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. Clearly, such a sheaf is *flasque*; hence, $\mathcal{S}(X, \mathcal{O}_X)$ possesses enough *flasque* sheaves.

If Z_1 is the cokernel of the canonical injection $\mathcal{F} \longrightarrow C^0(X, \mathcal{F})$, we define $C^1(X, \mathcal{F})$ to be the *flasque* sheaf $C^0(X, Z_1)$. In general, Z_n is the cokernel of the injection $Z_{n-1} \longrightarrow C^0(X, Z_{n-1})$, and $C^n(X, \mathcal{F})$ is the *flasque* sheaf $C^0(X, Z_n)$. Putting all this information together, we obtained the desired *flasque* resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(X, \mathcal{F}) \longrightarrow C^1(X, \mathcal{F}) \longrightarrow C^2(X, \mathcal{F}) \longrightarrow \cdots . \qquad \square$$

Remark: The resolution of $\mathcal{F}$ constructed in Proposition B.9 will be called the *canonical flasque resolution of* $\mathcal{F}$ or the *Godement resolution of* $\mathcal{F}$.

Here is the principal property of *flasque* sheaves.

Theorem B.10 *Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume $\mathcal{F}'$ is flasque. Then this sequence is exact as a sequence of presheaves. If both $\mathcal{F}'$ and $\mathcal{F}$ are flasque, so is $\mathcal{F}''$. Finally, any direct factor of a flasque sheaf is flasque.*

Proof. Given any open set, U, we must prove that

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

is exact. Of course, the sole problem is to prove that $\mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$ is surjective. By restricting attention to U, we may assume $U = X$; hence, we are going to prove that a global section of $\mathcal{F}''$ may be lifted to a global section of $\mathcal{F}$. Let s'' be a global section of $\mathcal{F}''$, then, locally, s'' may be lifted to sections of $\mathcal{F}$. Let T be the family of all pairs (U, σ) where U is an open in X, and σ is a section of $\mathcal{F}$ over U whose image, σ'', in $\mathcal{F}''(U)$ equal $\rho^X_{\mathcal{F}''(U)}(s'')$. Partially order T in the obvious way, and observe that T is inductive. Zorn's lemma provides us with a maximal lifting of s'' to a section $\sigma \in \mathcal{F}(U_0)$.

Were U_0 not X, there would exist $x \in X - U_0$, a neighborhood, V, of x, and a section τ of $\mathcal{F}$ over V which is a local lifting of $\rho^X_V(s'')$. The sections $\rho^{U_0}_{U_0 \cap V}(\sigma)$, $\rho^V_{U_0 \cap V}(\tau)$ have the same image in $\mathcal{F}''(U_0 \cap V)$. Consequently, there is a section t of $\mathcal{F}'(U_0 \cap V)$ such that

$$\rho^{U_0}_{U_0 \cap V}(\sigma) = \rho^V_{U_0 \cap V}(\tau) + t.$$

Since $\mathcal{F}'$ is *flasque*, the section t is the restriction of a section $t' \in \mathcal{F}'(V)$. Upon replacing τ by $\tau + t'$ (which does not affect the image in $\mathcal{F}''(V)$), we may assume that $\rho^{U_0}_{U_0\cap V}(\sigma) = \rho^{V}_{U_0\cap V}(\tau)$; that is, that σ and τ agree on the overlap $U_0 \cap V$. Clearly, we may extend σ (*via* τ) to $U_0 \cup V$, contradicting the maximality of (U_0, σ); hence, $U_0 = X$.

Now suppose that $\mathcal{F}'$ and $\mathcal{F}$ are *flasque*. If $s'' \in \mathcal{F}''(U)$, then by the above, there is a section $s \in \mathcal{F}(U)$ mapping onto s''. Since $\mathcal{F}$ is also *flasque*, we may lift s to a global section, t, of $\mathcal{F}$. The image, t'', of t in $\mathcal{F}''(X)$ is the required extension of s'' to a global section of $\mathcal{F}''$.

Finally, assume that $\mathcal{F}$ is *flasque*, and that $\mathcal{F}''$ is a direct factor of $\mathcal{F}$. Let $\mathcal{F}'$ be the kernel of the map $\mathcal{F} \longrightarrow \mathcal{F}''$, then by hypothesis, $\mathcal{F}$ is the product of the sheaves $\mathcal{F}'$ and $\mathcal{F}''$. It follows from this that the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

is exact for every open set U of X. (The presheaf direct product is already a sheaf!) Precisely the same argument as in the paragraph above, now shows that $\mathcal{F}''$ is a *flasque* $\mathcal{O}_X$-module. $\square$

Proposition B.11 *If $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and if $\mathcal{F}$ is a flasque $\mathcal{O}_X$-module, then $\varphi_*\mathcal{F}$ is a flasque $\mathcal{O}_Y$-module.*

Proof. Let V be open in Y, we must show that $\varphi_*\mathcal{F}(Y) \longrightarrow \varphi_*\mathcal{F}(V)$ is surjective. But

$$\varphi_*\mathcal{F}(Y) = \mathcal{F}(|\varphi|^{-1}(Y)) = \mathcal{F}(X) \quad \text{and} \quad \varphi_*\mathcal{F}(V) = \mathcal{F}(|\varphi|^{-1}(V)).$$

Since $\mathcal{F}$ is *flasque*, we are done. $\square$

Proposition B.12 *Let $(X, \mathcal{O}_X)$ be a ringed space, and assume that X is irreducible. If $\mathcal{F}$ is the $\mathcal{O}_X$-module associated to a constant presheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}$ is flasque.*

Proof. By Exercise 3 of Section A.4, $\mathcal{F}$ is a constant presheaf on X. Hence, $\mathcal{F}(X) = \mathcal{F}(U)$ for every open U in X, which ends the proof. $\square$

A topological space is called *noetherian* if and only if it has the descending chain condition on closed sets. Equivalently, a space is noetherian if and only if each open subset is compact. (An algebraic variety is a noetherian space as we have seen in Chapter 1, Section 1.2).

Theorem B.13 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $(\mathcal{F}_\lambda)$ be an inductive system of sheaves of sets on X. Let $\mathcal{F}_0$ be the presheaf direct limit of the family $(\mathcal{F}_\lambda)$, and let $\theta\colon \mathcal{F}_0 \to \mathcal{F}_0^{(+)}$ be the canonical mapping introduced in Section A.2.*

(a) *If X is compact, then $\theta(X)\colon \mathcal{F}_0(X) \to \mathcal{F}_0^{(+)}(X)$ is bijective and $\mathcal{F}_0(X) \to \mathcal{F}_0^{\#}(X)$ is injective.*

(b) *If X is noetherian, then θ is an isomorphism and, therefore, $\mathcal{F}_0$ is a sheaf.*

(c) *If X is noetherian, the direct limit of flasque $\mathcal{O}_X$-modules is a flasque $\mathcal{O}_X$-module.*

Proof. (a) By definition, $\mathcal{F}_0^{(+)}$ is the direct limit

$$\varinjlim_{\{U_i \to X\}} H^0(\{U_i \to X\}, \mathcal{F}_0).$$

Since X is compact, the *finite* coverings $\{U_i \to X\}$ form a cofinal subset of the family of all coverings of X. Consequently, in the above limit, we may always assume the given covering $\{U_i \to X\}$ to be finite. This being said, consider the exact sequence

$$\mathcal{F}_\lambda(X) \longrightarrow \prod_i \mathcal{F}_\lambda(U_i) \rightrightarrows \prod_{ij} \mathcal{F}_\lambda(U_i \cap U_i)$$

of sets. Since direct limits commute with *finite* products, we obtain the *exact sequence*

$$\mathcal{F}_0(X) \xrightarrow{\theta(X)} \prod_i \mathcal{F}_0(U_i) \rightrightarrows \prod_{ij} \mathcal{F}_0(U_i \cap U_i).$$

From passing to the limit over the coverings $\{U_i \to X\}$, we obtain the bijection $\theta(X)\colon \mathcal{F}_0(X) \to \mathcal{F}_0^{(+)}(X)$. Now we know from the proof of Theorem A.3 that the natural map $\mathcal{F}_0^{(+)}(X) \to \mathcal{F}_0^{\#}(X)$ is always injective; this yields the second statement of (a).

(b) To say that X is noetherian is to say that every open subset, U, of X, is compact. By (a), the mappings $\theta(U)\colon \mathcal{F}_0(U) \to \mathcal{F}_0^{(+)}(U)$ are all bijective; hence, the presheaves $\mathcal{F}_0$ and $\mathcal{F}_0^{(+)}$ are isomorphic.

(c) By (b), the presheaf direct limit of the family of *flasque* $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module (i.e., a sheaf!). The limiting $\mathcal{O}_X$-module is *flasque* because the functor $\varinjlim$ is exact. $\square$

Remarks:

(1) The proof of Theorem B.13 is another instance of the technical superiority of one method of defining associated sheaves to another–in this case the double limit method is superior to the stalk method. A proof *via* the stalk method may be found in (Grothendieck [21], page 162), and the reader should compare this proof with ours.

(2) It is true that if X is compact and *Hausdorff*, then $\mathcal{F}_0(X) \to \mathcal{F}_0^{\#}(X)$ is a bijection–but the proof (??, page 162) requires a deeper investigation of the local equality of sections on normal spaces (Grothendieck [21], page 158 ff).

Problem B.1 Let $(X, \mathcal{O}_X)$ be a ringed space and let Y be a closed subspace. Suppose that $\mathcal{Q}$ is an injective $\mathcal{O}_Y$ $(= \mathcal{O}_X \restriction Y)$-module. Show that $\mathcal{Q}$ extended by zero outside Y is an injective $\mathcal{O}_X$-module. Show by specific example that the restriction of an injective $\mathcal{O}_X$-module to the closed set Y need *not* be an injective $\mathcal{O}_Y$-module. If Y is open in X, need the extension of an injective $\mathcal{O}_Y$-module by zero outside Y be an injective $\mathcal{O}_X$-module? Proof or counterexample. Answer the same questions for *flasque* sheaves.

Problem B.2 Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules. Prove that the sequence

$$0 \longrightarrow C^n(X, \mathcal{F}') \longrightarrow C^n(X, \mathcal{F}) \longrightarrow C^n(X, \mathcal{F}'') \longrightarrow 0$$

is exact for all $n \geq 0$. In other words, prove that $\mathcal{F} \rightsquigarrow C^n(X, \mathcal{F})$ is an exact functor on $\mathcal{O}_X$-modules.

B.2 Cohomology of Sheaves

As in Section B.1, $(X, \mathcal{O}_X)$ will be a ringed space, and we will be interested in the category, $\mathcal{S}(X, \mathcal{O}_X)$, of $\mathcal{O}_X$-modules. However, we will start (as in B.1) with a more general situation.

Let $\mathbf{C}$ and $\mathbf{C}'$ be abelian categories (the reader is urged to think of the case $\mathbf{C} = \mathcal{S}(X, \mathcal{O}_X)$, $\mathbf{C}' = \mathcal{AB}$–which is the important one for what follows). In $\mathbf{C}'$, we have the notion of a *cochain complex*, that is, a sequence

$$Q_0 \xrightarrow{\delta^0} Q_1 \xrightarrow{\delta^1} Q_2 \xrightarrow{\delta^2} Q_3 \xrightarrow{\delta^3} \cdots \qquad (\dagger)$$

of objects of $\mathbf{C}'$ having the property $\delta^{i+1} \circ \delta^i = 0$ for every $i \geq 0$. Of course, this condition is equivalent to the existence of a *monomorphism*

$$B^i = \mathrm{Im}\ \delta^{i-1} \longrightarrow \mathrm{Ker}\ \delta^i = Z^i.$$

The object B^i (where $i \geq 1$) is called *the object of* i^{th} *coboundaries* (in case $\mathbf{C}' = \mathcal{AB}$, we speak of the *group of* i^{th} *coboundaries*), and the object Z^i is called *the object of* i^{th} *cocycles* (resp. *group of* i^{th} *cocycles*). If necessary, we write $B^i(Q)$ or $Z^i(Q)$, where (Q) denotes sequence (†). Since there is a monomorphism $B^i \longrightarrow Z^i$ for $i \geq 1$, the quotient object $H^i = Z_i/B_i$ exists. (We set H^0 equal to Z^0.) Now the general idea is that the objects H^i for $i \geq 1$ measure the *defect from exactness* of the sequence (†)–in fact, $H^i = (0)$ for $i \geq 1$ if and only if (†) is exact in the i^{th} place. In view of this, the objects H^i are important invariants of the complex (†); H^i is called the i^{th} *cohomology object* (resp. *group* when $\mathbf{C}' = \mathcal{AB}$) of the complex (†).

The reader may well ask: Where do complexes (†) arise naturally, and what significance does the exactness or non-exactness of (†) have for the situation from which it arose? The answer is that such complexes arise in all parts of mathematics–originally having been recognized in algebraic topology–and most often the exactness (or lack of it) in (†) is of profound significance. To give concrete examples of many such phenomena would take too much space. We hope that the use of these techniques in the book will convince the doubting reader of their importance, and that further reading in the literature will show the ubiquity of such situations in mathematics.

The most typical case in which (†) arises is the following: Let T be a left-exact functor from $\mathbf{C}$ to $\mathbf{C}'$ (think of $\mathbf{C} = \mathcal{S}(X, \mathcal{O}_X)$, $\mathbf{C}' = \mathcal{AB}$, and T is the functor $\mathcal{F} \rightsquigarrow \mathcal{F}(X)$). If F is an object of $\mathbf{C}$, and if $\mathbf{C}$ possesses enough injectives, then F has an injective resolution

$$0 \longrightarrow F \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \cdots .$$

If we apply T to the injective resolution of F, we get

$$0 \longrightarrow T(F) \longrightarrow T(Q_0) \longrightarrow T(Q_1) \longrightarrow T(Q_2) \longrightarrow \cdots .$$

Now, the sequence $T(Q_0) \longrightarrow T(Q_1) \longrightarrow T(Q_2) \longrightarrow \cdots$ is a complex, so (in at least one way) we have associated with each left-exact functor on $\mathbf{C}$ to $\mathbf{C}'$, and each object F of $\mathbf{C}$, a complex in $\mathbf{C}'$. (Of course, we assume $\mathbf{C}'$ possesses enough injectives.) Why do we need an injective resolution? Any acyclic resolution will do, and any complex over F will do! The point is this: The cohomology objects of the complex $T(Q_0) \longrightarrow \cdots$ depend *a priori* on T *and the resolution* of F. If we resolve F with injectives, then the cohomology objects will depend *only upon* T *and* F (NOT upon the particular resolution employed to obtain them); hence, they may be considered invariants of T and F. This uniqueness property is a consequence of the quasi-uniqueness of injective resolutions (Section B.1, Theorem B.2), and is the main content of

Proposition B.14 *Let $\mathbf{C}$ and $\mathbf{C}'$ be abelian categories, and assume $\mathbf{C}$ possesses enough injectives. If T is a left exact functor from $\mathbf{C}$ to $\mathbf{C}'$ and if F is any object of $\mathbf{C}$, then the objects, $(R^nT)(F)$, of $\mathbf{C}'$, defined by*

$$R^nT(F) = H^n(T(Q)),$$

where $T(Q)$ is the complex

$$T(Q_0) \longrightarrow T(Q_1) \longrightarrow T(Q_2) \longrightarrow \cdots$$

arising from any injective resolution of F, depend only upon T and F. Each R^nT is a functor from $\mathbf{C}$ to $\mathbf{C}'$, and we have an isomorphism of functors $T \xrightarrow{\sim} R^0T$.

Proof. Choose two injective resolutions of F

$$0 \longrightarrow F \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \cdots \qquad (\alpha)$$

and

$$0 \longrightarrow F \longrightarrow Q_0' \longrightarrow Q_1' \longrightarrow Q_2' \longrightarrow \cdots \qquad (\beta)$$

and form the complexes

$$T(Q_0) \longrightarrow T(Q_1) \longrightarrow T(Q_2) \longrightarrow \cdots \qquad (T(\alpha))$$

and

$$T(Q'_0) \longrightarrow T(Q'_1) \longrightarrow T(Q'_2) \longrightarrow \cdots. \qquad (T(\beta))$$

By Theorem B.2, resolutions (α) and (β) are of the same homotopy type, so that there are maps (ξ_i), (η_i) from (α) to (β) (resp. (β) to (α)) over the identity whose compositions are homotopic to the identity. Since T is a functor, those maps induce maps of complexes $(T(\xi_i)) = (\xi_i^*)$, etc. Moreover, in virtue of the commutativity requirement in the definition of a map of resolutions, the maps ξ_i^*, etc., induce maps (again denoted ξ_i^*, etc.) on the cohomology of the complexes $T(\alpha)$, $T(\beta)$

$$\begin{aligned} \xi_i^*\colon H^i(T(Q)) &\longrightarrow H^i(T(Q')) \\ \eta_i^*\colon H^i(T(Q')) &\longrightarrow H^i(T(Q)). \end{aligned}$$

Since $\xi \circ \eta_i \sim \text{id}$ and $\eta_i \circ \xi_i \sim \text{id}$ (here, $\sim$ denotes "homotopic to"), our first conclusion will follow from the statement: *Homotopic maps of complexes induce the same map on cohomology.*

To obtain the italicized statement, let (u_i) and (v_i) be homotopic maps of complexes; let let u_i^* and v_i^* denote the induced maps on the i^{th} cohomology object, for all i. If (s_i) is the homotopy, we have

$$\begin{aligned} v_i - u_i &= (\delta')^{i-1} \circ s_i + s_{i+1} \circ \delta^i, \quad i > 0 \\ v_0 - u_0 &= s_1 \circ \delta^0. \end{aligned}$$

Now, u_i^* and v_i^* are defined on Ker δ^i modulo Im δ^{i-1}, so it follows immediately from the above equations that $u_i^* = v_i^*$ for all $i \geq 0$.

Suppose $F \xrightarrow{f} G$ is a map of objects in $\mathbf{C}$. According to Theorem B.2, for any injective resolution of F and G there is a map over f. This morphism induces a map on cohomology, and hence we obtain a map $(R^nT)(F) \longrightarrow (R^nT)(G)$. The map on cohomology is independent of the resolution and the particular lifting of f to a map of resolutions. (Theorem B.2 shows that any two liftings are homotopic, and the italicized statement above gives us the uniqueness.) The axioms for a functor are trivially verified by the same methods.

Finally, as T is left-exact, the sequence

$$0 \longrightarrow T(F) \longrightarrow T(Q_0) \xrightarrow{\delta^0} T(Q_1) \longrightarrow \cdots$$

shows that $T(F) \longrightarrow \text{Ker } \delta^0 = (R^0T)(F)$ is an isomorphism, and it is obviously functorial. $\square$

The functors R^nT are called the *right derived* functors of T. When T is fixed, the objects (groups in case $\mathbf{C}' = \mathcal{AB}$) $(R^nT)(F)$ are important invariants of the object F.

Proposition B.15 *Let*

$$Q_0' \longrightarrow Q_1' \longrightarrow Q_2' \longrightarrow \cdots \qquad (Q')$$
$$Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \cdots \qquad (Q)$$
$$Q_0'' \longrightarrow Q_1'' \longrightarrow Q_2'' \longrightarrow \cdots \qquad (Q'')$$

be three complexes, and assume that for each $i \geq 0$ we have an exact sequence

$$0 \longrightarrow Q_i' \longrightarrow Q_i \longrightarrow Q_i'' \longrightarrow 0.$$

Assume moreover that the diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_i' \arrow[r] \arrow[d, "\delta'"] & Q_i \arrow[r] \arrow[d, "\delta"] & Q_i'' \arrow[r] \arrow[d, "\delta''"] & 0 \\
0 \arrow[r] & Q_{i+1}' \arrow[r] & Q_{i+1} \arrow[r] & Q_{i+1}'' \arrow[r] & 0
\end{tikzcd}$$

commutes for each $i \geq 0$. Then there is a map

$$\delta_i^* \colon H^i(Q'') \to H^{i+1}(Q')$$

for each $i \geq 0$, and the sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(Q') \arrow[r] & H^0(Q) \arrow[r] & H^0(Q'') \arrow[dll, rounded corners, "\delta_0^*"] \\
 & H^1(Q') \arrow[r] & \cdots \arrow[r] & \cdots \arrow[dll, rounded corners] \\
 & \cdots \arrow[r] & \cdots \arrow[r] & H^{n-1}(Q'') \arrow[dll, rounded corners, "\delta_{n-1}^*"] \\
 & H^n(Q') \arrow[r] & H^n(Q) \arrow[r] & H^n(Q'') \arrow[r] & \cdots
\end{tikzcd}$$

is exact.

Remark: One compresses the hypotheses of Proposition B.15 by saying that $0 \longrightarrow (Q') \longrightarrow (Q) \longrightarrow (Q'') \longrightarrow 0$ is an *exact sequence of complexes*. The maps δ_n^* are *connecting homomorphisms*, and the exact sequence of the conclusion is called the *cohomology sequence* or the *long exact sequence of cohomology*.

Proof. An application of the snake lemma to the commutative diagram in the hypothesis yields a map

$$\text{Ker } (\delta'')^i \longrightarrow \text{Coker } (\delta')^{i+1}.$$

However, a moment's thought shows that this map factors through $H^i(Q'')$ and $H^{i+1}(Q')$; so it yields the connecting homomorphism δ_i^*. The exactness of the cohomology sequence is straight-forward and will be left as an exercise. $\square$

Corollary B.16 *Let* $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ *be an exact sequence in the abelian category* $\mathbf{C}$*, and let* $T\colon \mathbf{C} \to \mathbf{C}'$ *be a left-exact functor. Then for every* $n \geq 0$*, there is a map*

$$(R^nT)(F'') \xrightarrow{\delta_n^*} (R^{n+1}T)(F'),$$

and the sequence

$$\begin{aligned}
0 &\longrightarrow T(F') \longrightarrow T(F) \longrightarrow T(F'') \longrightarrow \\
&\longrightarrow (R^1T)(F') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
&\longrightarrow (R^nT)(F') \longrightarrow (R^nT)(F) \longrightarrow (R^nT)(F'') \longrightarrow \\
&\longrightarrow (R^{n+1}T)(F') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{aligned}$$

is exact. If $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ *is another exact sequence in* $\mathbf{C}$*, and if there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] & G \arrow[r] & G'' \arrow[r] & 0,
\end{tikzcd}$$

then the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & T(F') \arrow[r] \arrow[d] & T(F) \arrow[r] \arrow[d] & T(F'') \arrow[r] \arrow[d] & (R^1T)(F') \arrow[r] \arrow[d] & \cdots \arrow[r] & (R^nT)(F') \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & T(G') \arrow[r] & T(G) \arrow[r] & T(G'') \arrow[r] & (R^1T)(G') \arrow[r] & \cdots \arrow[r] & (R^nT)(G') \arrow[r] & \cdots
\end{tikzcd}$$

is also commutative.

Proof. This is an immediate consequence of Theorem B.2 and Propositions B.3 and B.15. $\square$

Remark: A sequence $\{T^n\}$ of functors having the properties expressed by the conclusions of the corollary is called an *exact, connected sequence of functors* or a *cohomological functor* (or a δ*-functor*).

Corollary B.17 *The functor* T *is exact if and only if* R^nT *is zero for* $n > 0$.

Proof. If T is exact then clearly R^nT vanishes for positive n. Conversely suppose R^nT vanishes for $n > 0$, then in particular R^1T is identically zero. Hence, for any exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

we obtain

$$0 \longrightarrow T(F') \longrightarrow T(F) \longrightarrow T(F'') \longrightarrow (R^1T)(F')$$

is exact. This completes the proof. □

Now if Q is any injective object of $\mathbf{C}$ then the sequence $0 \longrightarrow Q \xrightarrow{id} Q \longrightarrow 0$ is an injective resolution of Q; hence, for any functor T (left-exact, of course) we obtain $(R^nT)(Q) = (0)$ for $n > 0$. The three properties

(a) $R^0T = T$

(b) R^nT vanishes on injectives for $n > 0$

(c) $\{R^nT\}$ is a cohomological functor

serve to *characterize* the sequence $\{R^nT\}$. This is a straightforward argument, however, it also follows from a more general line of reasoning which will be useful later on. We now turn to this more general reasoning.

Suppose we have a functor $T\colon \mathbf{C} \to \mathbf{C}'$. We say that T is *effaçable in* $\mathbf{C}$ if and only if for each object, F, of $\mathbf{C}$ there is a monomorphism into an object, M_F, of $\mathbf{C}'$

$$u\colon F \to M_F$$

such that $T(u) = 0$. (In particular, this will be the case if $T(M_F)$ is the zero object of $\mathbf{C}'$.) Let $\{T^n\})$ be a cohomological functor. The sequence $\{T^n\}$ is a *universal cohomological functor* if and only if for any cohomological functor, $\{S^n\}$, and for any map, $T^0 \longrightarrow S^0$ of functors *there exists a unique family* of maps of functors $T^n \longrightarrow S^n$ (having appropriate commutativity conditions) for $n \geq 0$ which reduces, when $n = 0$, to the given map $T^0 \longrightarrow S^0$. An obvious consequence of this rather long condition is the fact that *two universal cohomological functors are isomorphic if and only if their zero*$^{\text{th}}$ *terms are isomorphic*.

Proposition B.18 *Let $\mathbf{C}, \mathbf{C}'$ be abelian categories and let $\{T^n\})$ be a cohomological functor from $\mathbf{C}$ to $\mathbf{C}'$. Suppose T^n is effaçable for $n > 0$. Then, $\{T^n\}$ is a universal cohomological functor. Consequently, properties (a), (b), (c) of the sequence $\{R^nT\}$ characterize it as a cohomological functor.*

Proof. The proof is by induction on n; we shall treat only the case $n = 1$ for the other cases are very similar. Let $u_0\colon T^0 \to S^0$ be the given map of functors. If F is an object of $\mathbf{C}$, the *effaçability* of T^1 shows that there is an exact sequence

$$0 \longrightarrow F \longrightarrow M_F \longrightarrow F'' \longrightarrow 0$$

such that the map δ in the induced sequence

$$T^0(M_f) \longrightarrow T^0(F'') \xrightarrow{\delta} T^1(F) \longrightarrow T^1(M_F)$$

is surjective. Were a map $u_1\colon T^1 \to S^1$ to exist, the commutative diagram

$$\begin{tikzcd}
T^0(M_F) \arrow[r] \arrow[d, "u_0(M_F)"] & T^0(F'') \arrow[r, "\delta"] \arrow[d, "u_0(F'')"] & T^1(F) \arrow[r] \arrow[d, "u_1"] & T^1(M_F) \\
S^0(M_F) \arrow[r] & S^0(F'') \arrow[r] & S^1(F) &
\end{tikzcd}$$

and the surjectivity of δ would show that u_1 is completely determined by the maps $u_0(M_F)$, $u_0(F'')$. This argument proves the uniqueness of the map u_1.

Now we need to prove the existence of a map u_1. Since the lower line of the above diagram is exact (all that is necessary is that it be a complex), this diagram with u_1 removed implies that the map $u_0(F'')$ induces a map $T^1(F) \longrightarrow S^1(F)$ in such a way that our diagram commutes. However, it is *a priori* possible that the induced map depends upon the choice of exact sequence

$$0 \longrightarrow F \longrightarrow M_F \longrightarrow F'' \longrightarrow 0.$$

We claim that this is not the case. Observe first that a simple argument establishes: If

$$0 \longrightarrow F \longrightarrow M'_F \longrightarrow G \longrightarrow 0$$

is another exact sequence, and *if this sequence dominates the former* in the sense that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & M_F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & M'_F \arrow[r] & G \arrow[r] & 0,
\end{tikzcd}$$

then the maps $u_1\colon T^1(F) \to S^1(F)$ induced by these sequences are the same. From this it follows that given two sequences

$$0 \longrightarrow F \longrightarrow M_F \longrightarrow F'' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \longrightarrow M'_F \longrightarrow G \longrightarrow 0,$$

we need only find a common dominant. If ξ is the composed map $F \longrightarrow M_F \longrightarrow M_F \coprod M'_F$ and η is the composed map $F \longrightarrow M'_F \longrightarrow M_F \coprod M'_F$, then $\xi - \eta$ is an injection of F into $M_F \coprod M'_F$. Let M be the cokernel of $\xi - \eta$, then the exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow F'' \coprod G,$$

is the required dominant. The verification that u_1 is a map of functors is now trivial and the proof is complete. $\square$

The main point of Proposition B.18 is that it does away with the need for injectives in establishing the uniqueness of a cohomological functor. Frequently, one will have two cohomological functors which agree in dimension zero. To establish that they are *isomorphic on the category* $\mathbf{C}$*, one need only be able to efface them in* $\mathbf{C}$.

We need just one more abstract proposition.

Proposition B.19 *Let T be a functor from the abelian category $\mathbf{C}$ to the abelian category $\mathbf{C}'$, and suppose that $\mathbf{C}$ has enough injectives. Let X be a class of objects in $\mathbf{C}$ which satisfies the following conditions:*

(i) $\mathbf{C}$ *possesses enough X-objects,*

(ii) If F is an object of $\mathbf{C}$ and F is a direct factor of some object in X, then F belongs to X,

(iii) If $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is exact and if F' belongs to X, then $0 \longrightarrow T(F') \longrightarrow T(F) \longrightarrow T(F'') \longrightarrow 0$ is exact, and if F also belongs to X, then F'' belongs to X.

Under these conditions, every injective object belongs to X, for each M in X we have $(R^nT)(M) = (0)$ for $n > 0$, and finally the functors R^nT may be computed by taking X-resolutions.

Proof. Let Q be an injective of $\mathbf{C}$. By (i), Q admits a monomorphism into some object M of the class X. As Q is injective, Q is a direct factor of M; hence (ii) implies Q lies in X. Let us now show that $(R^nT)(M) = (0)$ for $n > 0$ if M lies in X. Now, $\mathbf{C}$ possesses enough injectives, so we have (with obvious notations) the exact sequences

$$\begin{array}{l} 0 \longrightarrow M \longrightarrow Q_0 \longrightarrow Z_0 \longrightarrow 0 \\ 0 \longrightarrow Z_0 \longrightarrow Q_1 \longrightarrow Z_1 \longrightarrow 0 \\ 0 \longrightarrow Z_1 \longrightarrow Q_2 \longrightarrow Z_2 \longrightarrow 0 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \longrightarrow Z_n \longrightarrow Q_{n+1} \longrightarrow Z_{n+1} \longrightarrow 0 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

Here, each Q_i is injective, so lies in X. As M belongs to X, (iii) show that Z_0 lies in X. By induction, Z_i belongs to X for every $i \geq 0$. Again, by (iii), the sequences

$$\begin{array}{l} 0 \longrightarrow T(M) \longrightarrow T(Q_0) \longrightarrow T(Z_0) \longrightarrow 0 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \longrightarrow T(Z_n) \longrightarrow T(Q_{n+1}) \longrightarrow T(Z_{n+1}) \longrightarrow 0 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

are exact. Consequently, the sequence

$$0 \longrightarrow T(M) \longrightarrow T(Q_0) \longrightarrow T(Q_1) \longrightarrow T(Q_2) \longrightarrow \cdots$$

is exact, and this proves that $(R^nT)(M) = (0)$ for positive n. Finally, we must show that the functors R^nT may be computed from arbitrary X-resolutions (which exist by (i)). Given

F, we construct an X-resolution *via* the usual exact sequences

$$\begin{gathered}0 \longrightarrow F \longrightarrow M_0 \longrightarrow Z_0 \longrightarrow 0\\ 0 \longrightarrow Z_0 \longrightarrow M_1 \longrightarrow Z_1 \longrightarrow 0\\ \vdots\end{gathered}$$

If we apply the sequence of cohomology to each of these sequences and make use of $(R^nT)(M) = (0)$ for $n > 0$, we deduce

$$(R^mT)(Z_n) = (R^pT)(Z_{m+n-p}) = (R^{m+n-1}T)(F)$$

for $m > 0$. Consequently, we deduce exact sequences

$$\begin{gathered}0 \longrightarrow T(F) \longrightarrow T(M_0) \longrightarrow T(Z_0) \longrightarrow (R^1T)(F) \longrightarrow 0\\ 0 \longrightarrow T(Z_0) \longrightarrow T(M_1) \longrightarrow T(Z_1) \longrightarrow (R^2T)(F) \longrightarrow 0\\ 0 \longrightarrow T(Z_1) \longrightarrow T(M_2) \longrightarrow T(Z_2) \longrightarrow (R^3T)(F) \longrightarrow 0,\end{gathered}$$

etc. These sequences prove that the cohomology of

$$0 \longrightarrow T(F) \longrightarrow T(M_0) \longrightarrow T(M_1) \longrightarrow T(M_2) \longrightarrow \cdots$$

is exactly $\{(R^nT))(F)\}_{n=0}^{\infty}$, as required. $\square$

At last we forsake arbitrary abelian categories and assume $\mathbf{C} = \mathcal{S}(X, \mathcal{O}_X)$ and $\mathbf{C}' = \mathcal{AB}$.

Definition B.4 Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let U be an open set of X. By the n^{th}–*cohomology group of U with coefficients in $\mathcal{F}$* we mean the abelian group $(R^n\Gamma_U)(\mathcal{F})$, where Γ_U is the functor: $\mathcal{O}(X, \mathcal{O}_X) \longrightarrow \mathcal{AB}$ given by $\mathcal{F} \rightsquigarrow \mathcal{F}(U)$. In particular, the cohomology groups of X with coefficients in $\mathcal{F}$ are the values of the derived functors $R^n\Gamma$ on $\mathcal{F}$, where Γ is the "global section" functor, i.e., $\Gamma(\mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

The notation for the n^{th} cohomology group of U with coefficients in $\mathcal{F}$ is $H^n(U, \mathcal{F})$.

Upon putting together Propositions B.9, B.14, B.15, B.18, B.19, and Theorems B.2 and B.10, we may state the following grand theorem.

Theorem B.20 *Cohomology groups of a ringed space X in any $\mathcal{O}_X$-module exist. If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ and $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ are exact sequences of $\mathcal{O}_X$-modules, and the diagram*

$$\begin{tikzcd}0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0\\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0\end{tikzcd}$$

is commutative then we have a commutative diagram of cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] \arrow[d] & H^0(X,\mathcal{F}) \arrow[r] \arrow[d] & H^0(X,\mathcal{F}'') \arrow[r] \arrow[d] & H^1(X,\mathcal{F}') \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & H^0(X,\mathcal{G}') \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{G}'') \arrow[r] & H^1(X,\mathcal{G}') \arrow[r] & \cdots .
\end{tikzcd}$$

The positive dimensional cohomology groups of X vanish for flasque (in particular, for injective) sheaves, and the cohomology groups are characterized by the above properties. One can compute the cohomology groups of $\mathcal{F}$ using **any** *flasque resolution of $\mathcal{F}$; in particular, they may be computed from the Godement resolution of $\mathcal{F}$. For any $\mathcal{O}_X$-module, $\mathcal{F}$, the zero*$^{\text{th}}$ *cohomology group, $H^0(X,\mathcal{F})$, is exactly the group of global sections, $\mathcal{F}(X) = \Gamma(X,\mathcal{F})$, of $\mathcal{F}$.*

Another easy consequence of our results is the following theorem which is of use in algebraic geometry.

Theorem B.21 *Let $(X,\mathcal{O}_X)$ be a ringed space, and assume X is noetherian. If $(\mathcal{F}_\lambda)$ is a direct mapping family of $\mathcal{O}_X$-modules and $\mathcal{F}$ is its limit, then we have the canonical isomorphism*

$$\varinjlim_{\lambda} H^n(X,\mathcal{F}_\lambda) \xrightarrow{\sim} H^n(X,\mathcal{F})$$

for every $n \geq 0$. In other words, cohomology commutes with direct limits of sheaves over noetherian spaces.

Proof. For each $\mathcal{F}_\lambda$ construct its Godement resolution. According to Problem 2 of Section B.1, the Godement resolution of a sheaf is a functor of that sheaf; consequently, we have a direct family of *flasque* resolutions

$$\left\{0 \longrightarrow \mathcal{F}_\lambda \longrightarrow C^0(X,\mathcal{F}_\lambda) \longrightarrow C^1(X,\mathcal{F}_\lambda) \longrightarrow \cdots\right\}.$$

The direct limit of these resolutions is a *flasque* resolution of $\mathcal{F}$ by Theorem B.13. Since direct limits of complexes commute with cohomology of complexes our theorem now follows immediately from Theorem B.20. $\square$

We now wish to examine the cohomology of sheaves concentrated on closed subspaces of X. For this we need

Lemma B.22 *Let Y be a closed subspace of X and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the canonical map*

$$\gamma\colon \Gamma(X,\mathcal{F}_Y) \to \Gamma(Y,\mathcal{F} \restriction Y)$$

is an isomorphism.

Proof. The map γ is the restriction of sections from X to Y. From the definition of $\mathcal{F}_Y$ each element of the lefthand group has support contained in Y; this proves that γ is injective. Conversely as Y is *closed* in X, each section of $\mathcal{F} \restriction Y$ has support closed in X; hence, the map ϵ which extends a section by zero outside Y is an inverse to γ. $\square$

Obviously if $\mathcal{G}$ is an $\mathcal{O}_Y$-module ($\mathcal{O}_Y = \mathcal{O}_X \restriction Y$) then Lemma B.22 yields the isomorphism

$$\Gamma(Y, \mathcal{G}) \xrightarrow{\sim} \Gamma(X, \overline{\mathcal{G}}).$$

Theorem B.23 *Let Y be closed in X and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then there exist canonical isomorphisms*

$$H^n(Y, \mathcal{G}) \xrightarrow{\sim} H^n(X, \overline{\mathcal{G}})$$

for every $n \geq 0$. In particular for an $\mathcal{O}_X$-module, $\mathcal{F}$, there are isomorphisms

$$H^n(Y, \mathcal{F} \restriction Y) \xrightarrow{\sim} H^n(X, \mathcal{F}_Y).$$

Moreover, we have the exact relative cohomology sequence

$$\begin{aligned}
0 &\longrightarrow H^0(X, \mathcal{F}_{X-Y}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(Y, \mathcal{F} \restriction Y) \longrightarrow \\
&\longrightarrow H^1(X, \mathcal{F}_{X-Y}) \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
&\longrightarrow H^n(X, \mathcal{F}_{X-Y}) \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(Y, \mathcal{F} \restriction Y) \longrightarrow \\
&\longrightarrow H^{n+1}(X, \mathcal{F}_{X-Y}) \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{aligned}$$

Proof. The functor $\mathcal{G} \rightsquigarrow \overline{\mathcal{G}}$ is exact, and when composed with the cohomological functor $\{H^n(X, -)\}$ it gives a cohomological functor

$$\mathcal{G} \rightsquigarrow H^n(X, \overline{\mathcal{G}})$$

on $\mathcal{S}(Y, \mathcal{O}_Y)$. Now there is a universal cohomological functor on $\mathcal{S}(Y, \mathcal{O}_Y)$, namely $\mathcal{G} \rightsquigarrow \{H^n(Y, \mathcal{G})\}$. Since we have a map $H^0(Y, \mathcal{G}) \longrightarrow H^0(X, \overline{\mathcal{G}})$ we deduce the canonical isomorphisms

$$H^n(Y, \mathcal{G}) \longrightarrow H^n(X, \overline{\mathcal{G}}) \quad \text{for } n \geq 0. \tag{α}$$

Each $\mathcal{O}_Y$-module, $\mathcal{G}$, is embeddable in an injective $\mathcal{O}_Y$-module $\mathcal{Q}$. By Problem 1 of Section B.1 (see also Proposition B.6), $\overline{\mathcal{Q}}$ is an injective $\mathcal{O}_X$-module. Hence, $\mathcal{G}$ admits a monomorphism into a sheaf $\mathcal{Q}$ for which $H^n(X, \overline{\mathcal{Q}})$ vanishes when $n > 0$. That is, *the functors $\mathcal{G} \rightsquigarrow H^n(X, \overline{\mathcal{G}})$ are effaçable* for $n > 0$. It follows from Proposition B.18 that

$\mathcal{G} \rightsquigarrow \{H^n(X, \overline{\mathcal{G}})\}$ is a universal cohomological functor; so the homomorphisms (α) are isomorphisms. From Theorem A.12, we deduce the exact sequence

$$0 \longrightarrow \mathcal{F}_{X-Y} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0. \tag{β}$$

Upon writing $\mathcal{G} = \mathcal{F} \upharpoonright Y$, using the cohomology sequence associated to (β), and employing the isomorphisms (α), we deduce the exact relative cohomology sequence. $\square$

There is an interesting application of Theorem B.23 to the notion of cohomological dimension of an algebraic variety. We say that a cohomological space X has *cohomological dimension less than or equal to n* if for every sheaf, $\mathcal{F}$, of abelian groups on X, the group $H^r(X, \mathcal{F})$ vanishes when $r > n$. The following theorem (due to Grothendieck [21]) provides a test for cohomological dimension on noetherian spaces.

Theorem B.24 *Let X be a noetherian space. In order that X have a cohomological dimension less that or equal to n, it is necessary and sufficient that*

$$H^r(X, \mathcal{Z}_U) = (0)$$

for all $r > n$ and all open U in X.

Proof. Every sheaf is a homomorphic image of a direct sum of sheaves, $\mathcal{Z}_{U_i}$, for open sets U_i (the $\mathcal{Z}_U$ are generators of $\mathcal{S}(X, \mathcal{O}_X)$). Such a direct sum is a direct limit of finite direct sums, and, since X is noetherian, Theorem B.21 show that we may restrict attention to homomorphic images of finite direct sums of $\mathcal{Z}_U$'s.

If $\mathcal{F}$ is such a sheaf, it follows immediately that $\mathcal{F}$ possesses a composition series

$$(0) = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_m = \mathcal{F}$$

whose factors $\mathcal{F}_j/\mathcal{F}_{j-1}$ are homomorphic images of the generating $\mathcal{Z}_U$. Were the theorem true for sheaves of composition length at most $m-1$, we could prove it for sheaves of composition length m as follows: The sequence $0 \longrightarrow \mathcal{F}_{m-1} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_{m-1} \longrightarrow 0$ yields the cohomology sequence

$$\cdots \longrightarrow H^r(X, \mathcal{F}_{m-1}) \longrightarrow H^r(X, \mathcal{F}) \longrightarrow H^r(X, \mathcal{F}/\mathcal{F}_{m-1}) \longrightarrow \cdots.$$

Since the extremes of the latter sequence vanish for $r > n$ by the induction hypothesis, the middle term also vanishes, as required. *Thus, in proving the theorem, we may assume that the sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0$$

is exact for some open set U of X. By Theorem A.16, the sheaf has a composition series

$$\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{K}_\nu = \mathcal{K}$$

whose quotients have the form $\mathcal{Z}_Y$ for locally closed subspaces Y of X. We need to prove $H^r(X, \mathcal{K}) = (0)$ for $r > n$, and the same argument as above shows that we may assume $\mathcal{K} = \mathcal{Z}_Y$ for Y locally closed in X.

Now, Y has the form $U - V$ for open sets $V \subseteq U$ in X, and it follows from Theorem A.12 that the sequence

$$0 \longrightarrow \mathcal{Z}_V \longrightarrow \mathcal{Z}_U \longrightarrow \mathcal{Z}_Y \longrightarrow 0$$

is exact. But an application of the cohomology sequence and the hypothesis to the last exact sequence yields

$$H^r(X, \mathcal{Z}_Y) = (0) \quad \text{for } r > n. \qquad \square$$

Let X be a noetherian space. We say that X has *combinatorial dimension less than or equal to* n if the least upper bound of the integers r for which there exists a strictly increasing chain

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_r = X$$

of *closed, nonempty, irreducible* subsets of X is less than or equal to n. When X is an affine variety in the sense of Chapter 1, dimension theory shows that the combinatorial dimension of X equals its dimension as a variety.

Theorem B.25 *Let X be a noetherian space. Then the cohomological dimension of X is bounded by its combinatorial dimension.*

Proof. We prove this by induction on the combinatorial dimension of X. When X has dimension zero it is a finite union of points and the theorem is trivial. Suppose X has dimesnion n and the theorem has been proved for smaller combinatorial dimensions. Let X_j, with $j = 1, \cdots, r$ be the irreducible components of X (= maximal irreducible closed subspaces of X, which exist as X is noetherian), let $\mathcal{F}$ be a sheaf on X, and let $\mathcal{F}_j = \mathcal{F}_{X_j}$. As each X_j is closed in X, we have a homomorphism $\mathcal{F} \longrightarrow \mathcal{F}_j$ for each j; consequently we deduce the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_j \mathcal{F}_j \longrightarrow \mathcal{G} \longrightarrow 0$$

on X. Now $\mathcal{G}$ is zero outside of the union $\bigcup_{i \neq j}(X_i \cap X_j) = Y$, and Y has combinatorial dimension at most $n - 1$. It follows from the induction hypothesis and Theorem B.23 that $H^m(X, \mathcal{G}) = (0)$ for $m > n - 1$. The cohomology sequence then shows that we need prove only that

$$H^m(X, \mathcal{F}_j) = (0), \quad \text{for all } j \text{ and all } m > n.$$

However, by Theorem B.23 again, $H^m(X_j, \mathcal{F}_j) = H^m(X, \mathcal{F}_j)$; hence, we may assume that X is *irreducible*.

Let U be open (nonempty) in X, and consider the exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{Z}_U \longrightarrow \mathcal{Z} \longrightarrow \mathcal{Z}_{X-U} \longrightarrow 0.$$

From the cohomology sequence and Theorem B.23, we obtain

$$H^{m-1}(X-U, \mathcal{Z}) \longrightarrow H^m(X, \mathcal{Z}_U) \longrightarrow H^m(X, \mathcal{Z}).$$

As X is irreducible, the combinatorial dimension of $X - U$ is strictly smaller than n, and the constant sheaf $\mathcal{Z}$ on X is *flasque* (see Proposition B.12). It follows from this that the extremes of our sequence vanish for $m > n$; hence, an application of Theorem B.24 completes the proof. $\square$

Problem B.3 (Buchsbaum). Let T be a left-exact functor from the abelian category $\mathbf{C}$ to $\mathcal{AB}$. We say that T is *locally effaçable* if and only if for every F in $\mathbf{C}$ and every $\xi \in T(F)$, there exists an object, $M_{F,\xi}$, of $\mathbf{C}$ and a *monomorphism* $u\colon F \to M_{F,\xi}$ such that $T(u)(\xi) = 0$. Prove that a cohomological functor $\{T^n\}$ on $\mathbf{C}$ to $\mathcal{AB}$ is universal if T^n is *locally effaçable* for all $n > 0$.

Problem B.4 Suppose $\mathbf{C}$ is an abelian category with enough injectives and $\{T^n\}$ is a universal cohomological functor from $\mathbf{C}$ to another abelian category $\mathbf{C}'$. If Q is an injective object of $\mathbf{C}$ prove that $T^n(Q) = (0)$ for $n > 0$.

Problem B.5 Let X be an irreducible affine variety and let A be its coordinate ring. Let $(Y, \mathcal{O}_Y)$ be the ringed space associated to A (Example 3 of Section A.6), and let $\mathcal{M}$ be the sheaf of meromorphic functions on Y. If $\mathcal{D}$ denotes the sheaf of germs of divisors on Y (i.e., the quotient sheaf $\mathcal{M}^*/\mathcal{O}_Y^*$), then as we shall show later on, $\mathcal{D}$ is *flasque*. Assume this fact and establish

(a) An isomorphism $\Gamma(Y, \mathcal{D})/\Gamma(Y, \mathcal{M}^*) \xrightarrow{\sim} H^1(Y, \mathcal{O}_Y^*)$, i.e., a classification of the classes of divisors on Y as elements of the first cohomology group of Y with coefficients in the germs of invertible holomorphic functions, and

(b) $H^n(Y, \mathcal{O}_Y^*) = (0)$ for $n \geq 2$.

Problem B.6 Give an example of a noetherian space X of zero cohomological dimension but infinite (or arbitrarily large) combinatorial dimension.
Hint. Let X be a well-ordered set and let a set $S \subseteq X$ be closed if and only if

$$S = S_x \quad \text{where} \quad S_x = \{y \in X \mid y < x\}.$$

B.3 Čech Cohomology

The cohomology groups of Section B.2 are apparently still hard to compute. Even more important is the question: What are they good for? We shall develop a cohomology theory for presheaves which is, in a mild way computable. In some instances it agrees with the cohomology of Section B.2. However, its biggest advantage is that the groups lend themselves to geometric interpretation as classifying groups for certain structures and constructs on a ringed space. An important example of this phenomenon is given in Section 4.4 (Theorem 5.11).

Let $\mathcal{F}$ be a presheaf of abelian groups on X, and let $\{U_i \to X\}$ be a covering. Given indices $i_0, \ldots, i_n$, we have $n+1$ maps

$$U_{i_0} \cap \cdots \cap U_{i_n} \longrightarrow U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_n}$$

(where $\widehat{U_{i_j}}$ means "omit U_{i_j}") corresponding to the omission of each one of the $n+1$ sets U_{i_j}, with $j = 0, \ldots, n$. Let δ_j^n denote the map from $U_{i_0} \cap \cdots \cap U_{i_n}$ to $U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_n}$, then our array of sets and maps may be schematically depicted as follows:

$$X \longleftarrow U_{i_0} \;\substack{\longleftarrow \\ \longleftarrow}\; U_{i_0} \cap U_{i_1} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; U_{i_0} \cap U_{i_1} \cap U_{i_2} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \cdots .$$

If we let $C^n(\{U_i \to X\}, \mathcal{F})$ be the product $\prod \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n})$ taken over all tuples $(i_0, \ldots, i_n)$ of $n+1$ indices, then we obtain a diagram

$$C^0(\{U_i \to X\}, \mathcal{F}) \;\substack{\longrightarrow \\ \longrightarrow}\; C^1(\{U_i \to X\}, \mathcal{F}) \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; C^2(\{U_i \to X\}, \mathcal{F}) \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots .$$

Each of the arrows in this diagram has the form $\mathcal{F}(\delta_j^n)$ for some n and some j, with $0 \le j \le n$. Given n, upon defining $\delta_{\mathcal{F}}^n$ by the formula

$$\delta_{\mathcal{F}}^n = \sum_{j=0}^{n+1} (-1)^j \mathcal{F}(\delta_j^{n+1}),$$

we get the diagram

$$C^0(\{U_i \to X\}, \mathcal{F}) \xrightarrow{\delta_{\mathcal{F}}^0} C^1(\{U_i \to X\}, \mathcal{F}) \xrightarrow{\delta_{\mathcal{F}}^1} C^2(\{U_i \to X\}, \mathcal{F}) \cdots . \tag{$*$}$$

As an exercise, the reader should prove that the diagram $(*)$ is a *complex* for every presheaf $\mathcal{F}$ and every covering $\{U_i \to X\}$ (DX).

As an aid in understanding the complicated symbolism above (and in doing the exercise!), let us render more explicit the nature of the groups $C^n(\{U_i \to X\}, \mathcal{F})$ and the maps $\delta_{\mathcal{F}}^n$. An element of $C^n(\{U_i \to X\}, \mathcal{F})$ is a collection of objects

$$f(i_0, \ldots, i_n)$$

each lying in $\mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_n})$. If I is the index set for the covering $\{U_i \to X\}$, then such an element is obviously a "function" on I^{n+1} with "value" at $(i_0,\ldots,i_{n+1})$ in $\mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_{n+1}})$. By tracing through the definition of $\delta^n_{\mathcal{F}}$, we find

$$(\delta^n_{\mathcal{F}} f)(i_0,\ldots,i_{n+1}) = \sum_{j=0}^{n+1}(-1)^j f(i_0,\ldots,\widehat{i_j},\ldots,i_{n+1}), \qquad (**)$$

where $\widehat{i_j}$ means "omit i_j." In particular

$$\begin{aligned}(\delta^0_{\mathcal{F}} f)(i,j) &= f(j) - f(i)\\ (\delta^1_{\mathcal{F}} f)(i,j,k) &= f(j,k) - f(i,k) + f(i,j), \quad \text{etc.}\end{aligned}$$

The group $C^n(\{U_i \to X\},\mathcal{F})$ is called the *group of Čech n-cochains for the covering $\{U_i \to X\}$ with values in $\mathcal{F}$*, the mappings $\delta^n_{\mathcal{F}}$ are called *coboundary operators*, and the cohomology groups of the complex $(*)$ are called the *Čech cohomology groups of $\mathcal{F}$ for the covering* $\{U_i \to X\}$. Observe that $H^0(\{U_i \to X\},\mathcal{F})$ is exactly the group introduced in Section A.2; hence, *when $\mathcal{F}$ is a sheaf we have the isomorphism*

$$\mathcal{F}(X) \simeq H^0(\{U_i \to X\},\mathcal{F}).$$

Also, when $\mathcal{F}$ is a sheaf, a Čech n-cochain of $\mathcal{F}$ for the covering $\{U_i \to X\}$ may be considered as a family $\{s(i_0,\ldots,i_n)\}$ of sections of $\mathcal{F}$, each section $s(i_0,\ldots,i_n)$ being defined over $U_{i_0}\cap\cdots\cap U_{i_n}$.

Now, the correspondence $\mathcal{F} \rightsquigarrow C^n(\{U_i \to X\},\mathcal{F})$ is obviously functorial, and moreover a trivial check shows that $\mathcal{F} \rightsquigarrow C^n(\{U_i \to X\},\mathcal{F})$ is an exact functor from $\mathcal{P}(X)$ to $\mathcal{AB}$. It follows that

$$\mathcal{F} \rightsquigarrow H^n(\{U_i \to X\},\mathcal{F}) \qquad (\dagger)$$

is a cohomological functor on $\mathcal{P}(X)$.

Theorem B.26 *The functors $\mathcal{F} \rightsquigarrow H^n(\{U_i \to X\},\mathcal{F})$ are effaçable for $n > 0$; consequently, the cohomological functor $(\dagger)$ is universal. The functors $H^n(\{U_i \to X\},-)$ are the right derived functors of the left-exact functor $H^0(\{U_i \to X\},-)$.*

Proof. Let $\mathbb{Z}_U$ (for open U) be the presheaf introduced in Example 4 of Section A.1. If $\mathcal{F}$ is an injective presheaf, we must show that the complex

$$C^0(\{U_i \to X\},\mathcal{F}) \xrightarrow{\delta^0_{\mathcal{F}}} C^1(\{U_i \to X\},\mathcal{F}) \xrightarrow{\delta^1_{\mathcal{F}}} C^2(\{U_i \to X\},\mathcal{F}) \longrightarrow \cdots$$

is acyclic (i.e., exact). Now for any open set U, $\mathrm{Hom}\,(\mathbb{Z}_U,\mathcal{F}) = \mathcal{F}(U)$; hence, our complex amounts to

$$\prod_i \mathrm{Hom}\,(\mathbb{Z}_{U_i},\mathcal{F}) \longrightarrow \prod_{i,j} \mathrm{Hom}\,(\mathbb{Z}_{U_i\cap U_j},\mathcal{F}) \longrightarrow \cdots.$$

But we know
$$\prod_\alpha \mathrm{Hom}\,(\mathbb{Z}_{U_\alpha}, \mathcal{F}) = \mathrm{Hom}\,(\coprod_\alpha \mathbb{Z}_{U_\alpha}, \mathcal{F});$$
hence our complex becomes:
$$\mathrm{Hom}\,(\coprod_i \mathbb{Z}_{U_i}, \mathcal{F}) \longrightarrow \mathrm{Hom}\,(\coprod_{i,j} \mathbb{Z}_{U_i\cap U_j}, \mathcal{F}) \longrightarrow \cdots.$$
As $\mathcal{F}$ is injective, the latter will be acyclic if we can prove that
$$\coprod_i \mathbb{Z}_{U_i} \longleftarrow \coprod_{i,j} \mathbb{Z}_{U_i\cap U_j} \longleftarrow \coprod_{i,j,k} \mathbb{Z}_{U_i\cap U_j\cap U_k} \longleftarrow \cdots$$
is exact.

Now, by definition of $\mathbb{Z}_{U_i}$, etc., we see that it is necessary to prove that the sequence
$$\coprod_i \Bigg(\coprod_{\mathrm{Hom}\,(U,U_i)} \mathbb{Z}\Bigg) \longleftarrow \coprod_{i,j} \Bigg(\coprod_{\mathrm{Hom}\,(U,U_i\cap U_j)} \mathbb{Z}\Bigg) \longleftarrow \cdots$$
is exact. However, the last sequence is induced by the maps of the diagram
$$\coprod_i \mathrm{Hom}\,(U, U_i) \;\substack{\longleftarrow\\ \longleftarrow}\; \coprod_{i,j} \mathrm{Hom}\,(U, U_i\cap U_j) \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots \tag{††}$$
(as follows from the definition of $\mathbb{Z}_{U_j}$ *as a presheaf*). If we let $J = \coprod_i \mathrm{Hom}\,(U, U_i)$, then (††) becomes the diagram
$$J \;\substack{\longleftarrow\\ \longleftarrow}\; J\times J \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; J\times J\times J \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; J\times J\times J\times J \cdots$$
and here the maps really are obvious: Consequently, we are reduced to proving that a diagram of the form
$$\coprod_J \mathbb{Z} \;\substack{\longleftarrow\\ \longleftarrow}\; \coprod_{J^2} \mathbb{Z} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{J^3} \mathbb{Z} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{J^4} \mathbb{Z} \cdots$$
is exact.

An element of $\coprod_{J^n} \mathbb{Z}$ is a function f on J^n to $\mathbb{Z}$ with *finite support*. Its image $\pi^n f$ in $\coprod_{J^{n-1}} \mathbb{Z}$ is given by
$$(\pi^n f)(\alpha_1, \ldots, \alpha_{n-1}) = \sum_{i=1}^n (-1)^i \sum_{\beta\in J} f(\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_i, \ldots, \alpha_{n-1}).$$

Consequently, f is a "cycle", i.e., $\pi^n f = 0$, when and only when

$$\sum_{i=1}^{n}(-1)^i \sum_{\beta\in J} f(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_i,\ldots,\alpha_{n-1}) = 0.$$

We must show that every cycle is a "boundary," i.e., $\pi^n f = 0$ implies $f = \pi^{n+1} g$ for some g. To this end, given f, define f^* by

$$f^*(\alpha_1,\ldots,\alpha_{n+1}) = \begin{cases} 0 & \text{if } \alpha_{n+1} \neq \lambda \\ f(\alpha_1,\ldots,\alpha_n) & \text{if } \alpha_{n+1} = \lambda \end{cases}$$

where λ is a chosen element of J, fixed once and for all. (we may assume $J \neq \emptyset$.) Then, $\pi^{n+1} f^*$ has the form

$$\begin{aligned}(\pi^{n+1}f^*)(\alpha_1,\ldots,\alpha_n) &= \sum_{i=1}^{n+1}(-1)^i \sum_{\beta\in J} f^*(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_i,\ldots,\alpha_n) \\ &= \sum_{i=1}^{n}(-1)^i \sum_{\beta\in J} f^*(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_i,\ldots,\alpha_n) + (-1)^{n+1} f(\alpha_1,\ldots,\alpha_n).\end{aligned}$$

When $\alpha_n \neq \lambda$, we deduce

$$(\pi^{n+1}f^*)(\alpha_1,\ldots,\alpha_n) = (-1)^{n+1} f(\alpha_1,\ldots,\alpha_n).$$

So, assume $\alpha_n = \lambda$, then

$$\begin{aligned}(\pi^{n+1}f^*)(\alpha_1,\ldots,\alpha_n) &= \sum_{i=1}^{n}(-1)^i \sum_{\beta\in J} f(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_i,\ldots,\alpha_{n-1}) + (-1)^{n+1} f(\alpha_1,\ldots,\alpha_n) \\ &= (\pi^n f)(\alpha_1,\ldots,\alpha_{n-1}) + (-1)^{n+1} f(\alpha_1,\ldots,\alpha_n) \\ &= (-1)^{n+1} f(\alpha_1,\ldots,\alpha_n),\end{aligned}$$

since we are assuming $\pi^n f = 0$. This completes the proof. $\square$

Let $\{U_i \to X\}$ be a refinement of $\{U'_\lambda \to X\}$, say by the map ϵ. As described in Section A.2, the map ϵ induces a map

$$\epsilon_n^*\colon C^n(\{U'_\lambda \to X\},\mathcal{F}) \to C^n(\{U_i \to X\},\mathcal{F})$$

for each $n \geq 0$. One easily checks that the mapping ϵ_n^* commute with the coboudary operators $\delta_{\mathcal{F}}$; hence, we obtain mappings

$$H^n(\{U'_\lambda \to X\},\mathcal{F}) \longrightarrow H^n(\{U_i \to X\},\mathcal{F})$$

for every $n \geq 0$.

Since the functor $H^n(\{U_i \to X\},\mathcal{F})$ is universal, any map from $H^0(\{U'_\lambda \to X\},\mathcal{F})$ to $H^0(\{U_i \to X\},\mathcal{F})$ may be *extended uniquely* to maps of the H^n. However, Lemma A.2 of Section A.2 shows that the map ϵ_0^* induced by ϵ is *independent* of ϵ. We have proven

Proposition B.27 *Let* $\{U'_\lambda \to X\} \xrightarrow{\epsilon} \{U_i \to X\}$ *be a refinement of coverings. Then, for each* $n \geq 0$, ϵ *induces a mapping*

$$\epsilon_n^*\colon H^n(\{U'_\lambda \to X\}, \mathcal{F}) \longrightarrow H^n(\{U_i \to X\}, \mathcal{F}).$$

Any two refining maps between the same coverings induce the same map on cohomology.

Just as in Section A.2, we may now pass to the direct limit over the family of all coverings partially ordered by domination. When this is done we obtain

Definition B.5 The *Čech cohomology groups of* X *with coefficients in the presheaf* $\mathcal{F}$, denoted $\check{H}^n(X, \mathcal{F})$, are defined by

$$\check{H}^n(X, \mathcal{F}) = \varinjlim_{\{U_i \to X\}} H^n(\{U_i \to X\}, \mathcal{F})$$

(the direct limit over all coverings $\{U_i \to X\}$.)

Observe that $\mathcal{F} \rightsquigarrow \check{H}^0(X, \mathcal{F})$ is a left-exact functor from $\mathcal{P}(X)$ to $\mathcal{AB}$, and that $\mathcal{F} \rightsquigarrow \check{H}^n(X, \mathcal{F})$ is its n^{th} right-derived functor (DX).

Čech cohomology groups lend themselves to geometric interpretation on ringed spaces. A good example of this phenomenon is given in Section 4.4. If $(X, \mathcal{O}_X)$ is a ringed space, it is proved in Theorem 5.11 that the *Picard group* $\mathrm{Pic}(X)$ (the group of isomorphism classes of invertible sheaves on $(X, \mathcal{O}_X)$) is isomorphic to $\check{H}^1(X, \mathcal{O}_X^*)$.

It turns out that the set, $\mathcal{LF}_n(\mathcal{O}_X)$, of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank n on a ringed space $(X, \mathcal{O}_X)$ is also classified by a Čech cohomology object, namely $\check{H}^1(X, \mathbb{GL}(n))$ (see Corollary 5.12). However, $\check{H}^1$ is a *set*, and not a group. Thus, we need to develop nonabelian cohomology. Here is how we proceed for Čech cohomology, in the special case of $\check{H}^1$, which is the only convenient case.

Let $\mathcal{G}$ be a sheaf of nonabelian groups and let $\mathcal{U} = \{U_\alpha \to X\}$ be a cover of X. We define the sets $C^0(\{U_\alpha \to X\}, \mathcal{G})$ and $C^1(\{U_\alpha \to X\}, \mathcal{G})$ by

$$C^0(\{U_\alpha \to X\}, \mathcal{G}) = \prod_\alpha G(U_\alpha),$$

and

$$C^1(\{U_\alpha \to X\}, \mathcal{G}) = \prod_{\alpha,\beta} G(U_\alpha \cap U_\beta)^{\text{alt}},$$

which means that $g_\alpha^\beta = [g_\beta^\alpha]^{-1}$ for every $(g_\alpha^\beta) \in C^1(\{U_\alpha \to X\}, \mathcal{G})$. Then, we define the 1-cocycles by

$$Z^1(\{U_\alpha \to X\}, \mathcal{G}) = \{(g_\alpha^\beta) \in C^1(\{U_\alpha \to X\}, \mathcal{G}) \mid g_\alpha^\gamma = g_\beta^\gamma \cdot g_\alpha^\beta\}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. The equivalence relation $\sim$ on $Z^1(\{U_\alpha \to X\}, \mathcal{G})$ is defined as follows: Given (g_α^β) and $(\widetilde{g}_\alpha^\beta)$ in $Z^1(\{U_\alpha \to X\}, \mathcal{G})$, we say that (g_α^β) and $(\widetilde{g}_\alpha^\beta)$ are *cohomologous*, denoted by

$$(g_\alpha^\beta) \sim (\widetilde{g}_\alpha^\beta),$$

iff there exist some 0-cocycle (h_α) in $C^0(\{U_\alpha \to X\}, \mathcal{G})$ so that

$$\widetilde{g}_\alpha^\beta = h_\beta \cdot g_\alpha^\beta \cdot h_\alpha^{-1}$$

for all α, β. Then, set

$$H^1(\{U_\alpha \to X\}, \mathcal{G}) = Z^1(\{U_\alpha \to X\}, \mathcal{G}) / \sim .$$

Note that $H^1(\{U_\alpha \to X\}, \mathcal{G})$ contains a distinguished element, namely, the equivalence class of 1, i.e.,

$$\{(g_\alpha^\beta) \mid g_\alpha^\beta = h_\beta \cdot h_\alpha^{-1}\}.$$

Under refinements of covers, all maps work correctly, and by taking the inductive limit, we obtain the Čech cohomology set, $\check{H}^1(X, \mathcal{G})$, given by

$$\check{H}^1(X, \mathcal{G}) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G}),$$

where $\mathcal{U}$ ranges over all covers of X.

Problem B.7 A complex

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

is *homotopically trivial* if and only if the identity map is homotopic to the zero map. (Any homotopy which effects this is usually called a *contracting homotopy*.) Show that if the above complex is homotopically trivial, it is acyclic. Show that the complex

$$\coprod_J \mathbb{Z} \;\leftleftarrows\; \coprod_{J^2} \mathbb{Z} \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; \coprod_{J^3} \mathbb{Z} \;\overset{\overset{\longleftarrow}{\longleftarrow}}{\underset{\underset{\longleftarrow}{\longleftarrow}}{}}\; \coprod_{J^4} \mathbb{Z} \cdots$$

is homotopically trivial. Does this imply that any two maps on the Čech cochain complexes are homotopic? Is the converse true?

Problem B.8 Let A be a commutative ring with unity and let Spec A be the ringed space introduced in Section A.6 (Example 3) for A. If Spec $A = (X, \mathcal{O}_X)$, then we denote $\mathrm{Pic}(X)$ by $\mathrm{Pic}(A)$.

(a) Compute $\mathrm{Pic}(A)$ when A is a Dedekind domain.

(b) Compute $\mathrm{Pic}(A)$ when A is a noetherian local ring.

(c) Compute $\mathrm{Pic}(A)$ when A is a unique factorization domain.

Problem B.9 Let $A = k[T]$ be the polynomial ring in one variable. If $(X, \mathcal{O}_X) = \mathrm{Spec}\, A$, show that $H^1(X, \mathcal{O}_X) = (0)$.

Problem B.10 Show that for every covering $\{U_i \to X\}$, the mapping

$$H^1(\{U_i \to X\}, \mathcal{F}) \longrightarrow \check{H}^1(X, \mathcal{F})$$

is an injection if $\mathcal{F}$ is a sheaf.

Problem B.11 Let $\mathcal{F}$ be a *flasque* sheaf on X. Show that for every open covering $\{U_i \to U\}$ of the open U, we have

$$H^n(\{U_i \to U\}, \mathcal{F}) = (0)$$

for $n > 0$. Is the converse true?

B.4 Spectral Sequences

There is a non-trivial connection between the cohomology of Section B.2 and the Čech cohomology of Section B.3. This connection is in the form of a "limiting procedure" starting at one and ending at the other. The precise form of this "limiting (or approximating) procedure" is technical, and the procedure itself is called a spectral sequence. A spectral sequence is the device used most often to connect various cohomology theories.

Let $\mathbf{C}$ be an abelian category (think of $\mathbf{C}$ as $\mathcal{AB}$) and let A be an object of $\mathbf{C}$. A *decreasing filtration* on A is a family, $\{A^n \mid n \in \mathbb{Z}\}$, of subobjects of A with $A^{n+1} \subseteq A^n$ for every n. Let $A^{-\infty} = A$ and $A^{\infty} = (0)$. The object, A, together with a given filtration on it will be called a *filterered object* of $\mathbf{C}$. If we have a filtered object, say A, we can form its *associated graded object* $\mathrm{gr}(A)$ as follows:

$$\mathrm{gr}(A) = \coprod_n \mathrm{gr}(A)_n; \quad \mathrm{gr}(A)_n = A^n / A^{n+1}.$$

Typically, A is a complex in $\mathbf{C}$ considered an object of $\mathbf{C}$. That is, A is a sequence

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \longrightarrow \cdots,$$

and to make A an object of $\mathbf{C}$ we form the direct sum $\coprod_n A_n$ and call it A. (We assume that $\mathbf{C}$ possesses direct sums.) Now the object A is naturally graded by the A_n; hence if we have a filtration $\{A^p\}$ on A there must be a connection between it and the grading or chaos will reign. We shall say that the *filtration* $\{A^p\}$ *is compatible with the grading* $A = \coprod_n A_n$ if and only if, for every p,

$$A^p = \coprod_q A^p \cap A_{p+q}.$$

Moreover, we set $A^{p,q}$ equal to the intersection $A^p \cap A_{p+q}$. (We assume of our category $\mathbf{C}$ that intersections exist with the usual properties.) Note that if only a grading on A is given there is a natural filtration $\{A^p\}$ on A which is compatible with the grading. Namely, set

$$A^p = \coprod_{m \geq p} A_m.$$

Of course, this filtration is not the only one compatible with the grading.

Now A also possesses a natural "differentiation" (i.e., endomorphism whose square is zero), namely $d = \coprod_n d_n$. We assume of our filtration that it is compatible with d in the sense that $d(A^p) \subseteq A^p$, for every p. In this case, A^p forms a complex; hence, one can talk of the (co)homology $H^\bullet(A^p)$ of A^p as well as that of A. Moreover, there is an important class of filtrations—the ones most often arising in practice—which can be defined when $d(A^p) \subseteq A^p$. These are the *regular filtrations*. We say that a filtration is *regular* if and only if, for each n, there exists an integer, $\mu(n)$, such that

$$H^n(A_p) = (0) \quad \text{for } p > \mu(n).$$

Observe that a filtration is regular if we know that $A^p \cap A_n = (0)$ for $p > \mu(n)$ (this is the criterion which one commonly meets).

The inclusion map $A^p \longrightarrow A$ induces a map of (co)homology $H^\bullet(A^p) \longrightarrow H^\bullet(A)$, whose image will be denoted $H^\bullet(A)^p$. Thus,

$$\mathrm{Im}\,(H^\bullet(A^p) \longrightarrow H^\bullet(A)) = H^\bullet(A)^p.$$

The subobjects $H^\bullet(A)^p$ filter $H^\bullet(A)$; so we may form $\mathrm{gr}(H^\bullet(A))$. In general, in the graded and filtered case, we let

$$\mathrm{gr}(A)^{p,q} = A^{p,q}/A^{p+1,q-1}.$$

Observe that it is the sum $p + q$ which is the invariant in many of these doubly indexed objects. The reason for this will be apparent when we study double complexes. (The reader who is experiencing difficulty in keeping track of all the indices is advised not to worry about this matter but read just a little further when, we hope, matters will be clarified.) From the above terminology and notation, we obtain several objects:

(1) $\mathrm{gr}(A)$—bigraded *via* $\mathrm{gr}(A)^{p,q} = A^{p,q}/A^{p+1,q-1}$, where $A^{p,q} = A^p \cap A_{p+q}$.

(2) $H^\bullet(A)$–bigraded *via* $H^{p,q}(A) = H^\bullet(A)^p \cap H^{p+q}(A)$.

(3) $\mathrm{gr}(H^\bullet(A))$—bigraded *via* $\mathrm{gr}(H^\bullet(A))^{p,q} = H^{p,q}(A)/H^{p+1,q-1}(A)$.

(4) $H^\bullet(\mathrm{gr}(A))$—bigraded *via* $H^{p,q}(\mathrm{gr}(A)) = H^{p+q}(\mathrm{gr}(A)_p) = H^{p+q}(A^p/A^{p+1})$.

The central problem is the following: We are given A and wish to compute $H^\bullet(A)$. What we know is $\mathrm{gr}(A)$ and $H^\bullet(\mathrm{gr}(A))$. Hence we ask: Given $H^\bullet(\mathrm{gr}(A))$, can we "compute" $H^\bullet(A)$? The answer is that we cannot do precisely this, but in a large number of cases, we can "compute" $\mathrm{gr}(H^\bullet(A))$.[2] The object $\mathrm{gr}(H^\bullet(A))$ (bigraded as above) is the "limit" of a sequence (the spectral sequence) of objects each constructed from the previous one by passing to (co)homology and all starting from $H^\bullet(\mathrm{gr}(A))$.

Definition B.6 A (*cohomological*) *spectral sequence* is a system $\mathcal{E} = \langle E_r^{p,q}, d_r^{p,q}, \alpha_r^{p,q}, E, \beta^{p,q}\rangle$ formed of

(a) Objects $E_r^{p,q}$ of $\mathbf{C}$ for $p, q \geq 0$ and $r \geq 2$

(b) Morphisms $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p, q, r

(c) Isomorphisms $\alpha_r^{p,q}\colon (\mathrm{Ker}\ d_r^{p,q}/\mathrm{Im}\ d_r^{p-r,q+r-1}) \to E_{r+1}^{p,q}$

(d) $E = \coprod_n E_n$ is a graded and filtered object in $\mathbf{C}$ with decreasing filtration $\{E^p\}$, so that each E^p is graded by the $E^{p,q} = E^p \cap E_{p+q}$;

(e) $\beta^{p,q}\colon E_\infty^{p,q} \to \mathrm{gr}(E)^{p,q}$ is an isomorphism for all p, q, where

$$\mathrm{gr}(E)^{p,q} = (E^p \cap E_{p+q})/(E^{p+1} \cap E_{p+q}) = E^{p,q}/E^{p+1,q-1},$$

and $E_\infty^{p,q}$ is the common value of $E_r^{p,q}$ for large r.

(Observe that from (a) and (c) we have $E_r^{p,q} \simeq E_{r+1}^{p,q}$ for every $r > \max\{p, q+1\}$, hence the common value of the $E_r^{p,q}$ exists for large r. In particular, $E_2^{0,0} = E_\infty^{0,0}$ and $E_2^{1,0} = E_\infty^{1,0}$.)

The object $E = \coprod_n E_n$ is called the *ending of the spectral sequence* and the whole definition is written in the compact form

$$E_2^{p,q} \underset{p}{\Longrightarrow} E,$$

which means that there exists objects $E_r^{p,q}$, morphisms d's, α's, β's, etc. so that E with its filtration satisfies (e).

Remark: It is customary to define spectral sequence beginning from $r = 2$, even though the terms $E_i^{p,q}$ are often defined and meaningful for $r = 1$, and even for $r = 0$. However, in the case of double complexes, the natural starting point is indeed $r = 2$, as pointed out in Cartan and Eilenberg [8] (Chapter XV, page 332).

[2] Passing from $\mathrm{gr}(H^\bullet(A))$ to $H^\bullet(A)$ is the subject of "deformation theory,", see ??.

Theorem B.28 *Let A be a complex in the abelian category* $\mathbf{C}$ *and assume that A has a regular filtration compatible with both its grading and differentiation. Then there exists a spectral sequence*

$$E_2^{p,q} \Longrightarrow H^\bullet(A),$$

where $H^\bullet(A)$ is filtered as described above and $E_2^{p,q}$ is the homology of $H^\bullet(\mathrm{gr}(A))$—so that $E_1^{p,q} = H^{p,q}(\mathrm{gr}(A)) = H^{p+q}(A^p/A^{p+1})$.

In the course of proving Theorem B.28, we shall make heavy use of the following simple lemma whose proof will be left as an exercise (or see Cartan and Eilenberg [8], Chapter XV, Lemma 1.1):

Lemma B.29 *(Lemma L) Let*

$$\begin{tikzcd} & C \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ A' \arrow[ur] \arrow[r, "\varphi'"'] & A \arrow[r, "\eta"'] & A'' \end{tikzcd}$$

be a commutative diagram with exact bottom row. Then, η induces an isomorphism $\mathrm{Im}\,\varphi/\mathrm{Im}\,\varphi' \xrightarrow{\sim} \mathrm{Im}\,\psi$.

Proof of Theorem B.28. Consider the exact sequence

$$0 \longrightarrow A^p \longrightarrow A^{p-r+1} \longrightarrow A^{p-r+1}/A^p \longrightarrow 0.$$

Upon applying cohomology, we obtain

$$\cdots \longrightarrow H^{p+q-1}(A^{p-r+1}) \longrightarrow H^{p+q-1}(A^{p-r+1}/A^p) \xrightarrow{\delta^*} H^{p+q}(A^p) \longrightarrow \cdots$$

There is also the natural map $H^{p+q}(A^p) \longrightarrow H^{p+q}(A^p/A^{p+1})$ induced by the projection $A^p \longrightarrow A^p/A^{p+1}$. Moreover, we have the projection $A^p/A^{p+r} \longrightarrow A^p/A^{p+1}$, which induces a map on cohomology

$$H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q}(A^p/A^{p+1}).$$

Set

$$\begin{aligned} Z_r^{p,q} &= \mathrm{Im}\,(H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q}(A^p/A^{p+1})) \\ B_r^{p,q} &= \mathrm{Im}\,(H^{p+q-1}(A^{p-r+1}/A^p) \longrightarrow H^{p+q}(A^p/A^{p+1})), \end{aligned}$$

the latter map being the composition of δ^* and the projection (where $r \geq 1$).

The inclusion $A^{p-r+1} \subseteq A^{p-r}$ yields a map $A^{p-r+1}/A^p \longrightarrow A^{p-r}/A^p$; hence we obtain the inclusion relation $B_r^{p,q} \subseteq B_{r+1}^{p,q}$. In a similar way, the projection $A^p/A^{p+r+1} \longrightarrow A^p/A^{p+r}$ yields the inclusion $Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$. When $r = \infty$, the coboundary map yields the inclusion $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$ (remember, $A^\infty = (0)$). Consequently, we can write

$$\cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \cdots .$$

Set

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}, \quad \text{where } 1 \leq r \leq \infty.$$

When $r = 1$, $B_1^{p,q} = (0)$ and

$$Z_1^{p,q} = H^{p+q}(A^p/A^{p+1});$$

We obtain $E_1^{p,q} = H^{p+q}(A^p/A^{p+1}) = H^{p,q}(\mathrm{gr}(A))$. On the other hand, when $r = \infty$ (remember, $A^{-\infty} = A$), we get

$$\begin{aligned} Z_\infty^{p,q} &= \mathrm{Im}\,(H^{p+q}(A^p) \longrightarrow H^{p+q}(A^p/A^{p+1})) \\ B_\infty^{p,q} &= \mathrm{Im}\,(H^{p+q-1}(A/A^p) \longrightarrow H^{p+q}(A^p/A^{p+1})). \end{aligned}$$

Now the exact sequence $0 \longrightarrow A^p/A^{p+1} \longrightarrow A/A^{p+1} \longrightarrow A/A^p \longrightarrow 0$ yields the cohomology sequence

$$\cdots \longrightarrow H^{p+q-1}(A/A^p) \xrightarrow{\delta^*} H^{p+q}(A^p/A^{p+1}) \longrightarrow H^{p+q}(A/A^{p+1}) \longrightarrow \cdots$$

and the exact sequence $0 \longrightarrow A^p \longrightarrow A \longrightarrow A/A^p \longrightarrow 0$ gives rise to the connecting homomorphism

$$H^{p+q-1}(A/A^p) \xrightarrow{\delta^{*\prime}} H^{p+q}(A^p).$$

Consequently, we obtain the commutative diagram (with exact bottom row)

$$\begin{tikzcd} & H^{p+q}(A^p) \arrow[d] \arrow[dr] & \\ H^{p+q-1}(A/A^p) \arrow[ur, "\delta^{*\prime}"] \arrow[r, "\delta^*"] & H^{p+q}(A^p/A^{p+1}) \arrow[r] & H^{p+q}(A/A^{p+1}) \end{tikzcd}$$

and Lemma B.29 yields an isomorphism

$$\xi^{p,q}\colon E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} \longrightarrow \mathrm{Im}\,(H^{p+q}(A^p) \longrightarrow H^{p+q}(A/A^{p+1})).$$

But another application of Lemma B.29 to the diagram

$$\begin{tikzcd} & H^{p+q}(A^p) \arrow[d] \arrow[dr] & \\ H^{p+q}(A^{p+1}) \arrow[ur] \arrow[r] & H^{p+q}(A) \arrow[r] & H^{p+q}(A/A^{p+1}) \end{tikzcd}$$

gives us the isomorphism

$$\eta^{p,q}\colon \mathrm{gr}(H^\bullet(A))^{p,q} \longrightarrow \mathrm{Im}\,(H^{p+q}(A^p) \longrightarrow H^{p+q}(A/A^{p+1})).$$

Thus, $(\eta^{p,q})^{-1} \circ \xi^{p,q}$ is the isomorphism $\beta^{p,q}$ required by part (e) of Definition B.6.

Only two things remain to be proven to complete the proof of Theorem B.28. They are the verification of (b) and (c) of Definition B.6, and the observation that E_∞ in Definition

B.6 is equal to E_∞ as computed above. The verification of (b) and (c) depends upon Lemma B.29. Specifically, we have the two commutative diagrams (with obvious origins)

$$\begin{tikzcd}
 & H^{p+q}(A^p/A^{p+r}) \arrow[d, "j_2"] \arrow[dr, "\theta"] & \\
H^{p+q}(A^p/A^{p+r+1}) \arrow[ur, "j_1"] \arrow[r, "j"'] & H^{p+q}(A^p/A^{p+1}) \arrow[r, "\delta^*"'] & H^{p+q+1}(A^{p+1}/A^{p+r+1})
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^{p+q}(A^p/A^{p+r}) \arrow[d, "\varphi"] \arrow[dr, "\theta"] & \\
H^{p+q}(A^{p+1}/A^{p+r}) \arrow[ur, "i_1"] \arrow[r, "{\delta^*}'"'] & H^{p+q+1}(A^{p+r}/A^{p+r+1}) \arrow[r, "i_2"'] & H^{p+q+1}(A^{p+1}/A^{p+r+1})
\end{tikzcd}$$

Here, the map θ is the composition

$$H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+1}/A^{p+r+1}).$$

Now, Lemma B.29 yields the following facts:

$$\begin{aligned} Z_r^{p,q}/Z_{r+1}^{p,q} &\ \widetilde{\longrightarrow}\ \text{Im } \theta, \\ B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} &\ \widetilde{\longrightarrow}\ \text{Im } \theta, \end{aligned}$$

that is,

$$\delta_r^{p,q}\colon Z_r^{p,q}/Z_{r+1}^{p,q} \ \widetilde{\longrightarrow}\ B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

As $B_r^{p,q} \subseteq Z_s^{p,q}$ for every r and s, there is a surjection

$$\pi_r^{p,q}\colon E_r^{p,q} \longrightarrow Z_r^{p,q}/Z_{r+1}^{p,q}$$

with kernel $Z_{r+1}^{p,q}/B_r^{p,q}$; and there exists an injection

$$\sigma_{r+1}^{p+r,q-r+1}\colon B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \longrightarrow E_r^{p+r,q-r+1}.$$

The composition $\sigma_{r+1}^{p+r,q-r+1} \circ \delta_r^{p,q} \circ \pi_r^{p,q}$ is the map $d_r^{p,q}$ from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$ required by (b). Observe that,

$$\text{Im } d_r^{p-r,q+r-1} = B_{r+1}^{p,q}/B_r^{p,q} \subseteq Z_{r+1}^{p,q}/B_r^{p,q} = \text{Ker } d_r^{p,q};$$

hence

$$H^{p,q}(E_r^{p,q}) = \text{Ker } d_r^{p,q}/\text{Im } d_r^{p-r,q+r-1} = Z_{r+1}^{p,q}/B_{r+1}^{p,q} = E_{r+1}^{p,q},$$

as required by (c).

To prove that $E_\infty^{p,q}$ as defined above is the common value of $E_r^{p,q}$ for large enough r, we must make use of the regularity of our filtration. Consider then the commutative diagram

$$\begin{tikzcd}
 & H^{p+q}(A^p/A^{p+r}) \arrow[d] \arrow[dr, "\lambda"] & \\
H^{p+q}(A^p) \arrow[ur] \arrow[r] & H^{p+q}(A^p/A^{p+1}) \arrow[r] & H^{p+q+1}(A^{p+1})
\end{tikzcd}$$

where λ is the composition

$$H^{p+q}(A^p/A^{p+r}) \xrightarrow{\delta^*} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+1}).$$

By Lemma B.29, we have $Z_r^{p,q}/Z_\infty^{p,q} \xrightarrow{\sim} \mathrm{Im}\ \lambda$. However, if $p+r > \mu(p+q+1)-p$, then δ^* is the zero map. This shows $\mathrm{Im}\ \lambda = (0)$; hence, we have proven

$$Z_r^{p,q} = Z_\infty^{p,q} \quad \text{for } r > \mu(p+q+1)-p.$$

It is easy to see that $\bigcup_r B_r^{p,q} = B_\infty^{p,q}$; hence, we obtain maps

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} \longrightarrow Z_s^{p,q}/B_s^{p,q} = E_s^{p,q}$$

for $s \geq r > \mu(p+q+1)-p$, and these maps are surjective. (The maps are in fact induced by the $d_r^{p-r,q+r-1}$'s because of the equality

$$E_r^{p,q}/\mathrm{Im}\ d_r^{p-r,q+r-1} = (Z_r^{p,q}/B_r^{p,q})/(B_{r+1}^{p,q}/B_r^{p,q}) = E_{r+1}^{p,q}$$

for $r > \mu(p+q+1)-p$.) Obviously, the direct limit of the mapping family

$$E_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow \cdots \longrightarrow E_s^{p,q} \longrightarrow \cdots$$

is the object $Z_\infty^{p,q}/(\bigcup B_r^{p,q}) = E_\infty^{p,q}$, and this completes the proof. $\square$

Remark: Observe from the definition of $E_\infty^{p,q}$ that

$$E_\infty^{p,q} = H^{p+q}(A) \cap H^\bullet(A)^p/H^{p+q}(A)/H^\bullet(A)^{p+1},$$

so that, for $p+q=n$, the $E_\infty^{p,q} = E_\infty^{p,n-p}$ are the composition factors in the filtration

$$H^n(A) \supseteq H^n(A)^1 \supseteq H^n(A)^2 \supseteq \cdots \supseteq H^n(A)^\nu \supseteq \cdots.$$

A pictorial representation of the above situation is very convenient. In this representation, the groups $E_r^{p,q}$ (for fixed r) are represented as points in the pq plane, *viz*:

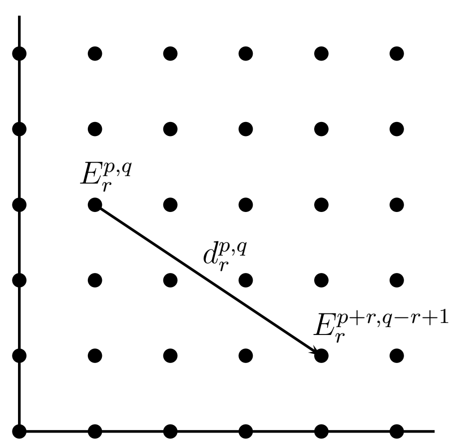

Figure B.1: The $E_r^{p,q}$ terms of a spectral sequence and the differentiation $d_r^{p,q}$ is represented as an arrow "going over r and down $r-1$." So, the situation above may be represented

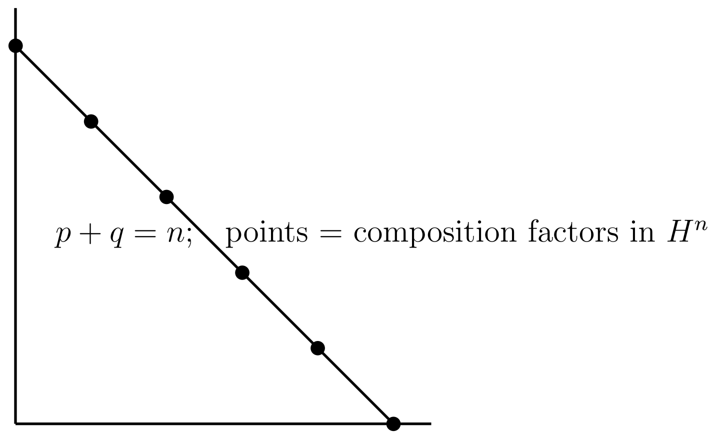

Figure B.2: The $E_\infty^{p,q}$ terms of a spectral sequence

Moreover, the index p tells us that we are looking at the p^{th} composition factor; consequently, p is usually called the *filtration index* (or *filtration degree*). The index $p+q$ is called the *total degree* and q is the *complementary degree*. If $p > n$ then $q < 0$; hence $E_\infty^{p,q} = (0)$ for $p > n$. This shows that $H^n(A)^{n+1} = H^n(A)^{n+2} = \cdots$, and since this filtration is regular, we deduce that $H^n(A)^{n+1} = (0)$. Consequently,

$$E_\infty^{n,0} = H^n(A)^n \subseteq H^n(A), \quad E_\infty^{n-1,1} = H^n(A)^{n-1}/H^n(A)^n, \ldots,$$

and

$$H_\infty^{0,n} = H^n(A)/H^n(A)^1, \quad \text{a homomorphic image of } H^n(A).$$

As an example, suppose we could show $E_\infty^{p,q} = (0)$ for $p+q > n$ and all p. It would follow that $H^r(A) = (0)$ for every $r > n$.

The notion of a morphism of spectral sequences is completely obvious; and if A and A' are graded and filtered complexes, one has the (also obvious) notion of a morphism from A to A'. It is trivial that a morphism of graded, filtered complexes, $f\colon A \to A'$ induces a morphism of their associated spectral sequences.

Theorem B.30 *Let A and A' be graded, filtered complexes whose filtrations are regular, and let $f\colon A \to A'$ be a morphism from A to A'. If f_k^* denotes the induced morphism*

$$f_k^*\colon E_k(A) \to E_k(A')$$

and if for some $k \geq 1$, the map f_k^* is an isomorphism, then for all r with $k \leq r \leq \infty$, f_r^* is an isomorphism. Moreover,

$$f^*\colon H^\bullet(A) \to H^\bullet(A')$$

is also an isomorphism.

Proof. Since $f_r^* d_r = d'_r f_{r'}^*$, the d_r cohomology of E_r is isomorphic (by f_{r+1}^*) to E_{r+1} if f_r^* is itself an isomorphism. Hence, induction proves the first statement of the theorem for $k \leq r < \infty$. When $r = \infty$, we have

$$\begin{aligned} Z_\infty^{p,q}/B_k^{p,q} &= \bigcap_{r \geq k}(Z_r^{p,q}/B_k^{p,q}) \\ B_\infty^{p,q}/B_k^{p,q} &= \bigcap_{r \geq k}(B_r^{p,q}/B_k^{p,q}) \end{aligned}$$

This implies the result for $r = \infty$. The interpretation of $E_\infty^{p,q}$ given in the remarks above shows that the case $r = \infty$ yields the isomorphism

$$H^n(A)^{p-1}/H^n(A)^p \xrightarrow{\sim} H^n(A')^{p-1}/H^n(A')^p$$

for all $p \geq 0$. By induction, we deduce the isomorphism

$$H^n(A)^{p-r}/H^n(A)^p \xrightarrow{\sim} H^n(A')^{p-r}/H^n(A')^p$$

for all $r \geq 1$; hence the isomorphism

$$H^n(A)/H^n(A)^p \xrightarrow{\sim} H^n(A')/H^n(A')^p.$$

As our filtration is regular, the theorem follows by choosing p large enough. $\square$

The conclusion of Theorem B.30 is in the nature of a uniqueness theorem; it says that the cohomology of A is "determined by" the spectral sequence associated to A. In certain special cases we can say more. For example, we say that a spectral sequence $\{E_r^{p,q}\}$ is *degenerate* (or *degenerates*) if and only if there exists an integer $r \geq 2$, such that for every n,

$$E_r^{n-q,q} = (0) \quad \text{if } q \neq q(n)$$

where $q(n)$ is an integer perhaps dependent upon n. The most common case is when $r = 2$ and $q(n) = 0$ for every n. Now, observe that for regular filtrations, Theorem B.28 shows that $E_r^{p,q} = (0)$ implies $E_s^{p,q} = (0)$ for $r \leq s \leq \infty$. Hence, for a degenerate spectral sequence, we may conclude that $E_\infty^{n-q,q} = (0)$ for all $q \neq q(n)$; in fact $E_r^{n-q,q} = E_\infty^{n-q,q}$ under these circumstances. However, $H^n(A)$ is filtered and its associated graded object is the direct sum $\coprod_q E_\infty^{n-q,q}$. As only one term is nonzero in this direct sum, we deduce that $H^n(A) = E_\infty^{n-q(n),q(n)}$. This proves

Proposition B.31 *If the filtration of A is regular and the spectral sequence*

$$E_2^{p,q} \Longrightarrow H^\bullet(A)$$

degenerates at r, then we have isomorphisms

$$E_r^{n-q(n),q(n)} \xrightarrow{\sim} H^n(A).$$

If there exist integers n, p_0, p_1 with $p_1 > p_0$ and $E_\infty^{\nu,n-\nu} = (0)$ for $\nu \neq p_0$ and $\nu \neq p_1$, then our description of E_∞ yields an exact sequence

$$0 \longrightarrow E_\infty^{p_1,n-p_1} \longrightarrow H^n \longrightarrow E_\infty^{p_0,n-p_0} \longrightarrow 0,$$

which should be viewed as a generalization of Proposition B.31.

We shall now give a series of technical propositions which will result in three exact sequences (I), (II), (III) below. These three sequences will then yield several important theorems about spectral sequences which are very useful for applications. The advantage of this method is that the important theorems are proved simultaneously, the disadvantage is that the treatment is technical and abstract. For this reason we advise the reader to skip the proof of Propositions B.32, B.33, B.34 on the first reading.

Proposition B.32 *Let $\{E_r^{p,q}\}$ be a spectral sequence with regular filtration. Assume that there exist integers r, p_0, p_1, n such that*

$$E_r^{u,v} = (0) \quad \textit{for} \quad \begin{cases} u+v=n,\ u \neq p_0, p_1 \\ u+v=n+1,\ u \geq p_1+r \\ u+v=n-1,\ u \leq p_0-r. \end{cases}$$

Then we have an exact sequence

$$E_r^{p_1,n-p_1} \longrightarrow H^n \longrightarrow E_r^{p_0,n-p_0}. \tag{I}$$

Proof. The first hypothesis on $E_r^{u,v}$ yields the exact sequence

$$0 \longrightarrow E_\infty^{p_1,n-p_1} \longrightarrow H^n \longrightarrow E_\infty^{p_0,n-p_0} \longrightarrow 0.$$

If $r \leq t < \infty$, then $B_{t+1}^{p_0,n-p_0}/B_t^{p_0,n-p_0} = \mathrm{Im}\ d_t^{p_0-t,n-p_0+t-1}$.
Since $u = p_0-t$, $v = n-p_0+t-1$ satisfies the third hypothesis on $E_r^{u,v}$, we deduce $B_{t+1}^{p_0,n-p_0} = B_t^{p_0,n-p_0}$; hence, $B_\infty^{p_0,n-p_0} = B_r^{p_0,n-p_0}$. There results a monomorphism $E_\infty^{p_0,n-p_0} \longrightarrow E_r^{p_0,n-p_0}$. Dually, the second hypothesis yields an epimorphism $E_r^{p_1,n-p_1} \longrightarrow E_\infty^{p_1,n-p_1}$, and completes the proof. ☐

Proposition B.33 *Let $\{E_r^{p,q}\}$ be a spectral sequence with regular filtration. Assume that there exist integers r, s, p, n such that $s \geq r$ and*

$$E_r^{u,v} = (0) \quad \textit{for} \quad \begin{cases} u+v=n-1,\ u \leq p-r \\ u+v=n,\ u \neq p,\ u \leq p+s-r \\ u+v=n+1,\ p+r \leq u \neq p+s. \end{cases}$$

Then we have an exact sequence

$$H^n \longrightarrow E_r^{p,n-p} \longrightarrow E_r^{p+s,(n+1)-(p+s)}. \tag{II}$$

Proof. We first claim that if $E_r^{p-s,n-p+s-1} = (0)$ then we have an exact sequence

$$0 \longrightarrow E_{s+1}^{p,n-p} \xrightarrow{\text{incl}} E_s^{p,n-p} \xrightarrow{d_s^{p,n-p}} E_s^{p+s,(n-p)-s+1}.$$

For, the assertion $E_r^{p-s,n-p+s-1} = (0)$ implies the assertion $E_s^{p-s,n-p+s-1} = (0)$ for all $s \geq r$. Hence $d_s^{p-s,n-p+s-1}$ vanishes, as does its image $B_{s+1}^{p,n-p}/B_s^{p,n-p}$. Consequently,

$$E_{s+1}^{p,n-p} = Z_{s+1}^{p,n-p}/B_{s+1}^{p,n-p} \hookrightarrow Z_s^{p,n-p}/B_{s+1}^{p,n-p} = Z_s^{p,n-p}/B_s^{p,n-p} = E_s^{p,n-p}.$$

The kernel of $d_s^{p,n-p}$ is $Z_{s+1}^{p,n-p}/B_s^{p,n-p} = E_{s+1}^{p,n-p}$, as required.

It follows from the first and third hypothesis on $E_r^{u,v}$ that

(a) $E_{s+1}^{p,n-p} = E_\infty^{p,n-p}$,

(b) $E_r^{p,n-p} = E_s^{p,n-p}$, and

(c) $0 \longrightarrow E_s^{p+s,(n+1)-(p+s)} \longrightarrow E_r^{p+s,(n+1)-(p+s)}$ is exact.

Hence, all that is necessary to prove (II) is the existence of a surjection $H^n \longrightarrow E_\infty^{p,n-p}$. This is trivial in virtue of the second hypothesis on $E_r^{u,v}$ and our remarks concerning the relationship of $E_\infty^{p,n-p}$ with the composition quotients of H^n. $\square$

In exactly the same manner, one proves

Proposition B.34 *Let $\{E_r^{p,q}\}$ be a spectral sequence with regular filtration. Assume that there exist integers $r, s\ (\geq r)$, p, n such*

$$E_r^{u,v} = (0) \quad \textit{for} \quad \begin{cases} u+v = n+1,\ u \geq p+r \\ u+v = n,\ p+r-s \leq u \neq p \\ u+v = n-1,\ p-s \neq u \leq p-r. \end{cases}$$

Then we have an exact sequence

$$E_r^{p-s,(n-1)-(p-s)} \longrightarrow E_r^{p,n-p} \longrightarrow H^n. \tag{III}$$

Here are the main applications of Propositions B.32, B.33, B.34.

Theorem B.35 *If $\{E_r^{p,q}\}$ is a spectral sequence with regular filtration and if there exist integers p_0, p_1, r with $r \geq 1$, $p_1 - p_0 \geq r$ such that $E_r^{u,v} = (0)$ for all $u \neq p_0, u \neq p_1$ then we have an exact sequence*

$$\cdots \longrightarrow E_r^{p_1,n-p_1} \longrightarrow H^n \longrightarrow E_r^{p_0,n-p_0} \longrightarrow E_r^{p_1,n+1-p_1} \longrightarrow H^{n+1} \longrightarrow \cdots.$$

Dually, if there exist integers q_0, q_1, r with $r \geq 2$ and $q_1 - q_0 \geq r-1$ such that $E_r^{u,v} = (0)$ for all $v \neq q_0, v \neq q_1$ then we have an exact sequence

$$\cdots \longrightarrow E_r^{n-q_0,q_0} \longrightarrow H^n \longrightarrow E_r^{n-q_1,q_1} \longrightarrow E_r^{n+1-q_0,q_0} \longrightarrow H^{n+1} \longrightarrow \cdots.$$

Proof. Let $s = p_1 - p_0 \geq r$, then Propositions B.32 and B.33 yield the required exact sequence. Let $s = 1 + q_1 - q_0 \geq r$, then again the hypothesis of Propositions B.32 and B.33 are satisfied; hence (I) and (II) yield the desired sequence. □

Theorem B.36 *Let $\{E_r^{p,q}\}$ be a spectral sequence with regular filtration. Assume $E_2^{p,q} = (0)$ for all q with $0 < q < n$ $(n > 0)$—no hypothesis if $n = 1$. Then $E_2^{r,0} \simeq H^r$ for $r = 0, 1, \ldots, n-1$ and*

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}$$

is exact. In particular, with no hypothesis on q, we have the "exact sequence of terms of low degree"

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2.$$

Proof. It follows easily from our hypotheses that $E_2^{r,0} \simeq E_\infty^{r,0}$ for $0 \leq r \leq n-1$, and we know that $E_\infty^{r,0} \simeq H^r$ under the hypotheses of the theorem. Moreover, one checks trivially that $E_2^{n,0} \simeq E_\infty^{n,0}$, so that there is a monomorphism

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n.$$

Exact sequence (I) applies to yield the exact sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n}.$$

Now take $s = n + 1 \geq 2$ and $p = n$. Then (II) yields

$$H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0}$$

is exact. Lastly, choose $s = n+1$, $p = n+1$ and apply (III). We obtain the exact sequence

$$E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1},$$

and a combination of all these sequences completes the proof. □

To use spectral sequences to connect cohomology theories depends on the notion of a double complex. By definition, a *double (cochain) complex* is a doubly graded object $C = \coprod_{p,q} C_{p,q}$, with $p, q \geq 0$, together with two *differentials* d' and d'', where

$$\begin{aligned} d'\colon \quad C_{p,q} &\longrightarrow C_{p+1,q} \\ d''\colon \quad C_{p,q} &\longrightarrow C_{p,q+1}; \end{aligned}$$

and we assume that $d' \circ d'' + d'' \circ d' = 0$. Under these conditions, if we write

$$C_n = \coprod_{p+q=n} C_{p,q}, \quad d = d' + d'',$$

then $C = \coprod_n C_n$ is a complex with differentiation d. The complex $C = \coprod_n C_n$ is called the *associated single complex* to $C = \coprod_{p,q} C_{p,q}$, and $d = d' + d''$ is called its *total differentiation*.

Given $p \geq 0$, let ${}'C^p = \coprod_{i \geq p,\, j} C_{ij}$, and given $q \geq 0$ let ${}''C^p = \coprod_{i,\, j \geq q} C_{ij}$. This gives us two filtrations of C considered as a single complex, each compatible with the total differential d. The former is called the *first filtration*; the latter is called the *second filtration*. By Theorem B.28, we get two spectral sequences

$$\begin{aligned} {}'E_2^{p,q} &\underset{p}{\Longrightarrow} H^\bullet(C) \\ {}''E_2^{p,q} &\underset{q}{\Longrightarrow} H^\bullet(C) \end{aligned}$$

with ending the cohomology of C under d, called the (first, second) spectral sequence of the double complex C.

Theorem B.37 *Let $C = \coprod_{p \geq 0, q \geq 0} C_{p,q}$ be a double complex. Then the ${}'E_2^{p,q}$ term of its first spectral sequence is canonically isomorphic to ${}'H^p({}''H^q(C))$, where ${}''H^q(C)$ means the q^{th} cohomology object of C with respect to d'', and ${}'H^p({}''H^q(C))$ means the p^{th} cohomology object of ${}''H^\bullet(C)$ considered as a complex with differentiation induced by d' from C. Hence, we obtain*

$${}'H^p({}''H^q(C)) \Longrightarrow H^\bullet(C).$$

Proof. Consider only the first filtration on C. The object ${}'\mathrm{gr}(C)$ is $\coprod_p {}'C^p/{}'C^{p+1}$; hence, is exactly $\coprod_p \coprod_q C_{p,q} = C$. Moreover, d induces the differentiation d'' on ${}'\mathrm{gr}(C)$, as one sees immediately. It follows from this that

$${}'E_1^{p,q} = H^{p,q}({}'\mathrm{gr}(C)) \; = \; {}''H^{p+q}(\coprod_j C_{p,j}).$$

Now, the differentiation $d_1^{p,q}$ on ${}'E_1^{p,q}$ is the connecting homomorphism for the cohomology of the exact sequence

$$0 \longrightarrow {}'C^{p+1}/{}'C^{p+2} \longrightarrow {}'C^p/{}'C^{p+2} \longrightarrow {}'C^p/{}'C^{p+1} \longrightarrow 0.$$

If ξ is an element of bidegree (p,q) in ${}''H({}'\mathrm{gr}(C))$, then ξ is represented by an element α of $C_{p,q}$. As $d''(\alpha) = 0$, we have $d(\alpha) = d'(\alpha)$; so, we see that d induces the differentiation d' on ${}''H({}'\mathrm{gr}(C))$. Upon recalling that the connecting homomorphism of cohomology is induced by the differentiation via the snake lemma, we obtain that $d_1^{p,q}$ on ${}'E_1^{p,q}$ corresponds exactly to the map induced on ${}''H({}'\mathrm{gr}(C))$ by the differentiation d' on C.

Therefore, the object ${}'E_2$ is canonically isomorphic to the cohomology, ${}'H({}''H(C))$, of ${}''H(C)$ with respect to the differentiation d' induced on it by C. From the above description of $d_1^{p,q}$ one finds that the elements of ${}'E_2^{p,q}$ are exactly those of ${}'H^p({}''H(C))$ which are represented by elements of ${}'C^p/{}'C^{p+1}$ of total degree $p+q$. These are the elements represented by the object $C_{p,q}$; and we finally obtain the isomorphism

$${}'E_2^{p,q} \simeq {}'H^p({}''H^q(C)). \qquad \square$$

The most frequently use corollary of Theorem B.37 is

Corollary B.38 *Let* $C = \coprod_{p\geq 0, q\geq 0} C_{p,q}$ *be a double (cochain) complex; suppose that for all* p
$$'H^p(''H^q(C)) = (0) \quad \textit{for all } q \geq 1.$$
Let D *be the subcomplex of* C *which consists of* $\coprod_p D_p$ *where* $D_p = \{\xi \in C_{p,0} \mid d_2(\xi) = 0\}$*. (Of course* d' *is the differentiation on* D*.) Then the cohomology of* C *with respect to* d *is isomorphic to the cohomology of* D *with respect to* d'*.*

Remark: This corollary is clearly a comparison theorem for cohomology.

Proof. By Theorem B.37 and Proposition B.31,
$$H^n(C) \simeq {'H^n}(''H^0(C)) = E_2^{n,0}(C).$$
The same is also true (for trivial reasons) for the complex D, where we obtain
$$H^n(D) = E_2^{n,0}(D).$$
Since $D \longrightarrow {''H^0}(C)$ is an isomorphism, we are done. $\square$

Actually, this corollary is usually applied under a slightly different guise. The double complex may be regarded as an array on the lattice points in the first quadrant of the plane as illustrated in Figure B.3:

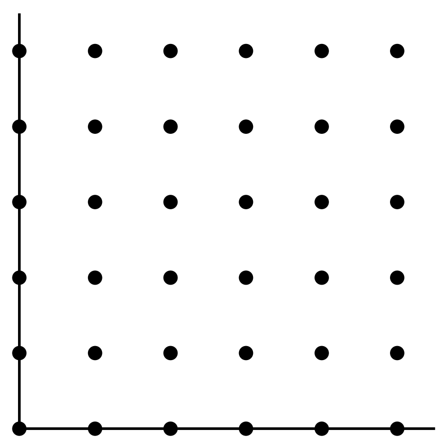

Figure B.3: A double complex

Suppose that the columns are acyclic. Then $''H^q(C)$ vanishes for every $q \geq 1$; consequently the corollary applies.

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