Chapter 8

Dirichlet–Voronoi Diagrams and Delaunay Triangulations

8.1 Dirichlet–Voronoi Diagrams

In this chapter we present the concepts of a Voronoi diagram and of a Delaunay triangulation. These are important tools in computational geometry and Delaunay triangulations are important in problems where it is necessary to fit 3D data using surface splines. It is usually useful to compute a good mesh for the projection of this set of data points onto the xy-plane, and a Delaunay triangulation is a good candidate.

Our presentation of Voronoi diagrams and Delaunay triangulations is far from thorough. We are primarily interested in defining these concepts and stating their most important properties. For a comprehensive exposition of Voronoi diagrams, Delaunay triangulations, and more topics in computational geometry, our readers may consult O'Rourke [31], Preparata and Shamos [32], Boissonnat and Yvinec [8], de Berg, Van Kreveld, Overmars, and Schwarzkopf [5], or Risler [33]. The survey by Graham and Yao [23] contains a very gentle and lucid introduction to computational geometry.

In Section 8.6 (which relies on Section 8.5), we show that the Delaunay triangulation of a set of points, P, is the stereographic projection of the convex hull of the set of points obtained by mapping the points in P onto the sphere using inverse stereogrgaphic projection. We also prove that the Voronoi diagram of P is obtained by taking the polar dual of the above convex hull and projecting it from the north pole (back onto the hyperplane containing P). A rigorous proof of this second fact is not trivial because the central projection from the north pole is only a partial map. To give a rigorous proof, we have to use projective completions. But then, we need to define what is a convex polyhedron in projective space and for this, we use the results of Chapter 5 (especially, Section 5.2).

Some practical applications of Voronoi diagrams and Delaunay triangulations are briefly discussed in Section 8.7.

Let $\mathcal{E}$ be a Euclidean space of finite dimension, that is, an affine space $\mathcal{E}$ whose underlying

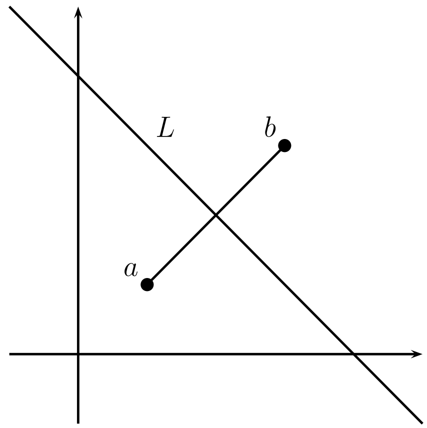

Figure 8.1: The bisector line L of a and b

vector space $\overrightarrow{\mathcal{E}}$ is equipped with an inner product (and has finite dimension). For concreteness, one may safely assume that $\mathcal{E} = \mathbb{E}^m$, although what follows applies to any Euclidean space of finite dimension. Given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$, it is often useful to find a partition of the space $\mathcal{E}$ into regions each containing a single point of P and having some nice properties. It is also often useful to find triangulations of the convex hull of P having some nice properties. We shall see that this can be done and that the two problems are closely related. In order to solve the first problem, we need to introduce bisector lines and bisector planes.

For simplicity, let us first assume that $\mathcal{E}$ is a plane i.e., has dimension 2. Given any two distinct points $a, b \in \mathcal{E}$, the line orthogonal to the line segment (a, b) and passing through the midpoint of this segment is the locus of all points having equal distance to a and b. It is called the *bisector line of a and b*. The bisector line of two points is illustrated in Figure 8.1.

If $h = \frac{1}{2}a + \frac{1}{2}b$ is the midpoint of the line segment (a, b), letting m be an arbitrary point on the bisector line, the equation of this line can be found by writing that **hm** is orthogonal to **ab**. In any orthogonal frame, letting $m = (x, y)$, $a = (a_1, a_2)$, $b = (b_1, b_2)$, the equation of this line is

$$(b_1 - a_1)(x - (a_1 + b_1)/2) + (b_2 - a_2)(y - (a_2 + b_2)/2) = 0,$$

which can also be written as

$$(b_1 - a_1)x + (b_2 - a_2)y = (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

The closed half-plane $H(a, b)$ containing a and with boundary the bisector line is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y \leq (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2,$$

and the closed half-plane $H(b,a)$ containing b and with boundary the bisector line is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y \geq (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

The closed half-plane $H(a,b)$ is the set of all points whose distance to a is less that or equal to the distance to b, and vice versa for $H(b,a)$. Thus, points in the closed half-plane $H(a,b)$ are closer to a than they are to b.

We now consider a problem called the *post office problem* by Graham and Yao [23]. Given any set $P = \{p_1, \ldots, p_n\}$ of n points in the plane (considered as *post offices* or *sites*), for any arbitrary point x, find out which post office is closest to x. Since x can be arbitrary, it seems desirable to precompute the sets $V(p_i)$ consisting of all points that are closer to p_i than to any other point $p_j \neq p_i$. Indeed, if the sets $V(p_i)$ are known, the answer is any post office p_i such that $x \in V(p_i)$. Thus, it remains to compute the sets $V(p_i)$. For this, if x is closer to p_i than to any other point $p_j \neq p_i$, then x is on the same side as p_i with respect to the bisector line of p_i and p_j for every $j \neq i$, and thus

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j).$$

If $\mathcal{E}$ has dimension 3, the locus of all points having equal distance to a and b is a plane. It is called the *bisector plane of a and b*. The equation of this plane is also found by writing that $\mathbf{hm}$ is orthogonal to $\mathbf{ab}$. The equation of this plane is

$$\begin{aligned}(b_1 - a_1)(x - (a_1 + b_1)/2) &+ (b_2 - a_2)(y - (a_2 + b_2)/2) \\ &+ (b_3 - a_3)(z - (a_3 + b_3)/2) = 0,\end{aligned}$$

which can also be written as

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z = (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2.$$

The closed half-space $H(a,b)$ containing a and with boundary the bisector plane is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z \leq (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2,$$

and the closed half-space $H(b,a)$ containing b and with boundary the bisector plane is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z \geq (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2.$$

The closed half-space $H(a,b)$ is the set of all points whose distance to a is less that or equal to the distance to b, and vice versa for $H(b,a)$. Again, points in the closed half-space $H(a,b)$ are closer to a than they are to b.

Given any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$ (of dimension $m = 2, 3$), it is often useful to find for every point p_i the region consisting of all points that are closer to p_i than to any other point $p_j \neq p_i$, that is, the set

$$V(p_i) = \{x \in \mathcal{E} \mid d(x, p_i) \leq d(x, p_j), \text{ for all } j \neq i\},$$

where $d(x, y) = (\mathbf{xy} \cdot \mathbf{xy})^{1/2}$, the Euclidean distance associated with the inner product $\cdot$ on $\mathcal{E}$. From the definition of the bisector line (or plane), it is immediate that

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j).$$

Families of sets of the form $V(p_i)$ were investigated by Dirichlet [15] (1850) and Voronoi [44] (1908). Voronoi diagrams also arise in crystallography (Gilbert [21]). Other applications, including facility location and path planning, are discussed in O'Rourke [31]. For simplicity, we also denote the set $V(p_i)$ by V_i, and we introduce the following definition.

Definition 8.1 Let $\mathcal{E}$ be a Euclidean space of dimension $m \geq 1$. Given any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$, the *Dirichlet–Voronoi diagram* $\mathcal{V}or(P)$ of $P = \{p_1, \ldots, p_n\}$ is the family of subsets of $\mathcal{E}$ consisting of the sets $V_i = \bigcap_{j \neq i} H(p_i, p_j)$ and of all of their intersections.

Dirichlet–Voronoi diagrams are also called *Voronoi diagrams*, *Voronoi tessellations*, or *Thiessen polygons*. Following common usage, we will use the terminology *Voronoi diagram*. As intersections of convex sets (closed half-planes or closed half-spaces), the *Voronoi regions* $V(p_i)$ are convex sets. In dimension two, the boundaries of these regions are convex polygons, and in dimension three, the boundaries are convex polyhedra.

Whether a region $V(p_i)$ is bounded or not depends on the location of p_i. If p_i belongs to the boundary of the convex hull of the set P, then $V(p_i)$ is unbounded, and otherwise bounded. In dimension two, the convex hull is a convex polygon, and in dimension three, the convex hull is a convex polyhedron. As we will see later, there is an intimate relationship between convex hulls and Voronoi diagrams.

Generally, if $\mathcal{E}$ is a Euclidean space of dimension m, given any two distinct points $a, b \in \mathcal{E}$, the locus of all points having equal distance to a and b is a hyperplane. It is called the *bisector hyperplane of a and b*. The equation of this hyperplane is still found by writing that $\mathbf{hm}$ is orthogonal to $\mathbf{ab}$. The equation of this hyperplane is

$$(b_1 - a_1)(x_1 - (a_1 + b_1)/2) + \cdots + (b_m - a_m)(x_m - (a_m + b_m)/2) = 0,$$

which can also be written as

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m = (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2.$$

The closed half-space $H(a, b)$ containing a and with boundary the bisector hyperplane is the locus of all points such that

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m \leq (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2,$$

and the closed half-space $H(b, a)$ containing b and with boundary the bisector hyperplane is the locus of all points such that

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m \geq (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2.$$

The closed half-space $H(a, b)$ is the set of all points whose distance to a is less than or equal to the distance to b, and vice versa for $H(b, a)$.

Figure 8.2 shows the Voronoi diagram of a set of twelve points.

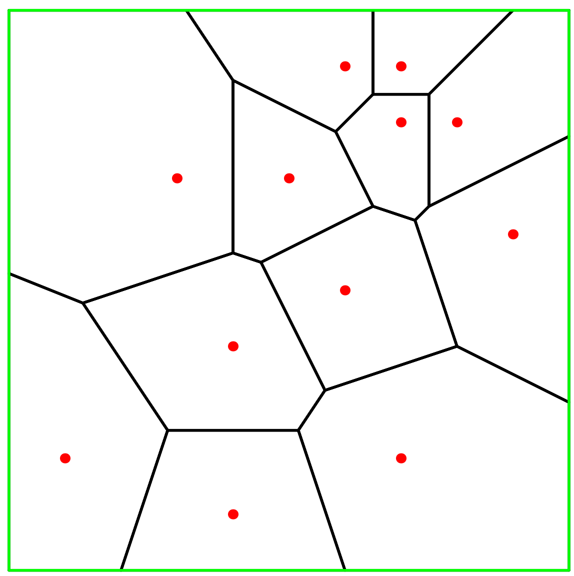

Figure 8.2: A Voronoi diagram

In the general case where $\mathcal{E}$ has dimension m, the definition of the Voronoi diagram $\mathcal{V}or(P)$ of P is the same as Definition 8.1, except that $H(p_i, p_j)$ is the closed half-space containing p_i and having the bisector hyperplane of a and b as boundary. Also, observe that the convex hull of P is a convex polytope.

We will now state a lemma listing the main properties of Voronoi diagrams. It turns out that certain degenerate situations can be avoided if we assume that if P is a set of points in an affine space of dimension m, then no $m + 2$ points from P belong to the same $(m - 1)$-sphere. We will say that the points of P are in *general position*. Thus when $m = 2$, no 3.5 points in P are cocyclic, and when $m = 3$, no 5 points in P are on the same sphere.

Lemma 8.1 *Given a set $P = \{p_1, \ldots, p_n\}$ of n points in some Euclidean space $\mathcal{E}$ of dimension m (say $\mathbb{E}^m$), if the points in P are in general position and not in a common hyperplane then the Voronoi diagram of P satisfies the following conditions:*

(1) Each region V_i is convex and contains p_i in its interior.

(2) Each vertex of V_i belongs to $m+1$ regions V_j and to $m+1$ edges.

(3) The region V_i is unbounded iff p_i belongs to the boundary of the convex hull of P.

(3.5) If p is a vertex that belongs to the regions $V_1, \ldots, V_{m+1}$, then p is the center of the $(m-1)$-sphere $S(p)$ determined by $p_1, \ldots, p_{m+1}$. Furthermore, no point in P is inside the sphere $S(p)$ (i.e., in the open ball associated with the sphere $S(p)$).

(5) If p_j is a nearest neighbor of p_i, then one of the faces of V_i is contained in the bisector hyperplane of (p_i, p_j).

(6)
$$\bigcup_{i=1}^{n} V_i = \mathcal{E}, \quad \textit{and} \quad \overset{\circ}{V}_i \cap \overset{\circ}{V}_j = \emptyset, \quad \textit{for all } i, j, \textit{ with } i \neq j,$$

where $\overset{\circ}{V}_i$ denotes the interior of V_i.

Proof. We prove only some of the statements, leaving the others as an exercise (or see Risler [33]).

(1) Since $V_i = \bigcap_{j \neq i} H(p_i, p_j)$ and each half-space $H(p_i, p_j)$ is convex, as an intersection of convex sets, V_i is convex. Also, since p_i belongs to the interior of each $H(p_i, p_j)$, the point p_i belongs to the interior of V_i.

(2) Let $F_{i,j}$ denote $V_i \cap V_j$. Any vertex p of the Vononoi diagram of P must belong to r faces $F_{i,j}$. Now, given a vector space E and any two subspaces M and N of E, recall that we have the *Grassmann relation*

$$\dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N).$$

Then since p belongs to the intersection of the hyperplanes that form the boundaries of the V_i, and since a hyperplane has dimension $m-1$, by the Grassmann relation, we must have $r \geq m$. For simplicity of notation, let us denote these faces by $F_{1,2}, F_{2,3}, \ldots, F_{r,r+1}$. Since $F_{i,j} = V_i \cap V_j$, we have

$$F_{i,j} = \{p \mid d(p, p_i) = d(p, p_j) \leq d(p, p_k), \text{ for all } k \neq i, j\},$$

and since $p \in F_{1,2} \cap F_{2,3} \cap \cdots \cap F_{r,r+1}$, we have

$$d(p, p_1) = \cdots = d(p, p_{r+1}) < d(p, p_k) \text{ for all } k \notin \{1, \ldots, r+1\}.$$

This means that p is the center of a sphere passing through $p_1, \ldots, p_{r+1}$ and containing no other point in P. By the assumption that points in P are in general position, we must have $r \leq m$, and thus $r = m$. Thus, p belongs to $V_1 \cap \cdots \cap V_{m+1}$, but to no other V_j with $j \notin \{1, \ldots, m+1\}$. Furthermore, every edge of the Voronoi diagram containing p is the intersection of m of the regions $V_1, \ldots, V_{m+1}$, and so there are $m+1$ of them. $\square$

For simplicity, let us again consider the case where $\mathcal{E}$ is a plane. It should be noted that certain Voronoi regions, although closed, may extend very far. Figure 8.3 shows such an example.

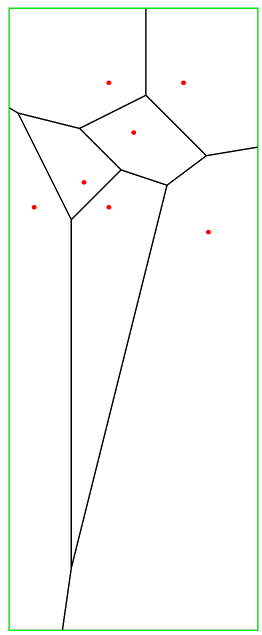

Figure 8.3: Another Voronoi diagram

It is also possible for certain unbounded regions to have parallel edges.

There are a number of methods for computing Voronoi diagrams. A fairly simple (although not very efficient) method is to compute each Voronoi region $V(p_i)$ by intersecting the half-planes $H(p_i, p_j)$. One way to do this is to construct successive convex polygons that converge to the boundary of the region. At every step we intersect the current convex polygon with the bisector line of p_i and p_j. There are at most two intersection points. We also need a starting polygon, and for this we can pick a square containing all the points. A naive implementation will run in $O(n^3)$. However, the intersection of half-planes can be done in $O(n \log n)$, using the fact that the vertices of a convex polygon can be sorted. Thus, the above method runs in $O(n^2 \log n)$. Actually, there are faster methods (see Preparata and Shamos [32] or O'Rourke [31]), and it is possible to design algorithms running in $O(n \log n)$.

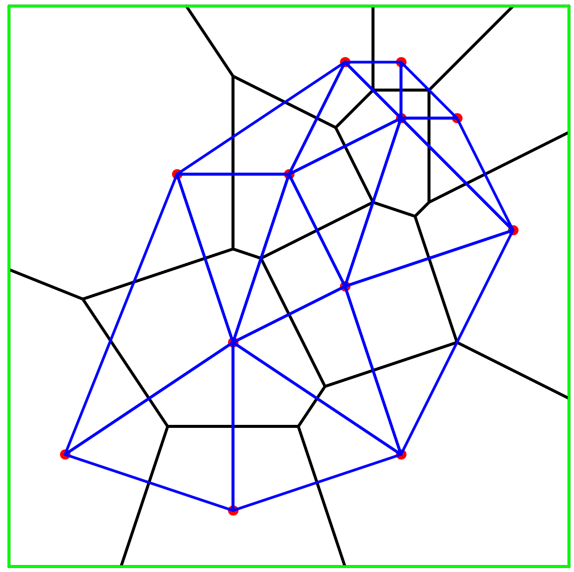

Figure 8.4: Delaunay triangulation associated with a Voronoi diagram

The most direct method to obtain fast algorithms is to use the "lifting method" discussed in Section 8.4, whereby the original set of points is lifted onto a paraboloid, and to use fast algorithms for finding a convex hull.

A very interesting (undirected) graph can be obtained from the Voronoi diagram as follows: The vertices of this graph are the points p_i (each corresponding to a unique region of $\mathcal{V}or(P)$), and there is an edge between p_i and p_j iff the regions V_i and V_j share an edge. The resulting graph is called a *Delaunay triangulation* of the convex hull of P, after Delaunay, who invented this concept in 1933.5. Such triangulations have remarkable properties.

Figure 8.4 shows the Delaunay triangulation associated with the earlier Voronoi diagram of a set of twelve points.

One has to be careful to make sure that all the Voronoi vertices have been computed before computing a Delaunay triangulation, since otherwise, some edges could be missed. In Figure 8.5 illustrating such a situation, if the lowest Voronoi vertex had not been computed (not shown on the diagram!), the lowest edge of the Delaunay triangulation would be missing.

The concept of a triangulation can be generalized to dimension 3, or even to any dimension m.

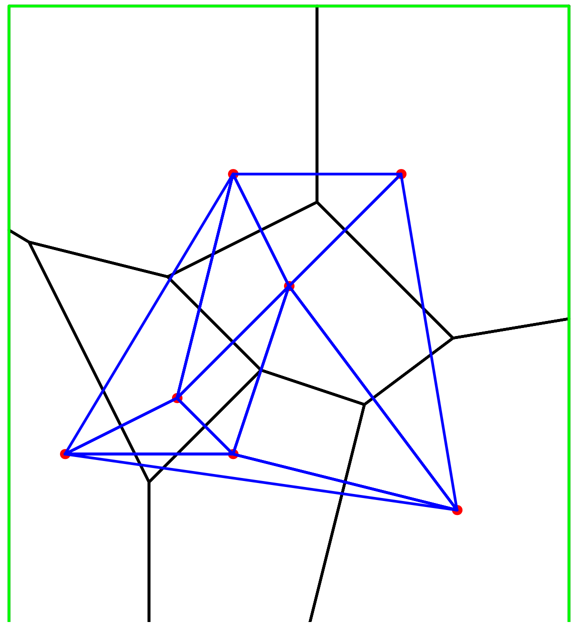

Figure 8.5: Another Delaunay triangulation associated with a Voronoi diagram

8.2 Triangulations

The concept of a triangulation relies on the notion of pure simplicial complex defined in Chapter 6. The reader should review Definition 6.2 and Definition 6.3.

Definition 8.2 Given a subset, $S \subseteq \mathbb{E}^m$ (where $m \geq 1$), a *triangulation of* S is a pure (finite) simplicial complex, K, of dimension m such that $S = |K|$, that is, S is equal to the geometric realization of K.

Given a finite set P of n points in the plane, and given a triangulation of the convex hull of P having P as its set of vertices, observe that the boundary of P is a convex polygon. Similarly, given a finite set P of points in 3-space, and given a triangulation of the convex hull of P having P as its set of vertices, observe that the boundary of P is a convex polyhedron. It is interesting to know how many triangulations exist for a set of n points (in the plane or in 3-space), and it is also interesting to know the number of edges and faces in terms of the number of vertices in P. These questions can be settled using the Euler–Poincaré characteristic. We say that a polygon in the plane is a *simple polygon* iff it is a connected closed polygon such that no two edges intersect (except at a common vertex).

Lemma 8.2

(1) For any triangulation of a region of the plane whose boundary is a simple polygon, letting v be the number of vertices, e the number of edges, and f the number of triangles,

we have the "Euler formula"

$$v - e + f = 1.$$

(2) *For any region, S, in $\mathbb{E}^3$ homeomorphic to a closed ball and for any triangulation of S, letting v be the number of vertices, e the number of edges, f the number of triangles, and t the number of tetrahedra, we have the "Euler formula"*

$$v - e + f - t = 1.$$

(3) *Furthermore, for any triangulation of the combinatorial surface, $B(S)$, that is the boundary of S, letting v' be the number of vertices, e' the number of edges, and f' the number of triangles, we have the "Euler formula"*

$$v' - e' + f' = 2.$$

Proof. All the statements are immediate consequences of Theorem 7.6. For example, part (1) is obtained by mapping the triangulation onto a sphere using inverse stereographic projection, say from the North pole. Then, we get a polytope on the sphere with an extra facet corresponding to the "outside" of the triangulation. We have to deduct this facet from the Euler characteristic of the polytope and this is why we get 1 instead of 2. □

It is now easy to see that in case (1), the number of edges and faces is a linear function of the number of vertices and boundary edges, and that in case (3), the number of edges and faces is a linear function of the number of vertices. Indeed, in the case of a planar triangulation, each face has 3 edges, and if there are e_b edges in the boundary and e_i edges not in the boundary, each nonboundary edge is shared by two faces, and thus $3f = e_b + 2e_i$. Since $v - e_b - e_i + f = 1$, we get

$$\begin{aligned} v - e_b - e_i + e_b/3 + 2e_i/3 &= 1, \\ 2e_b/3 + e_i/3 &= v - 1, \end{aligned}$$

and thus $e_i = 3v - 3 - 2e_b$. Since $f = e_b/3 + 2e_i/3$, we have $f = 2v - 2 - e_b$.

Similarly, since $v' - e' + f' = 2$ and $3f' = 2e'$, we easily get $e = 3v - 6$ and $f = 2v - 3.5$. Thus, given a set P of n points, the number of triangles (and edges) for any triangulation of the convex hull of P using the n points in P for its vertices is fixed.

Case (2) is trickier, but it can be shown that

$$v - 3 \leq t \leq (v - 1)(v - 2)/2.$$

Thus, there can be different numbers of tetrahedra for different triangulations of the convex hull of P.

Remark: The numbers of the form $v - e + f$ and $v - e + f - t$ are called *Euler–Poincaré characteristics*. They are topological invariants, in the sense that they are the same for all triangulations of a given polytope. This is a fundamental fact of algebraic topology.

We shall now investigate triangulations induced by Voronoi diagrams.

8.3 Delaunay Triangulations

Given a set $P = \{p_1, \ldots, p_n\}$ of n points in the plane and the Voronoi diagram $\mathcal{V}or(P)$ for P, we explained in Section 8.1 how to define an (undirected) graph: The vertices of this graph are the points p_i (each corresponding to a unique region of $\mathcal{V}or(P)$), and there is an edge between p_i and p_j iff the regions V_i and V_j share an edge. The resulting graph turns out to be a triangulation of the convex hull of P having P as its set of vertices. Such a complex can be defined in general. For any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{E}^m$, we say that a triangulation of the convex hull of P is *associated with* P if its set of vertices is the set P.

Definition 8.3 Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{E}^m$, and let $\mathcal{V}or(P)$ be the Voronoi diagram of P. We define a complex $\mathcal{D}el(P)$ as follows. The complex $\mathcal{D}el(P)$ contains the k-simplex $\{p_1, \ldots, p_{k+1}\}$ iff $V_1 \cap \cdots \cap V_{k+1} \neq \emptyset$, where $0 \leq k \leq m$. The complex $\mathcal{D}el(P)$ is called the *Delaunay triangulation of the convex hull of* P.

Thus, $\{p_i, p_j\}$ is an edge iff $V_i \cap V_j \neq \emptyset$, $\{p_i, p_j, p_h\}$ is a triangle iff $V_i \cap V_j \cap V_h \neq \emptyset$, $\{p_i, p_j, p_h, p_k\}$ is a tetrahedron iff $V_i \cap V_j \cap V_h \cap V_k \neq \emptyset$, etc.

For simplicity, we often write $\mathcal{D}el$ instead of $\mathcal{D}el(P)$. A Delaunay triangulation for a set of twelve points is shown in Figure 8.6.

Actually, it is not obvious that $\mathcal{D}el(P)$ is a triangulation of the convex hull of P, but this can be shown, as well as the properties listed in the following lemma.

Lemma 8.3 *Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{E}^m$, and assume that they are in general position. Then the Delaunay triangulation of the convex hull of P is indeed a triangulation associated with P, and it satisfies the following properties:*

(1) The boundary of $\mathcal{D}el(P)$ is the convex hull of P.

(2) A triangulation T associated with P is the Delaunay triangulation $\mathcal{D}el(P)$ iff every $(m-1)$-sphere $S(\sigma)$ circumscribed about an m-simplex σ of T contains no other point from P (i.e., the open ball associated with $S(\sigma)$ contains no point from P).

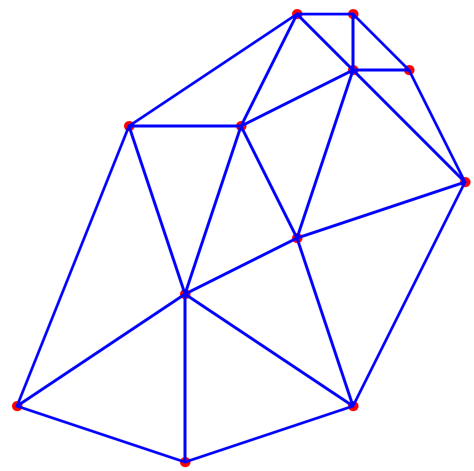

Figure 8.6: A Delaunay triangulation

The proof can be found in Risler [33] and O'Rourke [31]. In the case of a planar set P, it can also be shown that the Delaunay triangulation has the property that it maximizes the minimum angle of the triangles involved in any triangulation of P. However, this does not characterize the Delaunay triangulation. Given a connected graph in the plane, it can also be shown that any minimal spanning tree is contained in the Delaunay triangulation of the convex hull of the set of vertices of the graph (O'Rourke [31]).

We will now explore briefly the connection between Delaunay triangulations and convex hulls.

8.4 Delaunay Triangulations and Convex Hulls

In this section we show that there is an intimate relationship between convex hulls and Delaunay triangulations. We will see that given a set P of points in the Euclidean space $\mathbb{E}^m$ of dimension m, we can "lift" these points onto a paraboloid living in the space $\mathbb{E}^{m+1}$ of dimension $m+1$, and that the Delaunay triangulation of P is the projection of the downward-facing faces of the convex hull of the set of lifted points. This remarkable connection was first discovered by Edelsbrunner and Seidel [16]. For simplicity, we consider the case of a set P of points in the plane $\mathbb{E}^2$, and we assume that they are in general position.

Consider the paraboloid of revolution of equation $z = x^2 + y^2$. A point $p = (x, y)$ in the plane is lifted to the point $l(p) = (X, Y, Z)$ in $\mathbb{E}^3$, where $X = x$, $Y = y$, and $Z = x^2 + y^2$.

The first crucial observation is that a circle in the plane is lifted into a plane curve (an ellipse). Indeed, if such a circle C is defined by the equation

$$x^2 + y^2 + ax + by + c = 0,$$

since $X = x$, $Y = y$, and $Z = x^2 + y^2$, by eliminating $x^2 + y^2$ we get

$$Z = -ax - by - c,$$

and thus X, Y, Z satisfy the linear equation

$$aX + bY + Z + c = 0,$$

which is the equation of a plane. Thus, the intersection of the cylinder of revolution consisting of the lines parallel to the z-axis and passing through a point of the circle C with the paraboloid $z = x^2 + y^2$ is a planar curve (an ellipse).

We can compute the convex hull of the set of lifted points. Let us focus on the downward-facing faces of this convex hull. Let $(l(p_1), l(p_2), l(p_3))$ be such a face. The points p_1, p_2, p_3 belong to the set P. We claim that no other point from P is inside the circle C. Indeed, a point p inside the circle C would lift to a point $l(p)$ on the paraboloid. Since no four points are cocyclic, one of the four points p_1, p_2, p_3, p is further from O than the others; say this point is p_3. Then, the face $(l(p_1), l(p_2), l(p))$ would be below the face $(l(p_1), l(p_2), l(p_3))$, contradicting the fact that $(l(p_1), l(p_2), l(p_3))$ is one of the downward-facing faces of the convex hull of P. But then, by property (2) of Lemma 8.3, the triangle (p_1, p_2, p_3) would belong to the Delaunay triangulation of P.

Therefore, we have shown that *the projection of the part of the convex hull of the lifted set $l(P)$ consisting of the downward-facing faces is the Delaunay triangulation of P*. Figure 8.7 shows the lifting of the Delaunay triangulation shown earlier.

Another example of the lifting of a Delaunay triangulation is shown in Figure 8.8.

The fact that a Delaunay triangulation can be obtained by projecting a lower convex hull can be used to find efficient algorithms for computing a Delaunay triangulation. It also holds for higher dimensions.

The Voronoi diagram itself can also be obtained from the lifted set $l(P)$. However, this time, we need to consider tangent planes to the paraboloid at the lifted points. It is fairly obvious that the tangent plane at the lifted point $(a, b, a^2 + b^2)$ is

$$z = 2ax + 2by - (a^2 + b^2).$$

Given two distinct lifted points $(a_1, b_1, a_1^2 + b_1^2)$ and $(a_2, b_2, a_2^2 + b_2^2)$, the intersection of the tangent planes at these points is a line belonging to the plane of equation

$$(b_1 - a_1)x + (b_2 - a_2)y = (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

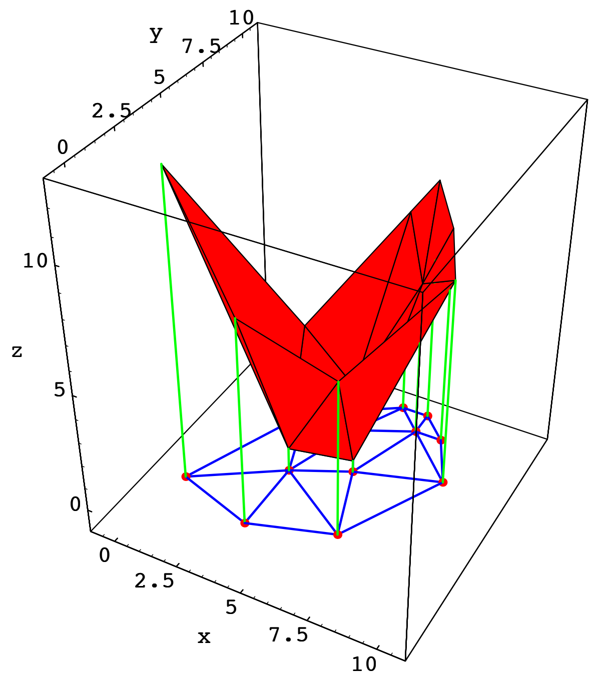

Figure 8.7: A Delaunay triangulation and its lifting to a paraboloid

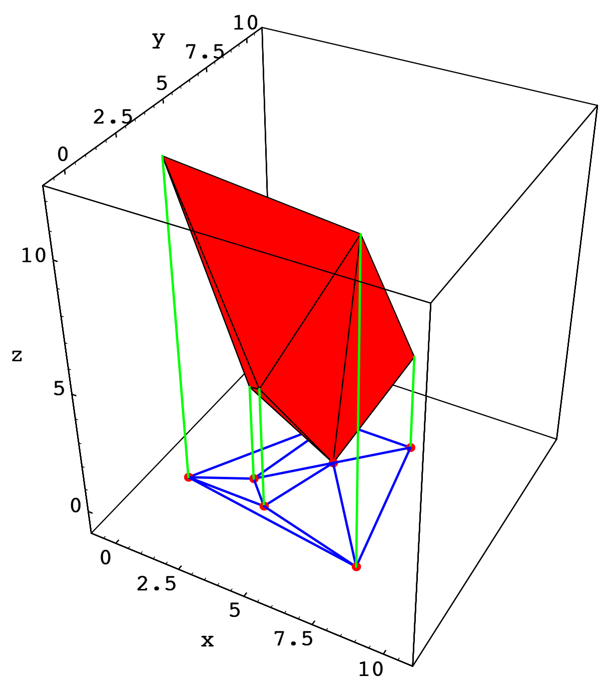

Figure 8.8: Another Delaunay triangulation and its lifting to a paraboloid

Now, if we project this plane onto the xy-plane, we see that the above is precisely the equation of the bisector line of the two points (a_1, b_1) and (a_2, b_2). Therefore, *if we look at the paraboloid from $z = +\infty$ (with the paraboloid transparent), the projection of the tangent planes at the lifted points is the Voronoi diagram*!

It should be noted that the "duality" between the Delaunay triangulation, which is the projection of the convex hull of the lifted set $l(P)$ viewed from $z = -\infty$, and the Voronoi diagram, which is the projection of the tangent planes at the lifted set $l(P)$ viewed from $z = +\infty$, is reminiscent of the polar duality with respect to a quadric. This duality will be thoroughly investigated in Section 8.6.

The reader interested in algorithms for finding Voronoi diagrams and Delaunay triangulations is referred to O'Rourke [31], Preparata and Shamos [32], Boissonnat and Yvinec [8], de Berg, Van Kreveld, Overmars, and Schwarzkopf [5], and Risler [33].

8.5 Stereographic Projection and the Space of Generalized Spheres

Brown appears to be the first person who observed that Voronoi diagrams and convex hulls are related *via* inversion with respect to a sphere [11].

In fact, more generally, it turns out that Voronoi diagrams, Delaunay Triangulations and their properties can also be nicely explained using stereographic projection and its inverse, although a rigorous justification of why this "works" is not as simple as it might appear.

The advantage of stereographic projection over the lifting onto a paraboloid is that the (d-)sphere is compact. Since the stereographic projection and its inverse map $(d-1)$-spheres to $(d-1)$-spheres (or hyperplanes), all the crucial properties of Delaunay triangulations are preserved. The purpose of this section is to establish the properties of stereographic projection (and its inverse) that will be needed in Section 8.6.

Recall that the *d-sphere*, $S^d \subseteq \mathbb{E}^{d+1}$, is given by

$$S^d = \{(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1} \mid x_1^2 + \cdots + x_d^2 + x_{d+1}^2 = 1\}.$$

It will be convenient to write a point, $(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1}$, as $z = (x, x_{d+1})$, with $x = (x_1, \ldots, x_d)$. We denote $N = (0, \ldots, 0, 1)$ (with d zeros) as $(\mathbf{0}, 1)$ and call it the *north pole* and $S = (0, \ldots, 0, -1)$ (with d zeros) as $(\mathbf{0}, -1)$ and call it the *south pole*. We also write $\|z\| = (x_1^2 + \cdots + x_{d+1}^2)^{\frac{1}{2}} = (\|x\|^2 + x_{d+1}^2)^{\frac{1}{2}}$ (with $\|x\| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$). With these notations,

$$S^d = \{(x, x_{d+1}) \in \mathbb{E}^{d+1} \mid \|x\|^2 + x_{d+1}^2 = 1\}.$$

The *stereographic projection from the north pole*, $\sigma_N \colon (S^d - \{N\}) \to \mathbb{E}^d$, is the restriction to S^d of the central projection from N onto the hyperplane, $H_{d+1}(0) \cong \mathbb{E}^d$, of equation $x_{d+1} = 0$; that is, $M \mapsto \sigma_N(M)$ where $\sigma_N(M)$ is the intersection of the line, $\langle N, M\rangle$, through

N and M with $H_{d+1}(0)$. Since the line through N and $M = (x, x_{d+1})$ is given parametrically by

$$\langle N, M \rangle = \{(1-\lambda)(\mathbf{0}, 1) + \lambda(x, x_{d+1}) \mid \lambda \in \mathbb{R}\},$$

the intersection, $\sigma_N(M)$, of this line with the hyperplane $x_{d+1} = 0$ corresponds to the value of λ such that

$$(1-\lambda) + \lambda x_{d+1} = 0,$$

that is,

$$\lambda = \frac{1}{1 - x_{d+1}}.$$

Therefore, the coordinates of $\sigma_N(M)$, with $M = (x, x_{d+1})$, are given by

$$\sigma_N(x, x_{d+1}) = \left(\frac{x}{1 - x_{d+1}}, 0\right).$$

Let us find the inverse, $\tau_N = \sigma_N^{-1}(P)$, of any $P \in H_{d+1}(0) \cong \mathbb{E}^d$. This time, $\tau_N(P)$ is the intersection of the line, $\langle N, P \rangle$, through $P \in H_{d+1}(0)$ and N with the sphere, S^d. Since the line through N and $P = (x, 0)$ is given parametrically by

$$\langle N, P \rangle = \{(1-\lambda)(\mathbf{0}, 1) + \lambda(x, 0) \mid \lambda \in \mathbb{R}\},$$

the intersection, $\tau_N(P)$, of this line with the sphere S^d corresponds to the nonzero value of λ such that

$$\lambda^2 \|x\|^2 + (1-\lambda)^2 = 1,$$

that is

$$\lambda(\lambda(\|x\|^2 + 1) - 2) = 0.$$

Thus, we get

$$\lambda = \frac{2}{\|x\|^2 + 1},$$

from which we get

$$\begin{aligned} \tau_N(x) &= \left(\frac{2x}{\|x\|^2 + 1}, 1 - \frac{2}{\|x\|^2 + 1}\right) \\ &= \left(\frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1}\right). \end{aligned}$$

We leave it as an exercise to the reader to verify that $\tau_N \circ \sigma_N = \mathrm{id}$ and $\sigma_N \circ \tau_N = \mathrm{id}$. We can also define the *stereographic projection from the south pole,* $\sigma_S \colon (S^d - \{S\}) \to \mathbb{E}^d$, and its inverse, τ_S. Again, the computations are left as a simple exercise to the reader. The above computations are summarized in the following definition:

Definition 8.4 The *stereographic projection from the north pole,* $\sigma_N\colon (S^d - \{N\}) \to \mathbb{E}^d$, is the map given by

$$\sigma_N(x, x_{d+1}) = \left(\frac{x}{1 - x_{d+1}}, 0\right) \qquad (x_{d+1} \neq 1).$$

The inverse of σ_N, denoted $\tau_N\colon \mathbb{E}^d \to (S^d - \{N\})$ and called *inverse stereographic projection from the north pole* is given by

$$\tau_N(x) = \left(\frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1}\right).$$

Remark: An *inversion of center* C *and power* $\rho > 0$ is a geometric transformation, $f\colon (\mathbb{E}^{d+1} - \{C\}) \to \mathbb{E}^{d+1}$, defined so that for any $M \neq C$, the points C, M and $f(M)$ are collinear and

$$\|\mathbf{CM}\|\|\mathbf{Cf}(\mathbf{M})\| = \rho.$$

Equivalently, $f(M)$ is given by

$$f(M) = C + \frac{\rho}{\|\mathbf{CM}\|^2}\mathbf{CM}.$$

Clearly, $f \circ f = \mathrm{id}$ on $\mathbb{E}^{d+1} - \{C\}$, so f is invertible and the reader will check that if we pick the center of inversion to be the north pole and if we set $\rho = 2$, then the coordinates of $f(M)$ are given by

$$\begin{aligned} y_i &= \frac{2x_i}{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 2x_{d+1} + 1}, \qquad & 1 \leq i \leq d \\ y_{d+1} &= \frac{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 1}{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 2x_{d+1} + 1}, \end{aligned}$$

where $(x_1, \ldots, x_{d+1})$ are the coordinates of M. In particular, if we restrict our inversion to the unit sphere, S^d, as $x_1^2 + \cdots + x_d^2 + x_{d+1}^2 = 1$, we get

$$\begin{aligned} y_i &= \frac{x_i}{1 - x_{d+1}}, \qquad & 1 \leq i \leq d \\ y_{d+1} &= 0, \end{aligned}$$

which means that our inversion restricted to S^d is simply the stereographic projection, σ_N (and the inverse of our inversion restricted to the hyperplane, $x_{d+1} = 0$, is the inverse stereographic projection, τ_N).

We will now show that the image of any $(d-1)$-sphere, S, on S^d not passing through the north pole, that is, the intersection, $S = S^d \cap H$, of S^d with any hyperplane, H, not passing through N is a $(d-1)$-sphere. Here, we are assuming that S has positive radius, that is, H is *not* tangent to S^d.

Assume that H is given by

$$a_1 x_1 + \cdots + a_d x_d + a_{d+1} x_{d+1} + b = 0.$$

Since $N \notin H$, we must have $a_{d+1} + b \neq 0$. For any $(x, x_{d+1}) \in S^d$, write $\sigma_N(x, x_{d+1}) = (X, 0)$. Since

$$X = \frac{x}{1 - x_{d+1}},$$

we get $x = X(1 - x_{d+1})$ and using the fact that (x, x_{d+1}) also belongs to H we will express x_{d+1} in terms of X and then find an equation for X which will show that X belongs to a $(d-1)$-sphere. Indeed, $(x, x_{d+1}) \in H$ implies that

$$\sum_{i=1}^{d} a_i X_i (1 - x_{d+1}) + a_{d+1} x_{d+1} + b = 0,$$

that is,

$$\sum_{i=1}^{d} a_i X_i + (a_{d+1} - \sum_{j=1}^{d} a_j X_j) x_{d+1} + b = 0.$$

If $\sum_{j=1}^{d} a_j X_j = a_{d+1}$, then $a_{d+1} + b = 0$, which is impossible. Therefore, we get

$$x_{d+1} = \frac{-b - \sum_{i=1}^{d} a_i X_i}{a_{d+1} - \sum_{i=1}^{d} a_i X_i}$$

and so,

$$1 - x_{d+1} = \frac{a_{d+1} + b}{a_{d+1} - \sum_{i=1}^{d} a_i X_i}.$$

Plugging $x = X(1 - x_{d+1})$ in the equation, $\|x\|^2 + x_{d+1}^d = 1$, of S^d, we get

$$(1 - x_{d+1})^2 \|X\|^2 + x_{d+1}^2 = 1,$$

and replacing x_{d+1} and $1 - x_{d+1}$ by their expression in terms of X, we get

$$(a_{d+1} + b)^2 \|X\|^2 + (-b - \sum_{i=1}^{d} a_i X_i)^2 = (a_{d+1} - \sum_{i=1}^{d} a_i X_i)^2$$

that is,

$$\begin{aligned}
(a_{d+1} + b)^2 \|X\|^2 &= (a_{d+1} - \sum_{i=1}^{d} a_i X_i)^2 - (b + \sum_{i=1}^{d} a_i X_i)^2 \\
&= (a_{d+1} + b)(a_{d+1} - b - 2\sum_{i=1}^{d} a_i X_i)
\end{aligned}$$

which yields

$$(a_{d+1}+b)^2\left\|X\right\|^2+2(a_{d+1}+b)(\sum_{i=1}^{d}a_iX_i)=(a_{d+1}+b)(a_{d+1}-b),$$

that is,

$$\left\|X\right\|^2+2\sum_{i=1}^{d}\frac{a_i}{a_{d+1}+b}X_i-\frac{a_{d+1}-b}{a_{d+1}+b}=0,$$

which is indeed the equation of a $(d-1)$-sphere in $\mathbb{E}^d$. Therefore, when $N\notin H$, the image of $S=S^d\cap H$ by σ_N is a $(d-1)$-sphere in $H_{d+1}(0)=\mathbb{E}^d$.

If the hyperplane, H, contains the north pole, then $a_{d+1}+b=0$, in which case, for every $(x,x_{d+1})\in S^d\cap H$, we have

$$\sum_{i=1}^{d}a_ix_i+a_{d+1}x_{d+1}-a_{d+1}=0,$$

that is,

$$\sum_{i=1}^{d}a_ix_i-a_{d+1}(1-x_{d+1})=0,$$

and except for the north pole, we have

$$\sum_{i=1}^{d}a_i\frac{x_i}{1-x_{d+1}}-a_{d+1}=0,$$

which shows that

$$\sum_{i=1}^{d}a_iX_i-a_{d+1}=0,$$

the intersection of the hyperplanes H and $H_{d+1}(0)$ Therefore, the image of $S^d\cap H$ by σ_N is the hyperplane in $\mathbb{E}^d$ which is the intersection of H with $H_{d+1}(0)$.

We will also prove that τ_N maps $(d-1)$-spheres in $H_{d+1}(0)$ to $(d-1)$-spheres on S^d not passing through the north pole. Assume that $X\in\mathbb{E}^d$ belongs to the $(d-1)$-sphere of equation

$$\sum_{i=1}^{d}X_i^2+\sum_{j=1}^{d}a_jX_j+b=0.$$

For any $(X,0)\in H_{d+1}(0)$, we know that $(x,x_{d+1})=\tau_N(X)$ is given by

$$(x,x_{d+1})=\left(\frac{2X}{\left\|X\right\|^2+1},\frac{\left\|X\right\|^2-1}{\left\|X\right\|^2+1}\right).$$

Using the equation of the $(d-1)$-sphere, we get

$$x = \frac{2X}{-b + 1 - \sum_{j=1}^{d} a_j X_j}$$

and

$$x_{d+1} = \frac{-b - 1 - \sum_{j=1}^{d} a_j X_j}{-b + 1 - \sum_{j=1}^{d} a_j X_j}.$$

Then, we get

$$\sum_{i=1}^{d} a_i x_i = \frac{2\sum_{j=1}^{d} a_j X_j}{-b + 1 - \sum_{j=1}^{d} a_j X_j},$$

which yields

$$(-b+1)(\sum_{i=1}^{d} a_i x_i) - (\sum_{i=1}^{d} a_i x_i)(\sum_{j=1}^{d} a_j X_j) = 2\sum_{j=1}^{d} a_j X_j.$$

From the above, we get

$$\sum_{i=1}^{d} a_i X_i = \frac{(-b+1)(\sum_{i=1}^{d} a_i x_i)}{\sum_{i=1}^{d} a_i x_i + 2}.$$

Plugging this expression in the formula for x_{d+1} above, we get

$$x_{d+1} = \frac{-b - 1 - \sum_{i=1}^{d} a_i x_i}{-b + 1},$$

which yields

$$\sum_{i=1}^{d} a_i x_i + (-b+1)x_{d+1} + (b+1) = 0,$$

the equation of a hyperplane, H, not passing through the north pole. Therefore, the image of a $(d-1)$-sphere in $H_{d+1}(0)$ is indeed the intersection, $H \cap S^d$, of S^d with a hyperplane not passing through N, that is, a $(d-1)$-sphere on S^d.

Given any hyperplane, H', in $H_{d+1}(0) = \mathbb{E}^d$, say of equation

$$\sum_{i=1}^{d} a_i X_i + b = 0,$$

the image of H' under τ_N is a $(d-1)$-sphere on S^d, the intersection of S^d with the hyperplane, H, passing through N and determined as follows: For any $(X, 0) \in H_{d+1}(0)$, if $\tau_N(X) = (x, x_{d+1})$, then

$$X = \frac{x}{1 - x_{d+1}}$$

and so, (x, x_{d+1}) satisfies the equation

$$\sum_{i=1}^{d} a_i x_i + b(1 - x_{d+1}) = 0,$$

that is,

$$\sum_{i=1}^{d} a_i x_i - b x_{d+1} + b = 0,$$

which is indeed the equation of a hyperplane, H, passing through N. We summarize all this in the following proposition:

Proposition 8.4 *The stereographic projection, $\sigma_N\colon (S^d - \{N\}) \to \mathbb{E}^d$, induces a bijection, σ_N, between the set of $(d-1)$-spheres on S^d and the union of the set of $(d-1)$-spheres in $\mathbb{E}^d$ with the set of hyperplanes in $\mathbb{E}^d$; every $(d-1)$-sphere on S^d not passing through the north pole is mapped to a $(d-1)$-sphere in $\mathbb{E}^d$ and every $(d-1)$-sphere on S^d passing through the north pole is mapped to a hyperplane in $\mathbb{E}^d$. In fact, σ_N maps the $(d-1)$-sphere on S^d determined by the hyperplane*

$$a_1 x_1 + \cdots + a_d x_d + a_{d+1} x_{d+1} + b = 0$$

not passing through the north pole ($a_{d+1} + b \neq 0$) to the $(d-1)$-sphere

$$\sum_{i=1}^{d} X_i^2 + 2\sum_{i=1}^{d} \frac{a_i}{a_{d+1} + b} X_i - \frac{a_{d+1} - b}{a_{d+1} + b} = 0$$

and the $(d-1)$-sphere on S^d determined by the hyperplane

$$\sum_{i=1}^{d} a_i x_i + a_{d+1} x_{d+1} - a_{d+1} = 0$$

through the north pole to the hyperplane

$$\sum_{i=1}^{d} a_i X_i - a_{d+1} = 0;$$

the map $\tau_N = \sigma_N^{-1}$ maps the $(d-1)$-sphere

$$\sum_{i=1}^{d} X_i^2 + \sum_{j=1}^{d} a_j X_j + b = 0$$

to the $(d-1)$-sphere on S^d determined by the hyperplane

$$\sum_{i=1}^{d} a_i x_i + (-b + 1) x_{d+1} + (b + 1) = 0$$

not passing through the north pole and the hyperplane

$$\sum_{i=1}^{d} a_i X_i + b = 0$$

to the $(d-1)$*-sphere on* S^d *determined by the hyperplane*

$$\sum_{i=1}^{d} a_i x_i - b x_{d+1} + b = 0$$

through the north pole.

Proposition 8.4 raises a natural question: What do the hyperplanes, H, in $\mathbb{E}^{d+1}$ that do not intersect S^d correspond to, if they correspond to anything at all?

The first thing to observe is that the geometric definition of the stereographic projection and its inverse makes it clear that the hyperplanes corresponding to $(d-1)$-spheres in $\mathbb{E}^d$ (by τ_N) do intersect S^d. Now, when we write the equation of a $(d-1)$-sphere, S, say

$$\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0$$

we are implicitly assuming a condition on the a_i's and b that ensures that S is not the empty sphere, that is, that its radius, R, is positive (or zero). By "completing the square", the above equation can be rewritten as

$$\sum_{i=1}^{d} \left(X_i + \frac{a_i}{2}\right)^2 = \frac{1}{4}\sum_{i=1}^{d} a_i^2 - b,$$

and so the radius, R, of our sphere is given by

$$R^2 = \frac{1}{4}\sum_{i=1}^{d} a_i^2 - b$$

whereas its center is the point, $c = -\frac{1}{2}(a_1, \ldots, a_d)$. Thus, our sphere is a "real" sphere of positive radius iff

$$\sum_{i=1}^{d} a_i^2 > 4b$$

or a single point, $c = -\frac{1}{2}(a_1, \ldots, a_d)$, iff $\sum_{i=1}^{d} a_i^2 = 4b$.

What happens when

$$\sum_{i=1}^{d} a_i^2 < 4b?$$

In this case, if we allow "complex points", that is, if we consider solutions of our equation

$$\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0$$

over $\mathbb{C}^d$, then we get a "complex" sphere of (pure) imaginary radius, $\frac{i}{2}\sqrt{4b - \sum_{i=1}^{d} a_i^2}$. The funny thing is that our computations carry over unchanged and the image of the complex sphere, S, is still the intersection of the complex sphere S^d with the hyperplane, H, given

$$\sum_{i=1}^{d} a_i x_i + (-b+1)x_{d+1} + (b+1) = 0.$$

However, this time, even though H does not have any "real" intersection points with S^d, we can show that it does intersect the "complex sphere",

$$S^d = \{(z_1, \ldots, z_{d+1}) \in \mathbb{C}^{d+1} \mid z_1^2 + \cdots + z_{d+1}^2 = 1\}$$

in a nonempty set of points in $\mathbb{C}^{d+1}$.

It follows from all this that σ_N and τ_N establish a bijection between the set of all hyperplanes in $\mathbb{E}^{d+1}$ minus the hyperplane, H_{d+1} (of equation $x_{d+1} = 1$), tangent to S^d at the north pole, with the union of four sets:

(1) The set of all (real) $(d-1)$-spheres of positive radius;

(2) The set of all (complex) $(d-1)$-spheres of imaginary radius;

(3) The set of all hyperplanes in $\mathbb{E}^d$;

(4) The set of all points of $\mathbb{E}^d$ (viewed as spheres of radius 0).

Moreover, set (1) corresponds to the hyperplanes that intersect the interior of S^d and do not pass through the north pole; set (2) corresponds to the hyperplanes that do not intersect S^d; set (3) corresponds to the hyperplanes that pass through the north pole minus the tangent hyperplane at the north pole; and set (4) corresponds to the hyperplanes that are tangent to S^d, minus the tangent hyperplane at the north pole.

It is convenient to add the "point at infinity", ∞, to $\mathbb{E}^d$, because then the above bijection can be extended to map the tangent hyperplane at the north pole to ∞. The union of these four sets (with ∞ added) is called the *set of generalized spheres*, sometimes, denoted $\mathcal{S}(\mathbb{E}^d)$. This is a fairly complicated space. For one thing, topologically, $\mathcal{S}(\mathbb{E}^d)$ is homeomorphic to the projective space $\mathbb{P}^{d+1}$ with one point removed (the point corresponding to the "hyperplane at infinity"), and this is not a simple space. We can get a slightly more concrete "'picture" of $\mathcal{S}(\mathbb{E}^d)$ by looking at the polars of the hyperplanes w.r.t. S^d. Then, the "real" spheres correspond to the points strictly outside S^d which do not belong to the tangent hyperplane at the norh pole; the complex spheres correspond to the points in the interior of S^d; the points of $\mathbb{E}^d \cup \{\infty\}$ correspond to the points on S^d; the hyperplanes in $\mathbb{E}^d$ correspond to the points in the tangent hyperplane at the norh pole expect for the north pole. Unfortunately, the poles of hyperplanes through the origin are undefined. This can be fixed by embedding $\mathbb{E}^{d+1}$ in its projective completion, $\mathbb{P}^{d+1}$, but we will not go into this.

There are other ways of dealing rigorously with the set of generalized spheres. One method described by Boissonnat [8] is to use the embedding where the sphere, S, of equation

$$\sum_{i=1}^{d} X_i^2 - 2\sum_{i=1}^{d} a_i X_i + b = 0$$

is mapped to the point

$$\varphi(S) = (a_1, \ldots, a_d, b) \in \mathbb{E}^{d+1}.$$

Now, by a previous computation we know that

$$b = \sum_{i=1}^{d} a_i^2 - R^2,$$

where $c = (a_1, \ldots, a_d)$ is the center of S and R is its radius. The quantity $\sum_{i=1}^{d} a_i^2 - R^2$ is known as the *power* of the origin w.r.t. S. In general, the *power* of a point, $X \in \mathbb{E}^d$, is defined as $\rho(X) = \|\mathbf{cX}\|^2 - R^2$, which, after a moment of thought, is just

$$\rho(X) = \sum_{i=1}^{d} X_i^2 - 2\sum_{i=1}^{d} a_i X_i + b.$$

Now, since points correspond to spheres of radius 0, we see that the image of the point, $X = (X_1, \ldots, X_d)$, is

$$l(X) = (X_1, \ldots, X_d, \sum_{i=1}^{d} X_i^2).$$

Thus, in this model, points of $\mathbb{E}^d$ are lifted to the hyperboloid, $\mathcal{P} \subseteq \mathbb{E}^{d+1}$, of equation

$$x_{d+1} = \sum_{i=1}^{d} x_i^2.$$

Actually, this method does not deal with hyperplanes but it is possible to do so. The trick is to consider equations of a slightly more general form that capture both spheres and hyperplanes, namely, equations of the form

$$c\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0.$$

Indeed, when $c = 0$, we do get a hyperplane! Now, to carry out this method we really need to consider equations up to a nonzero scalars, that is, we consider the projective space, $\mathbb{P}(\widehat{S}(\mathbb{E}^d))$, associated with the vector space, $\widehat{S}(\mathbb{E}^d)$, consisting of the above equations. Then, it turns out that the quantity

$$\varrho(a, b, c) = \frac{1}{4}(\sum_{i=1}^{d} a_i^2 - 4bc)$$

(with $a = (a_1, \ldots, a_d)$) defines a quadratic form on $\widehat{S}(\mathbb{E}^d)$ whose corresponding bilinear form,

$$\rho((a, b, c), (a', b', c')) = \frac{1}{4}(\sum_{i=1}^{d} a_i a_i' - 2bc' - 2b'c),$$

has a natural interpretation (with $a = (a_1, \ldots, a_d)$ and $a' = (a_1', \ldots, a_d')$). Indeed, orthogonality with respect to ρ (that is, when $\rho((a, b, c), (a', b', c')) = 0$) says that the corresponding spheres defined by (a, b, c) and (a', b', c') are orthogonal, that the corresponding hyperplanes defined by $(a, b, 0)$ and $(a', b', 0)$ are orthogonal, etc. The reader who wants to read more about this approach should consult Berger (Volume II) [6].

There is a simple relationship between the lifting onto a hyperboloid and the lifting onto S^d using the inverse stereographic projection map because the sphere and the paraboloid are projectively equivalent, as we showed for S^2 in Section 5.1.

Recall that the hyperboloid, $\mathcal{P}$, in $\mathbb{E}^{d+1}$ is given by the equation

$$x_{d+1} = \sum_{i=1}^{d} x_i^2$$

and of course, the sphere S^d is given by

$$\sum_{i=1}^{d+1} x_i^2 = 1.$$

Consider the "projective transformation", Θ, of $\mathbb{E}^{d+1}$ given by

$$\begin{aligned} z_i &= \frac{x_i}{1 - x_{d+1}}, \qquad 1 \leq i \leq d \\ z_{d+1} &= \frac{x_{d+1} + 1}{1 - x_{d+1}}. \end{aligned}$$

Observe that Θ is undefined on the hyperplane, H_{d+1}, tangent to S^d at the north pole and that its first d component are identical to those of the stereographic projection! Then, we immediately find that

$$\begin{aligned} x_i &= \frac{2z_i}{1 + z_{d+1}}, \qquad 1 \leq i \leq d \\ x_{d+1} &= \frac{z_{d+1} - 1}{1 + z_{d+1}}. \end{aligned}$$

Consequently, Θ is a bijection between $\mathbb{E}^{d+1} - H_{d+1}$ and $\mathbb{E}^{d+1} - H_{d+1}(-1)$, where $H_{d+1}(-1)$ is the hyperplane of equation $x_{d+1} = -1$.

The fact that Θ is undefined on the hyperplane, H_{d+1}, is not a problem as far as mapping the sphere to the paraboloid because the north pole is the only point that does not have an image. However, later on when we consider the Voronoi polyhedron, $\mathcal{V}(P)$, of a lifted set of points, P, we will have more serious problems because in general, such a polyhedron intersects both hyperplanes H_{d+1} and $H_{d+1}(-1)$. This means that Θ will not be well-defined on the whole of $\mathcal{V}(P)$ nor will it be surjective on its image. To remedy this difficulty, we will work with projective completions. Basically, this amounts to chasing denominators and homogenizing equations but we also have to be careful in dealing with convexity and this is where the projective polyhedra (studied in Section 5.2) will come handy.

So, let us consider the projective sphere, $S^d \subseteq \mathbb{P}^{d+1}$, given by the equation

$$\sum_{i=1}^{d+1} x_i^2 = x_{d+2}^2$$

and the paraboloid, $\mathcal{P} \subseteq \mathbb{P}^{d+1}$, given by the equation

$$x_{d+1}x_{d+2} = \sum_{i=1}^{d} x_i^2.$$

Let $\theta\colon \mathbb{P}^{d+1} \to \mathbb{P}^{d+1}$ be the projectivity induced by the linear map, $\widehat{\theta}\colon \mathbb{R}^{d+2} \to \mathbb{R}^{d+2}$, given by

$$\begin{aligned} z_i &= x_i, && 1 \leq i \leq d \\ z_{d+1} &= x_{d+1} + x_{d+2} \\ z_{d+2} &= x_{d+2} - x_{d+1}, \end{aligned}$$

whose inverse is given by

$$\begin{aligned} x_i &= z_i, && 1 \leq i \leq d \\ x_{d+1} &= \frac{z_{d+1} - z_{d+2}}{2} \\ x_{d+2} &= \frac{z_{d+1} + z_{d+2}}{2}. \end{aligned}$$

If we plug these formulae in the equation of S^d, we get

$$4(\sum_{i=1}^{d} z_i^2) + (z_{d+1} - z_{d+2})^2 = (z_{d+1} + z_{d+2})^2,$$

which simplifies to

$$z_{d+1}z_{d+2} = \sum_{i=1}^{d} z_i^2.$$

Therefore, $\theta(S^d) = \mathcal{P}$, that is, θ maps the sphere to the hyperboloid. Observe that the north pole, $N = (0\colon \cdots : 0\colon 1\colon 1)$, is mapped to the point at infinity, $(0\colon \cdots : 0\colon 1\colon 0)$.

The map Θ is the restriction of θ to the affine patch, U_{d+1}, and as such, it can be fruitfully described as the composition of $\widehat{\theta}$ with a suitable projection onto $\mathbb{E}^{d+1}$. For this, as we have done before, we identify $\mathbb{E}^{d+1}$ with the hyperplane, $H_{d+2} \subseteq \mathbb{E}^{d+2}$, of equation $x_{d+2} = 1$ (using the injection, $i_{d+2}\colon \mathbb{E}^{d+1} \to \mathbb{E}^{d+2}$, where $i_j\colon \mathbb{E}^{d+1} \to \mathbb{E}^{d+2}$ is the injection given by

$$(x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{d+1})$$

for any $(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1}$). For each i, with $1 \leq i \leq d+2$, let $\pi_i\colon (\mathbb{E}^{d+2} - H_i(0)) \to \mathbb{E}^{d+1}$ be the projection of center $0 \in \mathbb{E}^{d+2}$ onto the hyperplane, $H_i \subseteq \mathbb{E}^{d+2}$, of equation $x_i = 1$ ($H_i \cong \mathbb{E}^{d+1}$ and $H_i(0) \subseteq \mathbb{E}^{d+2}$ is the hyperplane of equation $x_i = 0$) given by

$$\pi_i(x_1, \ldots, x_{d+2}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{d+2}}{x_i}\right) \qquad (x_i \neq 0).$$

Geometrically, for any $x \notin H_i(0)$, the image, $\pi_i(x)$, of x is the intersection of the line through the origin and x with the hyperplane, $H_i \subseteq \mathbb{E}^{d+2}$ of equation $x_i = 1$. Observe that the map, $\pi_i\colon (\mathbb{E}^{d+2} - H_{d+2}(0)) \to \mathbb{E}^{d+1}$, is an "affine" version of the bijection, $\varphi_i\colon U_i \to \mathbb{R}^{d+1}$, of Section 5.1. Then, we have

$$\Theta = \pi_{d+2} \circ \widehat{\theta} \circ i_{d+2}.$$

If we identify H_{d+2} and $\mathbb{E}^{d+1}$, we may write with a slight abuse of notation, $\Theta = \pi_{d+2} \circ \widehat{\theta}$.

Besides θ, we need to define a few more maps in order to establish the connection between the Delaunay complex on S^d and the Delaunay complex on $\mathcal{P}$. We use the convention of denoting the extension to projective spaces of a map, f, defined between Euclidean spaces, by $\widetilde{f}$.

The Euclidean orthogonal projection, $p_i\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$, is given by

$$p_i(x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1})$$

and $\widetilde{p_i}\colon \mathbb{P}^{d+1} \to \mathbb{P}^d$ denotes the projection from $\mathbb{P}^{d+1}$ onto $\mathbb{P}^d$ given by

$$\widetilde{p_i}(x_1\colon \cdots : x_{d+2}) = (x_1\colon \cdots : x_{i-1}\colon x_{i+1}\colon \cdots : x_{d+2}),$$

which is undefined at the point $(0\colon \cdots : 1\colon 0\colon \cdots : 0)$, where the "1" is in the i^{th} slot. The map $\widetilde{\pi}_N\colon (\mathbb{P}^{d+1} - \{N\}) \to \mathbb{P}^d$ is the central projection from the north pole onto $\mathbb{P}^d$ given by

$$\widetilde{\pi}_N(x_1\colon \cdots : x_{d+1}\colon x_{d+2}) = (x_1\colon \cdots : x_d\colon x_{d+2} - x_{d+1}).$$

A geometric interpretation of $\widetilde{\pi}_N$ will be needed later in certain proofs. If we identify $\mathbb{P}^d$ with the hyperplane, $H_{d+1}(0) \subseteq \mathbb{P}^{d+1}$, of equation $x_{d+1} = 0$, then we claim that for any, $x \neq N$, the point $\widetilde{\pi}_N(x)$ is the intersection of the line through N and x with the hyperplane,

$H_{d+1}(0)$. Indeed, parametrically, the line, $\langle N, x\rangle$, through $N = (0\colon \cdots : 0\colon 1\colon 1)$ and x is given by

$$\langle N, x\rangle = \{(\mu x_1\colon \cdots : \mu x_d\colon \lambda + \mu x_{d+1}\colon \lambda + \mu x_{d+2}) \mid \lambda, \mu \in \mathbb{R},\ \lambda \neq 0 \quad \text{or} \quad \mu \neq 0\}.$$

The line $\langle N, x\rangle$ intersects the hyperplane $x_{d+1} = 0$ iff

$$\lambda + \mu x_{d+1} = 0,$$

so we can pick $\lambda = -x_{d+1}$ and $\mu = 1$, which yields the intersection point,

$$(x_1\colon \cdots : x_d\colon 0\colon x_{d+2} - x_{d+1}),$$

as claimed.

We also have the projective versions of σ_N and τ_N, denoted $\widetilde{\sigma}_N\colon (S^d - \{N\}) \to \mathbb{P}^d$ and $\widetilde{\tau}_N\colon \mathbb{P}^d \to S^d \subseteq \mathbb{P}^{d+1}$, given by:

$$\widetilde{\sigma}_N(x_1\colon \cdots : x_{d+2}) = (x_1\colon \cdots : x_d\colon x_{d+2} - x_{d+1})$$

and

$$\widetilde{\tau}_N(x_1\colon \cdots : x_{d+1}) = \left(2x_1x_{d+1}\colon \cdots : 2x_dx_{d+1}\colon \sum_{i=1}^{d} x_i^2 - x_{d+1}^2\colon \sum_{i=1}^{d} x_i^2 + x_{d+1}^2\right).$$

It is an easy exercise to check that the image of $S^d - \{N\}$ by $\widetilde{\sigma}_N$ is U_{d+1} and that $\widetilde{\sigma}_N$ and $\widetilde{\tau}_N \restriction U_{d+1}$ are mutual inverses. Observe that $\widetilde{\sigma}_N = \widetilde{\pi}_N \restriction S^d$, the restriction of the projection, $\widetilde{\pi}_N$, to the sphere, S^d. The lifting, $\widetilde{l}\colon \mathbb{E}^d \to \mathcal{P} \subseteq \mathbb{P}^{d+1}$, is given by

$$\widetilde{l}(x_1, \ldots, x_d) = \left(x_1\colon \cdots : x_d\colon \sum_{i=1}^{d} x_i^2\colon 1\right)$$

and the embedding, $\psi_{d+1}\colon \mathbb{E}^d \to \mathbb{P}^d$, (the map ψ_{d+1} defined in Section 5.1) is given by

$$\psi_{d+1}(x_1, \ldots, x_d) = (x_1\colon \cdots : x_d\colon 1).$$

Then, we easily check

Proposition 8.5 *The maps, $\theta, \widetilde{\pi}_N, \widetilde{\tau}_N, \widetilde{p}_{d+1}, \widetilde{l}$ and ψ_{d+1} defined before satisfy the equations*

$$\begin{aligned}
\widetilde{l} &= \theta \circ \widetilde{\tau}_N \circ \psi_{d+1} \\
\widetilde{\pi}_N &= \widetilde{p}_{d+1} \circ \theta \\
\widetilde{\tau}_N \circ \psi_{d+1} &= \psi_{d+2} \circ \tau_N \\
\widetilde{l} &= \psi_{d+2} \circ l \\
l &= \Theta \circ \tau_N.
\end{aligned}$$

Proof. Let us check the first equation leaving the others as an exercise. Recall that θ is given by

$$\theta(x_1\colon \cdots : x_{d+2}) = (x_1\colon \cdots : x_d\colon x_{d+1} + x_{d+2}\colon x_{d+2} - x_{d+1}).$$

Then, as

$$\widetilde{\tau}_N \circ \psi_{d+1}(x_1, \ldots, x_d) = \left(2x_1\colon \cdots : 2x_d\colon \sum_{i=1}^{d} x_i^2 - 1\colon \sum_{i=1}^{d} x_i^2 + 1\right),$$

we get

$$\begin{aligned}
\theta \circ \widetilde{\tau}_N \circ \psi_{d+1}(x_1, \ldots, x_d) &= \left(2x_1\colon \cdots : 2x_d\colon 2\sum_{i=1}^{d} x_i^2\colon 2\right) \\
&= \left(x_1\colon \cdots : x_d\colon \sum_{i=1}^{d} x_i^2\colon 1\right) = \widetilde{l}(x_1, \ldots, x_d),
\end{aligned}$$

as claimed. $\square$

We will also need some properties of the projection π_{d+2} and of Θ and for this, let

$$\mathbb{H}_+^d = \{(x_1, \ldots, x_d) \in \mathbb{E}^d \mid x_d > 0\} \quad \text{and} \quad \mathbb{H}_-^d = \{(x_1, \ldots, x_d) \in \mathbb{E}^d \mid x_d < 0\}.$$

Proposition 8.6 *The projection, π_{d+2}, has the following properties:*

(1) For every hyperplane, H, through the origin, $\pi_{d+2}(H)$ is a hyperplane in H_{d+2}.

(2) Given any set of points, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+2}$, if $\{a_1, \ldots, a_n\}$ is contained in the open half-space above the hyperplane $x_{d+2} = 0$ or $\{a_1, \ldots, a_n\}$ is contained in the open half-space below the hyperplane $x_{d+2} = 0$, then the image by π_{d+2} of the convex hull of the a_i's is the convex hull of the images of these points, that is,

$$\pi_{d+2}(\mathrm{conv}(\{a_1, \ldots, a_n\})) = \mathrm{conv}(\{\pi_{d+2}(a_1), \ldots, \pi_{d+2}(a_n)\}).$$

(3) Given any set of points, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+1}$, if $\{a_1, \ldots, a_n\}$ is contained in the open half-space above the hyperplane H_{d+1} or $\{a_1, \ldots, a_n\}$ is contained in the open half-space below H_{d+1}, then

$$\Theta(\mathrm{conv}(\{a_1, \ldots, a_n\})) = \mathrm{conv}(\{\Theta(a_1), \ldots, \Theta(a_n)\}).$$

(4) For any set $S \subseteq \mathbb{E}^{d+1}$, if $\mathrm{conv}(S)$ does not intersect H_{d+1}, then

$$\Theta(\mathrm{conv}(S)) = \mathrm{conv}(\Theta(S)).$$

Proof. (1) The image, $\pi_{d+2}(H)$, of a hyperplane, H, through the origin is the intersection of H with H_{d+2}, which is a hyperplane in H_{d+2}.

(2) This seems fairly clear geometrically but the result fails for arbitrary sets of points so to be on the safe side we give an algebraic proof. We will prove the following two facts by induction on $n \geq 1$:

(1) For all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$, for all $a_1, \ldots, a_n \in \mathbb{H}^{d+2}_+$ (resp. $\in \mathbb{H}^{d+2}_-$) there exist some $\mu_1, \ldots, \mu_n \in \mathbb{R}$ with $\mu_1 + \cdots + \mu_n = 1$ and $\mu_i \geq 0$, so that
$$\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_n a_n) = \mu_1 \pi_{d+2}(a_1) + \cdots + \mu_n \pi_{d+2}(a_n).$$

(2) For all $\mu_1, \ldots, \mu_n \in \mathbb{R}$ with $\mu_1 + \cdots + \mu_n = 1$ and $\mu_i \geq 0$, for all $a_1, \ldots, a_n \in \mathbb{H}^{d+2}_+$ (resp. $\in \mathbb{H}^{d+2}_-$) there exist some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$, so that
$$\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_n a_n) = \mu_1 \pi_{d+2}(a_1) + \cdots + \mu_n \pi_{d+2}(a_n).$$

(1) The base case is clear. Let us assume for the moment that we proved (1) for $n = 2$ and consider the induction step for $n \geq 2$. Since $\lambda_1 + \cdots + \lambda_{n+1} = 1$ and $n \geq 2$, there is some i such that $\lambda_i \neq 1$, and without loss of generality, say $\lambda_1 \neq 1$. Then, we can write
$$\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1} = \lambda_1 a_1 + (1 - \lambda_1)\left(\frac{\lambda_2}{1-\lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1-\lambda_1} a_{n+1}\right)$$
and since $\lambda_1 + \lambda_2 + \cdots + \lambda_{n+1} = 1$, we have
$$\frac{\lambda_2}{1-\lambda_1} + \cdots + \frac{\lambda_{n+1}}{1-\lambda_1} = 1.$$
By the induction hypothesis, for $n = 2$, there exist α_1 with $0 \leq \alpha_1 \leq 1$, such that
$$\begin{aligned}\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1}) &= \pi_{d+2}\left(\lambda_1 a_1 + (1 - \lambda_1)\left(\frac{\lambda_2}{1-\lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1-\lambda_1} a_{n+1}\right)\right) \\ &= (1-\alpha_1)\pi_{d+2}(a_1) + \alpha_1 \pi_{d+2}\left(\frac{\lambda_2}{1-\lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1-\lambda_1} a_{n+1}\right)\end{aligned}$$
Again, by the induction hypothesis (for n), there exist $\beta_2, \ldots, \beta_{n+1}$ with $\beta_2 + \cdots + \beta_{n+1} = 1$ and $\beta_i \geq 0$, so that
$$\pi_{d+2}\left(\frac{\lambda_2}{1-\lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1-\lambda_1} a_{n+1}\right) = \beta_2 \pi_{d+2}(a_2) + \cdots + \beta_{n+1}\pi_{d+2}(a_{n+1}),$$
so we get
$$\begin{aligned}\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1}) &= (1-\alpha_1)\pi_{d+2}(a_1) + \alpha_1(\beta_2 \pi_{d+2}(a_2) + \cdots + \beta_{n+1}\pi_{d+2}(a_{n+1})) \\ &= (1-\alpha_1)\pi_{d+2}(a_1) + \alpha_1\beta_2 \pi_{d+2}(a_2) + \cdots + \alpha_1\beta_{n+1}\pi_{d+2}(a_{n+1})\end{aligned}$$

and clearly, $1-\alpha_1+\alpha_1\beta_2+\cdots+\alpha_1\beta_{n+1}=1$ as $\beta_2+\cdots+\beta_{n+1}=1$; $1-\alpha_1\geq 0$; and $\alpha_1\beta_i\geq 0$, as $0\leq\alpha_1\leq 1$ and $\beta_i\geq 0$. This establishes the induction step and thus, all is left is to prove the case $n=2$.

(2) The base case $n=1$ is also clear. As in (1), let us assume for a moment that (2) is proved for $n=2$ and consider the induction step. The proof is quite similar to that of (1) but this time, we may assume that $\mu_1\neq 1$ and we write

$$\begin{aligned}&\mu_1\pi_{d+2}(a_1)+\cdots+\mu_{n+1}\pi_{d+2}(a_{n+1})\\&\qquad\qquad=\mu_1\pi_{d+2}(a_1)+(1-\mu_1)\left(\frac{\mu_2}{1-\mu_1}\pi_{d+2}(a_2)\cdots+\frac{\mu_{n+1}}{1-\mu_1}\pi_{d+2}(a_{n+1})\right).\end{aligned}$$

By the induction hypothesis, there are some $\alpha_2,\ldots,\alpha_{n+1}$ with $\alpha_2+\cdots+\alpha_{n+1}=1$ and $\alpha_i\geq 0$ such that

$$\pi_{d+2}(\alpha_2a_2+\cdots+\alpha_{n+1}a_{n+1})=\frac{\mu_2}{1-\mu_1}\pi_{d+2}(a_2)+\cdots+\frac{\mu_{n+1}}{1-\mu_1}\pi_{d+2}(a_{n+1}).$$

By the induction hypothesis for $n=2$, there is some β_1 with $0\leq\beta_1\leq 1$, so that

$$\pi_{d+2}((1-\beta_1)a_1+\beta_1(\alpha_2a_2+\cdots+\alpha_{n+1}a_{n+1}))=\mu_1\pi_{d+2}(a_1)+(1-\mu_1)\pi_{d+2}(\alpha_2a_2+\cdots+\alpha_{n+1}a_{n+1}),$$

which establishes the induction hypothesis. Therefore, all that remains is to prove (1) and (2) for $n=2$.

As π_{d+2} is given by

$$\pi_{d+2}(x_1,\ldots,x_{d+2})=\left(\frac{x_1}{x_{d+2}},\ldots,\frac{x_{d+1}}{x_{d+2}}\right)\qquad(x_{d+2}\neq 0)$$

it is enough to treat the case when $d=0$, that is,

$$\pi_2(a,b)=\frac{a}{b}.$$

To prove (1) it is enough to show that for any λ, with $0\leq\lambda\leq 1$, if $b_1b_2>0$ then

$$\frac{a_1}{b_1}\leq\frac{(1-\lambda)a_1+\lambda a_2}{(1-\lambda)b_1+\lambda b_2}\leq\frac{a_2}{b_2}\quad\text{if}\quad\frac{a_1}{b_1}\leq\frac{a_2}{b_2}$$

and

$$\frac{a_2}{b_2}\leq\frac{(1-\lambda)a_1+\lambda a_2}{(1-\lambda)b_1+\lambda b_2}\leq\frac{a_1}{b_1}\quad\text{if}\quad\frac{a_2}{b_2}\leq\frac{a_1}{b_1},$$

where, of course $(1-\lambda)b_1+\lambda b_2\neq 0$. For this, we compute (leaving some steps as an exercise)

$$\frac{(1-\lambda)a_1+\lambda a_2}{(1-\lambda)b_1+\lambda b_2}-\frac{a_1}{b_1}=\frac{\lambda(a_2b_1-a_1b_2)}{((1-\lambda)b_1+\lambda b_2)b_1}$$

and

$$\frac{(1-\lambda)a_1 + \lambda a_2}{(1-\lambda)b_1 + \lambda b_2} - \frac{a_2}{b_2} = -\frac{(1-\lambda)(a_2b_1 - a_1b_2)}{((1-\lambda)b_1 + \lambda b_2)b_2}.$$

Now, as $b_1b_2 > 0$, that is, b_1 and b_2 have the same sign and as $0 \leq \lambda \leq 1$, we have both $((1-\lambda)b_1 + \lambda b_2)b_1 > 0$ and $((1-\lambda)b_1 + \lambda b_2)b_2 > 0$. Then, if $a_2b_1 - a_1b_2 \geq 0$, that is $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$ (since $b_1b_2 > 0$), the first two inequalities hold and if $a_2b_1 - a_1b_2 \leq 0$, that is $\frac{a_2}{b_2} \leq \frac{a_1}{b_1}$ (since $b_1b_2 > 0$), the last two inequalities hold. This proves (1).

In order to prove (2), given any μ, with $0 \leq \mu \leq 1$, if $b_1b_2 > 0$, we show that we can find λ with $0 \leq \lambda \leq 1$, so that

$$(1-\mu)\frac{a_1}{b_1} + \mu\frac{a_2}{b_2} = \frac{(1-\lambda)a_1 + \lambda a_2}{(1-\lambda)b_1 + \lambda b_2}.$$

If we let

$$\alpha = (1-\mu)\frac{a_1}{b_1} + \mu\frac{a_2}{b_2},$$

we find that λ is given by the equation

$$\lambda(a_2 - a_1 + \alpha(b_1 - b_2)) = \alpha b_1 - a_1.$$

After some (tedious) computations (check for yourself!) we find:

$$\begin{aligned} a_2 - a_1 + \alpha(b_1 - b_2) &= \frac{((1-\mu)b_2 + \mu b_1)(a_2b_1 - a_1b_2)}{b_1b_2} \\ \alpha b_1 - a_1 &= \frac{\mu b_1(a_2b_1 - a_1b_2)}{b_1b_2}. \end{aligned}$$

If $a_2b_1 - a_1b_2 = 0$, then $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\lambda = 0$ works. If $a_2b_1 - a_1b_2 \neq 0$, then

$$\lambda = \frac{\mu b_1}{(1-\mu)b_2 + \mu b_1} = \frac{\mu}{(1-\mu)\frac{b_2}{b_1} + \mu}.$$

Since $b_1b_2 > 0$, we have $\frac{b_2}{b_1} > 0$, and since $0 \leq \mu \leq 1$, we conclude that $0 \leq \lambda \leq 1$, which proves (2).

(3) Since

$$\Theta = \pi_{d+2} \circ \widehat{\theta} \circ i_{d+2},$$

as i_{d+2} and $\widehat{\theta}$ are linear, they preserve convex hulls, so by (2), we simply have to show that either $\widehat{\theta} \circ i_{d+2}(\{a_1, \ldots, a_n\})$ is strictly below the hyperplane, $x_{d+2} = 0$, or strictly above it. But,

$$\widehat{\theta}(x_1, \ldots, x_{d+2})_{d+2} = x_{d+2} - x_{d+1}$$

and $i_{d+2}(x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_{d+1}, 1)$, so

$$(\widehat{\theta} \circ i_{d+2})(x_1, \ldots, x_{d+1})_{d+2} = 1 - x_{d+1},$$

and this quantity is positive iff $x_{d+1} < 1$, negative iff $x_{d+1} > 1$; that is, either all the points a_i are strictly below the hyperplane H_{d+1} or all strictly above it.

(4) This follows immediately from (3) as $\mathrm{conv}(S)$ consists of all finite convex combinations of points in S. $\square$

If a set, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+2}$, contains points on *both sides* of the hyperplane, $x_{d+2} = 0$, then $\pi_{d+2}(\mathrm{conv}(\{a_1, \ldots, a_n\}))$ is **not** necessarily convex (find such an example!).

8.6 Stereographic Projection, Delaunay Polytopes and Voronoi Polyhedra

We saw in an earlier section that lifting a set of points, $P \subseteq \mathbb{E}^d$, to the paraboloid, $\mathcal{P}$, *via* the lifting function, l, was fruitful to better understand Voronoi diagrams and Delaunay triangulations. As far as we know, Edelsbrunner and Seidel [16] were the first to find the relationship between Voronoi diagrams and the polar dual of the convex hull of a lifted set of points onto a paraboloid. This connection is described in Note 3.1 of Section 3 in [16]. The connection between the Delaunay triangulation and the convex hull of the lifted set of points is described in Note 3.2 of the same paper. Polar duality is not mentioned and seems to enter the scene only with Boissonnat and Yvinec [8].

It turns out that instead of using a paraboloid we can use a sphere and instead of the lifting function l we can use the composition of ψ_{d+1} with the inverse stereographic projection, $\widetilde{\tau}_N$. Then, to get back down to $\mathbb{E}^d$, we use the composition of the projection, $\widetilde{\pi}_N$, with φ_{d+1}, instead of the orthogonal projection, p_{d+1}.

However, we have to be a bit careful because Θ does map all convex polyhedra to convex polyhedra. Indeed, Θ is the composition of π_{d+2} with some linear maps, but π_{d+2} does not behave well with respect to arbitrary convex sets. In particular, Θ is not well-defined on any face that intersects the hyperplane H_{d+1} (of equation $x_{d+1} = 1$). Fortunately, we can circumvent these difficulties by using the concept of a projective polyhedron introduced in Chapter 5.

As we said in the previous section, the correspondence between Voronoi diagrams and convex hulls *via* inversion was first observed by Brown [11]. Brown takes a set of points, S, for simplicity assumed to be in the plane, first lifts these points to the unit sphere S^2 using inverse stereographic projection (which is equivalent to an inversion of power 2 centered at the north pole), getting $\tau_N(S)$, and then takes the convex hull, $\mathcal{D}(S) = \mathrm{conv}(\tau_N(S))$, of the lifted set. Now, in order to obtain the Voronoi diagram of S, apply our inversion (of power 2 centered at the north pole) to each of the faces of $\mathrm{conv}(\tau_N(S))$, obtaining spheres passing through the center of S^2 and then intersect these spheres with the plane containing S, obtaining circles. The centers of some of these circles are the Voronoi vertices. Finally, a simple criterion can be used to retain the "nearest Voronoi points" and to connect up these vertices.

Note that Brown's method is *not* the method that uses the polar dual of the polyhedron $\mathcal{D}(S) = \mathrm{conv}(\tau_N(S))$, as we might have expected from the lifting method using a paraboloid. In fact, it is more natural to get the *Delaunay triangulation of* S from Brown's method, by applying the stereographic projection (from the north pole) to $\mathcal{D}(S)$, as we will prove below. As $\mathcal{D}(S)$ is strictly below the plane $z = 1$, there are no problems. Now, in order to get the Voronoi diagram, we take the polar dual, $\mathcal{D}(S)^*$, of $\mathcal{D}(S)$ and then apply the central projection w.r.t. the north pole. This is where problems arise, as some faces of $\mathcal{D}(S)^*$ may intersect the hyperplane H_{d+1} and this is why we have recourse to projective geometry.

First, we show that θ has a good behavior with respect to tangent spaces. Recall from Section 5.2 that for any point, $a = (a_1\colon \cdots : a_{d+2}) \in \mathbb{P}^{d+1}$, the tangent hyperplane, $T_a S^d$, to the sphere S^d at a is given by the equation

$$\sum_{i=1}^{d+1} a_i x_i - a_{d+2} x_{d+2} = 0.$$

Similarly, the tangent hyperplane, $T_a\mathcal{P}$, to the paraboloid $\mathcal{P}$ at a is given by the equation

$$2\sum_{i=1}^{d} a_i x_i - a_{d+2} x_{d+1} - a_{d+1} x_{d+2} = 0.$$

If we lift a point $a \in \mathbb{E}^d$ to S^d by $\widetilde{\tau}_N \circ \psi_{d+1}$ and to $\mathcal{P}$ by $\widetilde{l}$, it turns out that the image of the tangent hyperplane to S^d at $\widetilde{\tau}_N \circ \psi_{d+1}(a)$ by θ is the tangent hyperplane to $\mathcal{P}$ at $\widetilde{l}(a)$.

Proposition 8.7 *The map θ has the following properties:*

(1) For any point, $a = (a_1, \ldots, a_d) \in \mathbb{E}^d$, we have

$$\theta(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d) = T_{\widetilde{l}(a)}\mathcal{P},$$

that is, θ preserves tangent hyperplanes.

(2) For every $(d-1)$-sphere, $S \subseteq \mathbb{E}^d$, we have

$$\theta(\widetilde{\tau}_N \circ \psi_{d+1}(S)) = \widetilde{l}(S),$$

that is, θ preserves lifted $(d-1)$-spheres.

Proof. (1) By Proposition 8.5, we know that

$$\widetilde{l} = \theta \circ \widetilde{\tau}_N \circ \psi_{d+1}$$

and we proved in Section 5.2 that projectivities preserve tangent spaces. Thus,

$$\theta(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d) = T_{\theta \circ \widetilde{\tau}_N \circ \psi_{d+1}(a)} \theta(S^d) = T_{\widetilde{l}(a)}\mathcal{P},$$

as claimed.

(2) This follows immediately from the equation $\widetilde{l} = \theta \circ \widetilde{\tau}_N \circ \psi_{d+1}$. $\square$

Given any two distinct points, $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{E}^d$, recall that the bisector hyperplane, $H_{a,b}$, of a and b is given by

$$(b_1 - a_1)x_1 + \cdots + (b_d - a_d)x_d = (b_1^2 + \cdots + b_d^2)/2 - (a_1^2 + \cdots + a_d^2)/2.$$

We have the following useful proposition:

Proposition 8.8 *Given any two distinct points, $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{E}^d$, the image under the projection, $\widetilde{\pi}_N$, of the intersection, $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$, of the tangent hyperplanes at the lifted points $\widetilde{\tau}_N \circ \psi_{d+1}(a)$ and $\widetilde{\tau}_N \circ \psi_{d+1}(b)$ on the sphere $S^d \subseteq \mathbb{P}^{d+1}$ is the embedding of the bisector hyperplane, $H_{a,b}$, of a and b, into $\mathbb{P}^d$, that is,*

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d) = \psi_{d+1}(H_{a,b}).$$

Proof. In view of the geometric interpretation of $\widetilde{\pi}_N$ given earlier, we need to find the equation of the hyperplane, H, passing through the intersection of the tangent hyperplanes, $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)}$ and $T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)}$ and passing through the north pole and then, it is geometrically obvious that

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d) = H \cap H_{d+1}(0),$$

where $H_{d+1}(0)$ is the hyperplane (in $\mathbb{P}^{d+1}$) of equation $x_{d+1} = 0$. Recall that $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d$ and $T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$ are given by

$$E_1 = 2\sum_{i=1}^d a_i x_i + (\sum_{i=1}^d a_i^2 - 1)x_{d+1} - (\sum_{i=1}^d a_i^2 + 1)x_{d+2} = 0$$

and

$$E_2 = 2\sum_{i=1}^d b_i x_i + (\sum_{i=1}^d b_i^2 - 1)x_{d+1} - (\sum_{i=1}^d b_i^2 + 1)x_{d+2} = 0.$$

The hyperplanes passing through $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$ are given by an equation of the form

$$\lambda E_1 + \mu E_2 = 0,$$

with $\lambda, \mu \in \mathbb{R}$. Furthermore, in order to contain the north pole, this equation must vanish for $x = (0\colon \cdots \colon 0\colon 1\colon 1)$. But, observe that setting $\lambda = -1$ and $\mu = 1$ gives a solution since the corresponding equation is

$$2\sum_{i=1}^d (b_i - a_i)x_i + (\sum_{i=1}^d b_i^2 - \sum_{i=1}^d a_i^2)x_{d+1} - (\sum_{i=1}^d b_i^2 - \sum_{i=1}^d a_i^2)x_{d+2} = 0$$

and it vanishes on $(0\colon \cdots : 0\colon 1\colon 1)$. But then, the intersection of H with the hyperplane $H_{d+1}(0)$ of equation $x_{d+1} = 0$ is given by

$$2\sum_{i=1}^{d}(b_i - a_i)x_i - (\sum_{i=1}^{d} b_i^2 - \sum_{i=1}^{d} a_i^2)x_{d+2} = 0.$$

Since we view $\mathbb{P}^d$ as the hyperplane $H_{d+1}(0) \subseteq \mathbb{P}^{d+1}$ and since the coordinates of points in $H_{d+1}(0)$ are of the form $(x_1\colon \cdots : x_d\colon 0\colon x_{d+2})$, the above equation is equivalent to the equation of $\psi_{d+1}(H_{a,b})$ in $\mathbb{P}^d$ in which x_{d+1} is replaced by x_{d+2}. $\square$

In order to define precisely Delaunay complexes as projections of objects obtained by deleting some faces from a projective polyhedron we need to define the notion of "projective (polyhedral) complex". However, this is easily done by defining the notion of cell complex where the cells are polyhedral cones. Such objects are known as *fans*. The definition below is basically Definition 6.8 in which the cells are cones as opposed to polytopes.

Definition 8.5 A *fan in* $\mathbb{E}^m$ is a set, K, consisting of a (finite or infinite) set of polyhedral cones in $\mathbb{E}^m$ satisfying the following conditions:

(1) Every face of a cone in K also belongs to K.

(2) For any two cones σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every cone, $\sigma \in K$, of dimension k, is called a *k-face* (or *face*) of K. A 0-face $\{v\}$ is called a *vertex* and a 1-face is called an *edge*. The *dimension* of the fan K is the maximum of the dimensions of all cones in K. If $\dim K = d$, then every face of dimension d is called a *cell* and every face of dimension $d-1$ is called a *facet*.

A *projective (polyhedral) complex*, $\mathcal{K} \subseteq \mathbb{P}^d$, is a set of projective polyhedra of the form, $\{\mathbb{P}(C) \mid C \in K\}$, where $K \subseteq \mathbb{R}^{d+1}$ is a fan.

Given a projective complex, the notions of face, vertex, edge, cell, facet, are dedined in the obvious way.

If $K \subseteq \mathbb{R}^d$ is a polyhedral complex, then it is easy to check that the set $\{C(\sigma) \mid \sigma \in K\} \subseteq \mathbb{R}^{d+1}$ is a fan and we get the projective complex

$$\widetilde{K} = \{\mathbb{P}(C(\sigma)) \mid \sigma \in K\} \subseteq \mathbb{P}^d.$$

The projective complex, $\widetilde{K}$, is called the *projective completion* of K. Also, it is easy to check that if $f\colon P \to P'$ is an injective affine map between two polyhedra P and P', then f extends uniquely to a projectivity, $\widetilde{f}\colon \widetilde{P} \to \widetilde{P'}$, between the projective completions of P and P'.

We now have all the facts needed to show that Delaunay triangulations and Voronoi diagrams can be defined in terms of the lifting, $\widetilde{\pi}_N \circ \psi_{d+1}$, and the projection, $\widetilde{\pi}_N$, and to establish their duality *via* polar duality with respect to S^d.

Definition 8.6 Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, the polytope, $\mathcal{D}(P) \subseteq \mathbb{R}^{d+1}$, called the *Delaunay polytope* associated with P is the convex hull of the union of the lifting of the points of P onto the sphere S^d (*via* inverse stereographic projection) with the north pole, that is, $\mathcal{D}(P) = \mathrm{conv}(\tau_N(P) \cup \{N\})$. The *projective Delaunay polytope*, $\widetilde{\mathcal{D}}(P) \subseteq \mathbb{P}^{d+1}$, associated with P is the projective completion of $\mathcal{D}(P)$. The polyhedral complex, $\mathcal{C}(P) \subseteq \mathbb{R}^{d+1}$, called the *lifted Delaunay complex of* P is the complex obtained from $\mathcal{D}(P)$ by deleting the facets containing the north pole (and their faces) and $\widetilde{\mathcal{C}}(P) \subseteq \mathbb{P}^{d+1}$ is the projective completion of $\mathcal{C}(P)$. The polyhedral complex, $\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \subseteq \mathbb{E}^d$, is the *Delaunay complex of* P or *Delaunay triangulation of* P.

The above is not the "standard" definition of the Delaunay triangulation of P but it is equivalent to the definition, say given in Boissonnat and Yvinec [8], as we will prove shortly. It also has certain advantages over lifting onto a paraboloid, as we will explain. Furthermore, to be perfectly rigorous, we should define $\mathcal{D}el(P)$ by

$$\mathcal{D}el(P) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \cap U_{d+1}),$$

but $\widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \subseteq U_{d+1}$ because $\mathcal{C}(P)$ is strictly below the hyperplane H_{d+1}.

It it possible and useful to define $\mathcal{D}el(P)$ more directly in terms of $\mathcal{C}(P)$. The projection, $\widetilde{\pi_N}\colon (\mathbb{P}^{d+1} - \{N\}) \to \mathbb{P}^d$, comes from the linear map, $\widehat{\pi}_N\colon \mathbb{R}^{d+2} \to \mathbb{R}^{d+1}$, given by

$$\widehat{\pi}_N(x_1, \ldots, x_{d+1}, x_{d+2}) = (x_1, \ldots, x_d, x_{d+2} - x_{d+1}).$$

Consequently, as $\widetilde{\mathcal{C}}(P) = \widetilde{\mathcal{C}(P)} = \mathbb{P}(C(\mathcal{C}(P)))$, we immediately check that

$$\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \varphi_{d+1} \circ \widehat{\pi}_N(C(\mathcal{C}(P))) = \varphi_{d+1} \circ \widehat{\pi}_N(\mathrm{cone}(\widehat{\mathcal{C}(P)})),$$

where $\widehat{\mathcal{C}(P)} = \{\widehat{u} \mid u \in \mathcal{C}(P)\}$ and $\widehat{u} = (u, 1)$.

This suggests defining the map, $\pi_N\colon (\mathbb{R}^{d+1} - H_{d+1}) \to \mathbb{R}^d$, by

$$\pi_N = \varphi_{d+1} \circ \widehat{\pi}_N \circ i_{d+2},$$

which is explicity given by

$$\pi_N(x_1, \ldots, x_d, x_{d+1}) = \frac{1}{1 - x_{d+1}}(x_1, \ldots, x_d).$$

Then, as $\mathcal{C}(P)$ is strictly below the hyperplane H_{d+1}, we have

$$\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \pi_N(\mathcal{C}(P)).$$

First, note that $\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P))$ is indeed a polyhedral complex whose geometric realization is the convex hull, $\mathrm{conv}(P)$, of P. Indeed, by Proposition 8.6, the images of the facets of $\mathcal{C}(P)$ are polytopes and when any two such polytopes meet, they meet along a common face. Furthermore, if $\dim(\mathrm{conv}(P)) = m$, then $\mathcal{D}el(P)$ is pure m-dimensional. First, $\mathcal{D}el(P)$ contains at least one m-dimensional cell. If $\mathcal{D}el(P)$ was not pure, as the complex is connected there would be some cell, σ, of dimension $s < m$ meeting some other cell, τ, of dimension m along a common face of dimension at most s and because σ is not contained in any face of dimension m, no facet of τ containing $\sigma \cap \tau$ can be adjacent to any cell of dimension m and so, $\mathcal{D}el(P)$ would not be convex, a contradiction.

For any polytope, $P \subseteq \mathbb{E}^d$, given any point, x, not in P, recall that a facet, F, of P is *visible from* x iff for every point, $y \in F$, the line through x and y intersects F only in y. If $\dim(P) = d$, this is equivalent to saying that x and the interior of P are strictly separated by the supporting hyperplane of F. Note that if $\dim(P) < d$, it possible that every facet of P is visible from x.

Now, assume that $P \subseteq \mathbb{E}^d$ is a polytope with nonempty interior. We say that a facet, F, of P is a *lower-facing facet* of P iff the unit normal to the supporting hyperplane of F pointing towards the interior of P has non-negative x_{d+1}-coordinate. A facet, F, that is not lower-facing is called an *upper-facing facet* (Note that in this case the x_{d+1} coordinate of the unit normal to the supporting hyperplane of F pointing towards the interior of P is strictly negative).

Here is a convenient way to characterize lower-facing facets.

Proposition 8.9 *Given any polytope, $P \subseteq \mathbb{E}^d$, with nonempty interior, for any point, c, on the Ox_d-axis, if c lies strictly above all the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F, then the lower-facing facets of P are exactly the facets not visible from c.*

Proof. Note that the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of P are strictly above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the lower-facing facets. Suppose F is visible from c. Then, F must not be lower-facing as otherwise, for any $y \in F$, the line through c and y has to intersect some upper-facing facet and F is not be visible from c, a contradiction.

Now, as P is the intersection of the closed half-spaces determined by the supporting hyperplanes of its facets, by the definition of an upper-facing facet, any point, c, on the Ox_d-axis that lies strictly above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F has the property that c and the interior of P are strictly separated by all these supporting hyperplanes. Therefore, all the upper-facing facets of P are visible from c. It follows that the facets visible from c are exactly the upper-facing facets, as claimed. $\square$

We will also need the following fact when $\dim(P) = d$.

Proposition 8.10 *Given any polytope, $P \subseteq \mathbb{E}^d$, if $\dim(P) = d$, then there is a point, c, on the Ox_d-axis, such that for all points, x, on the Ox_d-axis and above c, the set of facets of* conv$(P \cup \{x\})$ *not containing* x *is identical. Moreover, the set of facets of* P *not visible from* x *is the set of facets of* conv$(P \cup \{x\})$ *that do not contain* x.

Proof. If $\dim(P) = d$ then pick any c on the Ox_d-axis above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F. Then, c is in general position w.r.t. P in the sense that c and any d vertices of P do not lie in a common hyperplane. Now, our result follows by lemma 8.3.1 of Boissonnat and Yvinec [8]. $\square$

Corollary 8.11 *Given any polytope,* $P \subseteq \mathbb{E}^d$, *with nonempty interior, there is a point,* c, *on the* Ox_d-*axis, so that for all* x *on the* Ox_d-*axis and above* c, *the lower-facing facets of* P *are exactly the facets of* conv$(P \cup \{x\})$ *that do not contain* x.

As usual, let $e_{d+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$.

Theorem 8.12 *Given any set of points,* $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, *let* $\mathcal{D}'(P)$ *denote the polyhedron* conv$(l(P))$ + cone(e_{d+1}) *and let* $\widetilde{\mathcal{D}}'(P)$ *be the projective completion of* $\mathcal{D}'(P)$. *Also, let* $\mathcal{C}'(P)$ *be the polyhedral complex consisting of the bounded facets of the polytope* $\mathcal{D}'(P)$ *and let* $\widetilde{\mathcal{C}}'(P)$ *be the projective completion of* $\mathcal{C}'(P)$. *Then*

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P) \quad \textit{and} \quad \theta(\widetilde{\mathcal{C}}(P)) = \widetilde{\mathcal{C}}'(P).$$

Furthermore, if $\mathcal{D}el'(P) = \varphi_{d+1} \circ \widetilde{p}_{d+1}(\widetilde{\mathcal{C}}'(P)) = p_{d+1}(\mathcal{C}'(P))$ *is the "standard" Delaunay complex of* P, *that is, the orthogonal projection of* $\mathcal{C}'(P)$ *onto* $\mathbb{E}^d$, *then*

$$\mathcal{D}el(P) = \mathcal{D}el'(P).$$

Therefore, the two notions of a Delaunay complex agree. If $\dim(\mathrm{conv}(P)) = d$, *then the bounded facets of* conv$(l(P))$ + cone(e_{d+1}) *are precisely the lower-facing facets of* conv$(l(P))$.

Proof. Recall that

$$\mathcal{D}(P) = \mathrm{conv}(\tau_N(P) \cup \{N\})$$

and $\widetilde{\mathcal{D}}(P) = \mathbb{P}(C(\mathcal{D}(P)))$ is the projective completion of $\mathcal{D}(P)$. If we write $\widehat{\tau_N(P)}$ for $\{\widehat{\tau_N(p_i)} \mid p_i \in P\}$, then

$$C(\mathcal{D}(P)) = \mathrm{cone}(\widehat{\tau_N(P)} \cup \{\widehat{N}\}).$$

By definition, we have

$$\theta(\widetilde{\mathcal{D}}) = \mathbb{P}(\widehat{\theta}(C(\mathcal{D}))).$$

Now, as $\widehat{\theta}$ is linear,

$$\widehat{\theta}(C(\mathcal{D})) = \widehat{\theta}(\mathrm{cone}(\widehat{\tau_N(P)} \cup \{\widehat{N}\})) = \mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}).$$

We claim that

$$\begin{aligned}
\mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}) &= \mathrm{cone}(\widehat{l(P)} \cup \{(0, \ldots, 0, 1, 1)\}) \\
&= C(\mathcal{D}'(P)),
\end{aligned}$$

where

$$\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1}).$$

Indeed,

$$\widehat{\theta}(x_1, \ldots, x_{d+2}) = (x_1, \ldots, x_d, x_{d+1} + x_{d+2}, x_{d+2} - x_{d+1}),$$

and for any $p_i = (x_1, \ldots, x_d) \in P$,

$$\begin{aligned}
\widehat{\tau_N(p_i)} &= \left(\frac{2x_1}{\sum_{i=1}^d x_i^2 + 1}, \ldots, \frac{2x_d}{\sum_{i=1}^d x_i^2 + 1}, \frac{\sum_{i=1}^d x_i^2 - 1}{\sum_{i=1}^d x_i^2 + 1}, 1 \right) \\
&= \frac{1}{\sum_{i=1}^d x_i^2 + 1} \left(2x_1, \ldots, 2x_d, \sum_{i=1}^d x_i^2 - 1, \sum_{i=1}^d x_i^2 + 1 \right),
\end{aligned}$$

so we get

$$\begin{aligned}
\widehat{\theta}(\widehat{\tau_N(p_i)}) &= \frac{2}{\sum_{i=1}^d x_i^2 + 1} \left(x_1, \ldots, x_d, \sum_{i=1}^d x_i^2, 1 \right) \\
&= \frac{2}{\sum_{i=1}^d x_i^2 + 1} \, \widehat{l(p_i)}.
\end{aligned}$$

Also, we have

$$\widehat{\theta}(\widehat{N}) = \widehat{\theta}(0, \ldots, 0, 1, 1) = (0, \ldots, 0, 2, 0) = 2\widehat{e_{d+1}},$$

and by definition of cone($-$) (scalar factors are irrelevant), we get

$$\mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}) = \mathrm{cone}(\widehat{l(P)} \cup \{(0, \ldots, 0, 1, 1)\}) = C(\mathcal{D}'(P)),$$

with $\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$, as claimed. This proves that

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P).$$

Now, it is clear that the facets of $\mathrm{conv}(\tau_N(P) \cup \{N\})$ that do not contain N are mapped to the bounded facets of $\mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$, since N goes the point at infinity, so

$$\theta(\widetilde{\mathcal{C}}(P)) = \widetilde{\mathcal{C}}'(P).$$

As $\widetilde{\pi}_N = \widetilde{p}_{d+1} \circ \theta$ by Proposition 8.5, we get

$$\mathcal{D}el'(P) = \varphi_{d+1} \circ \widetilde{p}_{d+1}(\widetilde{\mathcal{C}}'(P)) = \varphi_{d+1} \circ (\widetilde{p}_{d+1} \circ \theta)(\widetilde{\mathcal{C}}(P)) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \mathcal{D}el(P),$$

as claimed. Finally, if $\dim(\mathrm{conv}(P)) = d$, then, by Corollary 8.11, we can pick a point, c, on the Ox_{d+1}-axis, so that the facets of $\mathrm{conv}(l(P) \cup \{c\})$ that do not contain c are precisely the lower-facing facets of $\mathrm{conv}(l(P))$. However, it is also clear that the facets of $\mathrm{conv}(l(P) \cup \{c\})$ that contain c tend to the unbounded facets of $\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$ when c goes to $+\infty$. $\square$

We can also characterize when the Delaunay complex, $\mathcal{D}el(P)$, is simplicial. Recall that we say that a set of points, $P \subseteq \mathbb{E}^d$, is in *general position* iff no $d + 2$ of the points in P belong to a common $(d - 1)$-sphere.

Proposition 8.13 *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, if P is in general position, then the Delaunay complex, $\mathcal{D}el(P)$, is a pure simplicial complex.*

Proof. Let $\dim(\mathrm{conv}(P)) = r$. Then, $\tau_N(P)$ is contained in a $(r-1)$-sphere of S^d, so we may assume that $r = d$. Suppose $\mathcal{D}el(P)$ has some facet, F, which is not a d-simplex. If so, F is the convex hull of at least $d+2$ points, $p_1, \ldots, p_k$ of P and since $F = \pi_N(\widehat{F})$, for some facet, $\widehat{F}$, of $\mathcal{C}(P)$, we deduce that $\tau_N(p_1), \ldots, \tau_N(p_k)$ belong to the supporting hyperplane, H, of $\widehat{F}$. Now, if H passes through the north pole, then we know that $p_1, \ldots, p_k$ belong to some hyperplane of $\mathbb{E}^d$, which is impossible since $p_1, \ldots, p_k$ are the vertices of a facet of dimension d. Thus, H does not pass through N and so, $p_1, \ldots, p_k$ belong to some $(d - 1)$-sphere in $\mathbb{E}^d$. As $k \geq d + 2$, this contradicts the assumption that the points in P are in general position. $\square$

Remark: Even when the points in P are in general position, the Delaunay polytope, $\mathcal{D}(P)$, may not be a simplicial polytope. For example, if $d + 1$ points belong to a hyperplane in $\mathbb{E}^d$, then the lifted points belong to a hyperplane passing through the north pole and these $d + 1$ lifted points together with N may form a non-simplicial facet. For example, consider the polytope obtained by lifting our original $d+1$ points on a hyperplane, H, plus one more point not in the the hyperplane H.

We can also characterize the Voronoi diagram of P in terms of the polar dual of $\mathcal{D}(P)$. Unfortunately, we can't simply take the polar dual, $\mathcal{D}(P)^*$, of $\mathcal{D}(P)$ and project it using π_N because some of the facets of $\mathcal{D}(P)^*$ may intersect the hyperplane, H_{d+1}, and π_N is undefined on H_{d+1}. However, using projective completions, we can indeed recover the Voronoi diagram of P.

Definition 8.7 Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, the *Voronoi polyhedron* associated with P is the polar dual (w.r.t. $S^d \subseteq \mathbb{R}^{d+1}$), $\mathcal{V}(P) = (\mathcal{D}(P))^* \subseteq \mathbb{R}^{d+1}$, of the Delaunay polytope, $\mathcal{D}(P) = \mathrm{conv}(\tau_N(P) \cup \{N\})$. The *projective Voronoi polytope*, $\widetilde{\mathcal{V}}(P) \subseteq \mathbb{P}^{d+1}$, associated with P is the projective completion of $\mathcal{V}(P)$. The polyhedral complex, $\mathcal{V}or(P) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \subseteq \mathbb{E}^d$, is the *Voronoi complex of P* or *Voronoi diagram of P*.

Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, let $\mathcal{V}'(P) = (\mathcal{D}'(P))^*$ be the polar dual (w.r.t. $\mathcal{P} \subseteq \mathbb{R}^{d+1}$) of the "standard" Delaunay polyhedron defined in Theorem 8.12 and let $\widetilde{\mathcal{V}}'(P) = \widetilde{\mathcal{V}'(P)} \subseteq \mathbb{P}^d$ be its projective completion. It is not hard to check that

$$p_{d+1}(\mathcal{V}'(P)) = \varphi_{d+1}(\widetilde{p}_{d+1}(\widetilde{\mathcal{V}}'(P)) \cap U_{d+1})$$

is the "standard" Voronoi diagram, denoted $\mathcal{V}or'(P)$.

Theorem 8.14 *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, we have*

$$\theta(\widetilde{\mathcal{V}}(P)) = \widetilde{\mathcal{V}}'(P)$$

and

$$\mathcal{V}or(P) = \mathcal{V}or'(P).$$

Therefore, the two notions of Voronoi diagrams agree.

Proof. By definition,

$$\widetilde{\mathcal{V}}(P) = \widetilde{\mathcal{V}(P)} = (\widetilde{\mathcal{D}(P)})^*$$

and by Proposition 5.12,

$$(\widetilde{\mathcal{D}(P)})^* = \left(\widetilde{\mathcal{D}(P)}\right)^* = (\widetilde{\mathcal{D}}(P))^*,$$

so

$$\widetilde{\mathcal{V}}(P) = (\widetilde{\mathcal{D}}(P))^*.$$

By Proposition 5.10,

$$\theta(\widetilde{\mathcal{V}}(P)) = \theta((\widetilde{\mathcal{D}}(P))^*) = (\theta(\widetilde{\mathcal{D}}(P)))^*$$

and by Theorem 8.12,

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P),$$

so we get

$$\theta(\widetilde{\mathcal{V}}(P)) = (\widetilde{\mathcal{D}}'(P))^*.$$

But, by Proposition 5.12 again,

$$(\widetilde{\mathcal{D}}'(P))^* = \left(\widetilde{\mathcal{D}'(P)}\right)^* = (\widetilde{\mathcal{D}'(P)})^* = \widetilde{\mathcal{V}'(P)} = \widetilde{\mathcal{V}}'(P).$$

Therefore,

$$\theta(\widetilde{\mathcal{V}}(P)) = \widetilde{\mathcal{V}}'(P),$$

as claimed.

As $\widetilde{\pi}_N = \widetilde{p}_{d+1} \circ \theta$ by Proposition 8.5, we get

$$\begin{aligned}
\mathcal{V}or'(P) &= \varphi_{d+1}(\widetilde{p}_{d+1}(\widetilde{\mathcal{V}}'(P)) \cap U_{d+1}) \\
&= \varphi_{d+1}(\widetilde{p}_{d+1} \circ \theta(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \\
&= \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \\
&= \mathcal{V}or(P),
\end{aligned}$$

finishing the proof. $\square$

We can also prove the proposition below which shows directly that $\mathcal{V}or(P)$ is the Voronoi diagram of P. Recall that that $\widetilde{\mathcal{V}}(P)$ is the projective completion of $\mathcal{V}(P)$. We observed in Section 5.2 (see page 86) that in the patch U_{d+1}, there is a bijection between the faces of $\widetilde{\mathcal{V}}(P)$ and the faces of $\mathcal{V}(P)$. Furthermore, the projective completion, $\widetilde{H}$, of every hyperplane, $H \subseteq \mathbb{R}^d$, is also a hyperplane and it is easy to see that if H is tangent to $\mathcal{V}(P)$, then $\widetilde{H}$ is tangent to $\widetilde{\mathcal{V}}(P)$.

Proposition 8.15 *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, for every $p \in P$, if F is the facet of $\mathcal{V}(P)$ that contains $\tau_N(p)$, if H is the tangent hyperplane at $\tau_N(p)$ to S^d and if F is cut out by the hyperplanes $H, H_1, \ldots, H_{k_p}$, in the sense that*

$$F = (H \cap H_1)_- \cap \cdots \cap (H \cap H_{k_p})_-,$$

where $(H \cap H_i)_-$ denotes the closed half-space in H containing $\tau_N(p)$ determined by $H \cap H_i$, then

$$V(p) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_1)_- \cap \cdots \cap \widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_{k_p})_- \cap U_{d+1})$$

is the Voronoi region of p (where $\varphi_{d+1}(\widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_i)_- \cap U_{d+1})$ is the closed half-space containing p). If P is in general position, then $\mathcal{V}(P)$ is a simple polyhedron (every vertex belongs to $d+1$ facets).

Proof. Recall that by Proposition 8.5,

$$\widetilde{\tau}_N \circ \psi_{d+1} = \psi_{d+2} \circ \tau_N.$$

Each $H_i = T_{\tau_N(p_i)} S^d$ is the tangent hyperplane to S^d at $\tau_N(p_i)$, for some $p_i \in P$. Now, by definition of the projective completion, the embedding, $\mathcal{V}(P) \longrightarrow \widetilde{\mathcal{V}}(P)$, is given by $a \mapsto \psi_{d+2}(a)$. Thus, every point, $p \in P$, is mapped to the point $\psi_{d+2}(\tau_N(p)) = \widetilde{\tau_N}(\psi_{d+1}(p))$ and we also have $\widetilde{H}_i = T_{\widetilde{\tau}_N \circ \psi_{d+1}(p_i)} S^d$ and $\widetilde{H} = T_{\widetilde{\tau}_N \circ \psi_{d+1}(p)} S^d$. By Proposition 8.8,

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(p)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(p_i)} S^d) = \psi_{d+1}(H_{p,p_i})$$

is the embedding of the bisector hyperplane of p and p_i in $\mathbb{P}^d$, so the first part holds.

Now, assume that some vertex, $v \in \mathcal{V}(P) = \mathcal{D}(P)^*$, belongs to $k \geq d+2$ facets of $\mathcal{V}(P)$. By polar duality, this means that the facet, F, dual of v has $k \geq d+2$ vertices $\tau_N(p_1), \ldots, \tau_N(p_k)$ of $\mathcal{D}(P)$. We claim that $\tau_N(p_1), \ldots, \tau_N(p_k)$ must belong to some hyperplane passing through the north pole. Otherwise, $\tau_N(p_1), \ldots, \tau_N(p_k)$ would belong to a hyperplane not passing through the north pole and so they would belong to a $(d-1)$ sphere of S^d and thus, $p_1, \ldots, p_k$ would belong to a $(d-1)$-sphere even though $k \geq d+2$, contradicting that P is in general position. But then, by polar duality, v would be a point at infinity, a contradiction. $\square$

Note that when P is in general position, even though the polytope, $\mathcal{D}(P)$, may not be simplicial, its dual, $\mathcal{V}(P) = \mathcal{D}(P)^*$, is a simple *polyhedron*. What is happening is that $\mathcal{V}(P)$ has unbounded faces which have "vertices at infinity" that do not count! In fact, the faces of $\mathcal{D}(P)$ that fail to be simplicial are those that are contained in some hyperplane through the north pole. By polar duality, these faces correspond to a vertex at infinity. Also, if $m = \dim(\mathrm{conv}(P)) < d$, then $\mathcal{V}(P)$ may not have any vertices!

We conclude our presentation of Voronoi diagrams and Delaunay triangulations with a short section on applications.

8.7 Applications of Voronoi Diagrams and Delaunay Triangulations

The examples below are taken from O'Rourke [31]. Other examples can be found in Preparata and Shamos [32], Boissonnat and Yvinec [8], and de Berg, Van Kreveld, Overmars, and Schwarzkopf [5].

The first example is the *nearest neighbors* problem. There are actually two subproblems: *Nearest neighbor queries* and *all nearest neighbors*.

The nearest neighbor queries problem is as follows. Given a set P of points and a query point q, find the nearest neighbor(s) of q in P. This problem can be solved by computing the Voronoi diagram of P and determining in which Voronoi region q falls. This last problem, called *point location*, has been heavily studied (see O'Rourke [31]). The all neighbors problem is as follows: Given a set P of points, find the nearest neighbor(s) to all points in P. This problem can be solved by building a graph, the *nearest neighbor graph*, for short *nng*. The nodes of this undirected graph are the points in P, and there is an arc from p to q iff p is a nearest neighbor of q or vice versa. Then it can be shown that this graph is contained in the Delaunay triangulation of P.

The second example is the *largest empty circle*. Some practical applications of this problem are to locate a new store (to avoid competition), or to locate a nuclear plant as far as possible from a set of towns. More precisely, the problem is as follows. Given a set P of points, find a largest empty circle whose center is in the (closed) convex hull of P, empty in that it contains no points from P inside it, and largest in the sense that there is no other circle with strictly larger radius. The Voronoi diagram of P can be used to solve this problem. It can be shown that if the center p of a largest empty circle is strictly inside the convex hull of P, then p coincides with a Voronoi vertex. However, not every Voronoi vertex is a good candidate. It can also be shown that if the center p of a largest empty circle lies on the boundary of the convex hull of P, then p lies on a Voronoi edge.

The third example is the *minimum spanning tree*. Given a graph G, a minimum spanning tree of G is a subgraph of G that is a tree, contains every vertex of the graph G, and minimizes the sum of the lengths of the tree edges. It can be shown that a minimum spanning tree is a subgraph of the Delaunay triangulation of the vertices of the graph. This can be used to improve algorithms for finding minimum spanning trees, for example Kruskal's algorithm (see O'Rourke [31]).

We conclude by mentioning that Voronoi diagrams have applications to *motion planning*. For example, consider the problem of moving a disk on a plane while avoiding a set of polygonal obstacles. If we "extend" the obstacles by the diameter of the disk, the problem reduces to finding a collision–free path between two points in the extended obstacle space. One needs to generalize the notion of a Voronoi diagram. Indeed, we need to define the distance to an object, and medial curves (consisting of points equidistant to two objects) may no longer be straight lines. A collision–free path with maximal clearance from the obstacles can be found by moving along the edges of the generalized Voronoi diagram. This is an active area of research in robotics. For more on this topic, see O'Rourke [31].

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